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ON METALLIC SEMI-SYMMETRIC METRIC F−CONNECTIONS

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Abstract. In this article, we generate a metallic semi-symmetric metric F-connection on a locally decomposable metallic Riemann manifold. Also, we examine some features of torsion and curvature tensor fields of this connection.

1. Introduction

The topic of connection with torsion on a Riemann manifold has been studied with great interest in literature. Firstly, Hayden defined the concept of metric connection with torsion [3]. For a linear connection $\widetilde{\nabla}$ with torsion on a Riemann manifold (M, g), if $\widetilde{\nabla} g = 0$, then linear connection $\widetilde{\nabla}$ is called a metric connection. Then, Yano constructed a connection whose torsion tensor has the form: $S(X,Y) = \omega(Y)X - \omega(X)Y$, where ω is a 1−form, [15] and named this connection as semi-symmetric connection.

In [11], Prvanovic has defined a product semi-symmetric F−connection on locally decomposable Riemann manifold and worked its curvature properties. A locally decomposable Riemann manifold is expressed by the triple (M, g, F) and the conditions $\nabla F = 0$ and $g(FX, Y) = g(X, FY)$ are provided, where F, g and ∇ are product structure, metric tensor and Riemann connection (or Levi-Civita connection) of g on manifold respectively. For further references, see [8, 9, 10, 12].

The positive root of the equation $x^2 - x - 1 = 0$ is the number $x_1 = \frac{1+\sqrt{5}}{2}$, which is called golden ratio. The golden ratio has many applications and has played an important role in mathematics. One of them is a golden Riemann manifold (M, g, φ) endowed with golden structure φ and Riemann metric tensor g. The golden structure φ created by Crasmareanu and Hretcanu is actually root of the equality $\varphi^2 - \varphi - I = 0$ [5]. In [2], the authors have defined golden semi-symmetric metric F−connections on a locally decomposable golden Riemann manifold and examined torsion, projective curvature, conharmonic curvature and curvature tensors of this connection. Also, the golden ratio has many important generalizations. One

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of the them is metallic proportions or metallic means family which was introduced by de Spinadel in [6, 7]. The positive root of the equation $x^2 - px - q = 0$ is called the metallic means family, where p and q are two positive integer. Also, the solution of the metallic means family is as follows

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

These numbers $\sigma_{p,q}$ are also named (p, q) metallic numbers. In the last equation,

- if $p = q = 1$, then the number $\sigma_{1,1} = \frac{1+\sqrt{5}}{2}$ is golden ratio;
- if $p = 2$ and $q = 1$, then the number $\sigma_{2,1} = 1 + \sqrt{2}$ is silver ratio, which is used for fractal and Cantorian geometry;
- if $p = 3$ and $q = 1$, then the number $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$ is bronze ratio, which plays an important role in dynamical systems and quasicrystals and so on.

Inspired by the metallic number family, Hretcanu and Crasmareanu was introduced metallic Riemann structure [4]. Indeed, a metallic structure is polynomial structure such that $F^2 - pF - qI = 0$, where F is $(1, 1)$-tensor field on manifold. Given a Riemann manifold (M, g) endowed with the metallic structure F, if

$$g(FX, Y) = g(X, FY)$$

or equivalently

$$g(FX, FY) = pg(FX, Y) + qg(X, Y)$$

for all vector fields X and Y on M, then the triple (M, g, F) is called a metallic Riemann manifold.

In [1], For almost product structures J and the Tachibana operator ϕ_F, the authors proved that the manifold (M, g, F) is a locally decomposable metallic Riemannian manifold iff $\phi_{J_\pm} g = 0$. In this article, we made a semi-symmetric metric $F-$connection with metallic structure F on a locally decomposable metallic Riemann manifold. Then we examine some properties related to its torsion and curvature tensors.

2. Preliminaries

Let M be an n-dimensional manifold. Throughout this paper, tensor fields, connections and all manifolds are always assumed to be differentiable of class C^∞

For a $(1, 1)-$tensor F and a $(r, s)-$tensor K, The tensor K is named as a pure tensor with regard to the tensor F, if the following condition is holds:

$$\begin{aligned} K^{j_1 \dots j_r}_{m i_2 \dots i_s} F^{\ m}_{i_1} &= K^{j_1 \dots j_r}_{i_1 m \dots i_s} F^{\ m}_{i_2} = \dots = K^{j_1 \dots j_r}_{i_1 i_2 \dots m} F^{\ m}_{i_s} = \\ K^{m j_2 \dots j_r}_{i_1 \dots i_s} F^{\ j_1}_{m} &= K^{j_1 m \dots j_r}_{i_1 \dots i_s} F^{\ j_2}_{m} = \dots = K^{j_1 j_2 \dots m}_{i_1 \dots i_s} F^{\ j_r}_{m}, \end{aligned}$$

where $K^{j_1 j_2 \dots j r}_{i_1 i_2 \dots i s}$ and $F_i^{\ j}$ is the components the tensor K and $(1, 1)-$tensor F respectively. Also, the Tachibana operator applied to a pure $(r, s)-$tensor K is given by

$$\begin{aligned}(\phi_F K)^{j_1\ldots j_r}_{ki_1\ldots i_s} &= F_k{}^m \partial_m t^{j_1\ldots j_r}_{i_1\ldots i_s} - \partial_k (K\circ F)^{j_1\ldots j_r}_{i_1\ldots i_s} \\ &+ \sum_{\lambda=1}^{s} (\partial_{i_\lambda} F_k{}^m)\ K^{j_1\ldots j_r}_{i_1\ldots m\ldots i_s} \\ &+ \sum_{\mu=1}^{r} \left(\partial_k F_m{}^{j_\mu} - \partial_m F_k{}^{j_\mu}\right)\ K^{i_1\ldots m\ldots i_r}_{j_1\ldots j_s},\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}(K\circ F)^{j_1\ldots j_r}_{i_1\ldots i_s} &= K^{j_1\ldots j_r}_{mi_2\ldots i_s} F_{i_1}^{\ m} = \ldots = K^{j_1\ldots j_r}_{i_1 i_2\ldots m} F_{i_s}^{\ m} \\ &= K^{mj_2\ldots j_r}_{i_1\ldots i_s} F_m^{\ j_1} = \ldots = K^{j_1 j_2\ldots m}_{i_1\ldots i_s} F_m^{\ j_r}.\end{aligned}$$

The equation (2.1) firstly defined by Tachibana [14] and the applications of this operator have been made by many authors [13, 16]. For the pure tensor K, if the condition $\phi_F K = 0$ holds, then K is called as a $\phi-$tensor. Specially, if the $(1,1)-$tensor F is a product structure, then K is a decomposable tensor [14].

A metallic Riemannian manifold is a manifold M equipped with a $(1,1)-$tensor field F and a Riemannian metric g which satisfy the following conditions:

$$F^2 - pF - qI = 0 \tag{2.2}$$

and

$$g(FX, Y) = g(X, FY) \tag{2.3}$$

Also, the equation (2.3) equal to $g(FX, FY) = pg(FX, Y) + qg(X, Y)$, where p, q are positive integers. The last two equations in local coordinates are as follows:

$$F_i{}^k F_k{}^j = pF_i{}^j + q\delta_i^j \tag{2.4}$$

and

$$F_i{}^k g_{kj} = F_j{}^k g_{ik}, \tag{2.5}$$

It is obvious that $F_i{}^k F_{kj} = pF_{ij} + qg_{ij}$ and $F_{ij} = F_{ji}$ (symmetry) from (2.4) and (2.5). The almost product structure J and metallic structure F on M are related to each other as follows [4],

$$J_\pm = \frac{p}{2} I \pm (\frac{2\sigma_{p,q} - p}{2}) F \tag{2.6}$$

or conversely

$$F_\pm = \pm\left(\frac{2}{2\sigma_{p,q} - p} J - \frac{p}{2\sigma_{p,q} - p} I\right), \tag{2.7}$$

where $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ which is the root of the (2.2). Also, it is obvious from (2.7) that a Riemann metric g is pure with regard to a metallic structure F if and only if the Riemann metric g is pure with regard to the almost product structure J. By using (2.7) and (2.1), we have

$$\phi_F K = \pm \frac{2}{2\sigma_{p,q} - p} \phi_J K \tag{2.8}$$

for any $(r,s)-$tensor K. We note that a metallic Riemann manifold (M,g,F) is a locally decomposable metallic Riemann manifold if and only if the Riemann metric g is a decomposable tensor, i.e., $(\phi_J g)_{kij} = 0$ and the condition $(\phi_J g)_{kij} = 0$ is equivalent to $\nabla_k J_i^{\ j} = 0$ [1].

3. The Metallic Semi-Symmetric metric $F-$connection

Let (M,g,F) be a locally decomposable metallic Riemann manifold. We consider an affine connection $\widetilde{\nabla}$ on M. If the affine connection $\widetilde{\nabla}$ holds

$$\begin{aligned} i)\ \widetilde{\nabla}_h g_{ij} &= 0 \\ ii)\ \widetilde{\nabla}_h F_i^{\ j} &= 0, \end{aligned} \tag{3.1}$$

then it is called a metric $F-$connection. In the special case, when the torsion tensor $\widetilde{S}_{ij}^k$ of $\widetilde{\nabla}$ is as following shape

$$\widetilde{S}_{ij}^k = \omega_j \delta_i^k - \omega_i \delta_j^k + \frac{1}{q}\left(\omega_t F_j^{\ t} F_i^{\ k} - \omega_t F_i^{\ t} F_j^{\ k}\right), \tag{3.2}$$

where ω_i are local ingredients of an $1-$form, we say that the affine connection $\widetilde{\nabla}$ is a metallic semi-symmetric metric connection.

Let $\widetilde{\Gamma}_{ij}^k$ be the ingredients of the metallic semi-symmetric metric connection $\widetilde{\nabla}$. If we put

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + T_{ij}^k, \tag{3.3}$$

where Γ_{ij}^k and T_{ij}^k are the ingredients of the Riemann connection ∇ of g and $(1,2)-$tensor field T on M respectively, then the torsion tensor $\widetilde{S}_{ij}^k$ of $\widetilde{\nabla}$ is as following form

$$\widetilde{S}_{ij}^k = \widetilde{\Gamma}_{ij}^k - \widetilde{\Gamma}_{ji}^k = T_{ij}^k - T_{ji}^k.$$

When the connection (3.3) provides the condition (i) of (3.1), by applying the method in [3], we get

$$T_{ij}^k = \omega_j \delta_i^k - \omega^k g_{ij} + \frac{1}{q}\left(\omega_t F_j^{\ t} F_i^{\ k} - \omega_t F^{kt} F_{ij}\right),$$

where $\omega^k = \omega_i g^{ik}$, $F^{kt} = F_i^{\ t} g^{ik}$ and $F_{ij} = F_j^{\ k} g_{ik}$. Hence the connection (3.3) becomes the following form

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \omega_j \delta_i^k - \omega^k g_{ij} + \frac{1}{q}\left(\omega_t F_j^{\ t} F_i^{\ k} - \omega_t F^{kt} F_{ij}\right). \tag{3.4}$$

Also, by using the connection (3.4), we obtain the following equation with a simple calculation:

$$\widetilde{\nabla}_k F_i^{\ j} = g_{ki}(\omega^t F_t^{\ j} - \omega_t F^{jt}) = 0.$$

Therefore, the connection $\widetilde{\nabla}$ given by (3.4) is named metallic semi-symmetric metric $F-$connection.

4. Curvature and Torsion properties of the Metallic Semi-Symmetric metric $F-$connection

In this section, we examine some properties associated with the torsion and curvature tensor of the connection (3.4).

Let (M, g, F) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). We say easily that the torsion tensor $\widetilde{S}$ of the connection (3.4) is pure. Indeed, by using (2.4) and (3.2), we get

$$\widetilde{S}^k_{im} F_j^{\ m} = \widetilde{S}^k_{mj} F_i^{\ m} = \widetilde{S}^m_{ij} F_m^{\ k}.$$

In [13], the author prove that a $F-$connection is pure iff torsion tensor of that connection is pure. Thus, the connection (3.4) provides the following condition:

$$\widetilde{\Gamma}^k_{mj} F_i^{\ m} = \widetilde{\Gamma}^k_{im} F_j^{\ m} = \widetilde{\Gamma}^m_{ij} F_m^{\ k}.$$

Theorem 4.1. *Let (M, g, F) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). If the $1-$form ω is a $\phi-$tensor,then the torsion tensor $\widetilde{S}$ of the connection (3.4) is a $\phi-$tensor and holds following equation:*

$$F_k^{\ m}(\nabla_m \widetilde{S}^l_{ij}) = F_i^{\ m}(\nabla_k \widetilde{S}^l_{mj}) = F_j^{\ m}(\nabla_k \widetilde{S}^l_{im}). \tag{4.1}$$

Proof. Let (M, g, F) be a locally decomposable metallic Riemann manifold. Since a zero tensor is pure, a $F-$connection with torsion-free is always pure. Hence, we can say that the Levi-Civita connection ∇ of g on M is always pure with respect to F.

If we implement the Tachibana operator ϕ_F to the torsion tensor $\widetilde{S}$ of the connection (3.4), then we have

$$\begin{aligned}(\phi_F \widetilde{S})_{kij}^{\ \ \ l} &= F_k^{\ m}(\partial_m \widetilde{S}^l_{ij}) - \partial_k(\widetilde{S}^l_{mj} F_i^{\ m}) \\ &= F_k^{\ m}(\nabla_m \widetilde{S}^l_{ij} + \Gamma^s_{mi}\widetilde{S}^l_{sj} + \Gamma^s_{mj}\widetilde{S}^l_{is} - \Gamma^l_{ms}\widetilde{S}^s_{ij}) \\ &\quad - F_i^{\ m}(\nabla_k \widetilde{S}^l_{mj} + \Gamma^s_{km}\widetilde{S}^l_{sj} + \Gamma^s_{kj}\widetilde{S}^l_{ms} - \Gamma^l_{ks}\widetilde{S}^s_{mj}).\end{aligned}$$

When the torsion tensor $\widetilde{S}$ and Levi-Civita connection ∇ are pure, the above relation reduces to

$$(\phi_F \widetilde{S})_{kij}^{\ \ \ l} = F_k^{\ m}(\nabla_m \widetilde{S}^l_{ij}) - F_i^{\ m}(\nabla_k \widetilde{S}^l_{mj}). \tag{4.2}$$

Substituting (3.2) into (4.2), we get

$$\begin{aligned}(\phi_F\widetilde{S})_{kij}{}^l &= [(\nabla_m\omega_j)F_k{}^m-(\nabla_k\omega_m)F_j{}^m]\delta_i^l \\ &-[(\nabla_m\omega_i)F_k{}^m-(\nabla_k\omega_m)F_i{}^m]\delta_j^l \\ &+[\frac{1}{q}(\nabla_m\omega_s)F_k{}^mF_j{}^s-\frac{p}{q}(\nabla_k\omega_s)F_j{}^s-\nabla_k\omega_j]F_i{}^l \\ &-[\frac{1}{q}(\nabla_m\omega_s)F_k{}^mF_i{}^s-\frac{p}{q}(\nabla_k\omega_s)F_i{}^s-\nabla_k\omega_i]F_j{}^l.\end{aligned} \tag{4.3}$$

Also, for the $1-$form ω, we calculate

$$\begin{aligned}(\phi_F\omega)_{kj} &= F_k{}^m(\partial_m\omega_j)-\partial_k(F_j{}^m\omega_m) \\ &= F_k{}^m(\nabla_m\omega_j+\Gamma^s_{mj}\omega_s)-F_j{}^m(\nabla_k\omega_m+\Gamma^s_{km}\omega_s) \\ &= F_k{}^m(\nabla_m\omega_j)-F_j{}^m(\nabla_k\omega_m).\end{aligned}$$

From last equation, we can say that the $1-$form ω is a $\phi-$tensor iff

$$F_k^m(\nabla_m p_j)=F_j^m(\nabla_k p_m). \tag{4.4}$$

Assuming that the $1-$form ω is a $\phi-$tensor, thanks to (2.4) the relation (4.3) becomes $(\phi_F\widetilde{S})_{kij}{}^l=0$, i.e., the torsion tensor $\widetilde{S}$ is a $\phi-$tensor. Also, from the equation (4.2) we get

$$F_k{}^m(\nabla_m\widetilde{S}^l_{ij})=F_i{}^m(\nabla_k\widetilde{S}^l_{mj})=F_j{}^m(\nabla_k\widetilde{S}^l_{im}).$$

The proof is complete. □

From the equation (2.8), it is obvious that the torsion tensor $\widetilde{S}$ of the connection (3.4) and the $1-$form ω are hold following equality

$$\phi_J\widetilde{S}=0 \text{ and } \phi_J\omega=0,$$

i.e., they are decomposable tensors, where J is the product structure associated with the metallic structure F. From on now, we shall consider $1-$form ω is a $\phi-$tensor (or decomposable tensor), i.e., the following conditions are provided:

$$F_k{}^m(\nabla_m\omega_j)=F_j{}^m(\nabla_k\omega_m)$$

and

$$J_k{}^m(\nabla_m\omega_j)=J_j{}^m(\nabla_k\omega_m).$$

It is well known that the curvature tensor $\widetilde{R}_{ijk}{}^l$ of the connection (3.4) is as follows:

$$\widetilde{R}_{ijk}{}^l=\partial_i\widetilde{\Gamma}^l_{jk}-\partial_j\widetilde{\Gamma}^l_{ik}+\widetilde{\Gamma}^l_{im}\widetilde{\Gamma}^m_{jk}-\widetilde{\Gamma}^l_{jm}\widetilde{\Gamma}^m_{ik}.$$

Then, the curvature tensor $\widetilde{R}_{ijk}{}^{l}$ can be expressed

$$\begin{aligned}\widetilde{R}_{ijk}{}^{l} &= R_{ijk}{}^{l}+\delta_j^l\mathcal{A}_{ik}-\delta_i^l\mathcal{A}_{jk}+g_{ik}\mathcal{A}_j{}^{l}-g_{jk}\mathcal{A}_i{}^{l} \\ &\quad +\frac{1}{q}(F_j{}^{l}F_k{}^{t}\mathcal{A}_{it}-F_i{}^{l}F_k{}^{t}\mathcal{A}_{jt}+F_{ik}F^{lt}\mathcal{A}_{jt}-F_{jk}F^{lt}\mathcal{A}_{it}),\end{aligned} \tag{4.5}$$

where $R_{ijk}{}^{l}$ are the ingredients of the Riemann curvature tensor of the Riemann connection ∇ and

$$\mathcal{A}_{jk}=\nabla_j\omega_k-\omega_j\omega_k+\frac{1}{2}\omega^m\omega_m g_{kj}-\frac{1}{q}\omega_m\omega_t F_k{}^{t}F_j{}^{m}+\frac{1}{2q}\omega^m\omega_t F_m{}^{t}F_{jk}. \tag{4.6}$$

It is clear that the tensor A provide $\mathcal{A}_{jk}-\mathcal{A}_{kj}=\nabla_j\omega_k-\nabla_k\omega_j=2(d\omega)_{jk}$, where the operator d is exterior differential on M. Thus, we say that $\mathcal{A}_{jk}-\mathcal{A}_{kj}=0$ if and only if $1-$form ω is closed.

Also, from the equation (4.5), we obtain

$$\begin{aligned}\widetilde{R}_{ijkl} &= R_{ijkl}+g_{jl}\mathcal{A}_{ik}-g_{il}\mathcal{A}_{jk}+g_{ik}\mathcal{A}_{jl}-g_{jk}\mathcal{A}_{il} \\ &\quad +\frac{1}{q}(F_{jl}F_k{}^{t}\mathcal{A}_{it}-F_{il}F_k{}^{t}\mathcal{A}_{jt}+F_{ik}F_l{}^{t}\mathcal{A}_{jt}-F_{jk}F_l{}^{t}\mathcal{A}_{it}).\end{aligned} \tag{4.7}$$

It is clear that the curvature tensor satisfies $\widetilde{R}_{ijkl}=-\widetilde{R}_{jikl}$ and $\widetilde{R}_{ijkl}=-\widetilde{R}_{ijlk}$.

For Ricci tensors of the connection (3.4) $\widetilde{R}_{jk}$, contracting (4.5) with respect to i and l, we have

$$\begin{aligned}\widetilde{R}_{jk} &= R_{jk}+(4-n)\,\mathcal{A}_{jk}-trace\mathcal{A}g_{jk} \\ &\quad +\frac{1}{q}\left(2p-F_l{}^{l}\right)F_k{}^{t}\mathcal{A}_{jt}-\frac{1}{q}F_{jk}F_l^t\mathcal{A}_t^l,\end{aligned} \tag{4.8}$$

where R_{jk} is Ricci tensors of the Riemann connection ∇ of g and

$$trace\mathcal{A}=\mathcal{A}_l{}^{l}=\nabla_l\omega^l+(\frac{n-4}{2})\omega_l\omega^l-\frac{1}{q}\omega_t\omega^m F_m{}^{t}(p-\frac{1}{2}F_l{}^{l}).$$

Contracting the last equation with g^{jk}, for the scalar curvature $\overline{\tau}$ of the connections (3.4), we get

$$\overline{\tau}=\tau+2\,(2-n)\,trace\mathcal{A}+\frac{2}{q}\left(p-F_l{}^{l}\right)F_l^t\mathcal{A}_t^l, \tag{4.9}$$

where τ is scalar curvature of Levi-Civita connection ∇ of g. From the equation (4.8), we can have

$$\widetilde{R}_{jk}-\widetilde{R}_{kj}=(n-4)\,(\mathcal{A}_{kj}-\mathcal{A}_{jk})+\frac{1}{q}\left(2p-F_l{}^{l}\right)F_k{}^{t}\,(\mathcal{A}_{jt}-\mathcal{A}_{tj}). \tag{4.10}$$

From the equation (4.10), we easily say that if the $1-$form ω is closed, then $\widetilde{R}_{jk}-\widetilde{R}_{kj}=0$.

Lemma 4.2. *Let (M, g, F) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). Then the tensor $\mathcal{A}$ given by (4.6) is a $\phi-$tensor (or decomposable tensor) and thus the following relation holds:*

$$(\nabla_m \mathcal{A}_{ij})\, F_k^{\ m} = (\nabla_k \mathcal{A}_{mj})\, F_i^{\ m} = (\nabla_k \mathcal{A}_{im})\, F_j^{\ m}.$$

Proof. The tensor $\mathcal{A}$ is pure with regard to F. Indeed

$$F_k^{\ t} \mathcal{A}_{it} - F_i^{\ t} \mathcal{A}_{tk} = (\nabla_i \omega_t)\, F_k^{\ t} - (\nabla_t \omega_k)\, F_i^{\ t} = 0.$$

If the Tachibana operator is applied to the tensor A, then we get

$$\begin{aligned} (\phi_F \mathcal{A})_{kij} &= F_k^{\ m}\, (\partial_m \mathcal{A}_{ij}) - \partial_k (\mathcal{A}_{mj} F_i^{\ m}) \\ &= F_k^{\ m} \left(\nabla_m \mathcal{A}_{ij} + \Gamma^s_{mi} \mathcal{A}_{sj} + \Gamma^s_{mj} \mathcal{A}_{is} \right) \\ &\quad - F_i^{\ m} (\nabla_k \mathcal{A}_{mj} + \Gamma^s_{km} \mathcal{A}_{sj} + \Gamma^s_{kj} \mathcal{A}_{ms}). \end{aligned}$$

From the purity of the Riemann connection ∇ and the tensor $\mathcal{A}$, we have

$$(\phi_F \mathcal{A})_{kij} = (\nabla_m \mathcal{A}_{ij})\, F_k^{\ m} - (\nabla_k \mathcal{A}_{mj})\, F_i^{\ m}. \tag{4.11}$$

Substituting (4.6) into (4.11), standard calculations give

$$(\phi_F \mathcal{A})_{kij} = (\nabla_m \nabla_i \omega_j)\, F_k^{\ m} - (\nabla_k \nabla_m \omega_j)\, F_i^{\ m}. \tag{4.12}$$

When we apply the Ricci identity to the $1-$form ω, we get

$$(\nabla_m \nabla_i \omega_j)\, F_k^{\ m} = (\nabla_i \nabla_m \omega_j)\, F_k^{\ m} - \frac{1}{2} \omega_s R_{mij}^{\ \ \ s} F_k^{\ m}$$

and

$$\begin{aligned} (\nabla_k \nabla_m \omega_j)\, F_i^{\ m} &= (\nabla_k \nabla_i \omega_m)\, F_j^{\ m} \\ &= (\nabla_i \nabla_k \omega_m)\, F_j^{\ m} - \frac{1}{2} \omega_s R_{kim}^{\ \ \ \ s} F_j^{\ m} \\ &= (\nabla_i \nabla_m \omega_k)\, F_j^{\ m} - \frac{1}{2} \omega_s R_{kim}^{\ \ \ \ s} F_j^{\ m} \end{aligned}$$

With the help of the last two equation, from (4.12), the equation (4.12) becomes as follows,

$$(\phi_F \mathcal{A})_{kij} = -\frac{1}{2} \omega_s (R_{mij}^{\ \ \ s} F_k^{\ m} - R_{kim}^{\ \ \ \ s} F_j^{\ m}).$$

In a locally decomposable metallic Riemann manifold (M, g, F), the Riemann curvature tensor R is pure [1]. This instantly gives $(\phi_F \mathcal{A})_{kij} = 0$. Hence, from (4.11) we can write

$$(\nabla_m \mathcal{A}_{ij})\, F_k^{\ m} = (\nabla_k \mathcal{A}_{mj})\, F_i^{\ m} = (\nabla_k \mathcal{A}_{im})\, F_j^{\ m}.$$

Also, with help of (2.8), we can say that $\phi_J A = 0$, i.e., the tensor A is decomposable, where J is the product structure associated with the metallic structure F. □

By using the purity of the tensor $\mathcal{A}$, standard calculations give

$$\widetilde{R}_{imk}{}^{l}F_{j}{}^{m}=\widetilde{R}_{ijm}{}^{l}F_{k}{}^{m}=\widetilde{R}_{ijk}{}^{m}F_{m}{}^{l}=\widetilde{R}_{mjk}{}^{l}F_{i}{}^{m},$$

i.e., the curvature tensor $\widetilde{R}$ is pure with respect to metallic structure F.

If Tachibana operator ϕ_F is applied to the curvature tensor $\widetilde{R}$, then we get

$$\begin{aligned}(\phi_F\widetilde{R})_{kijl}{}^{t} &= F_k{}^m(\partial_m\widetilde{R}_{ijl}{}^{t})-\partial_k(\widetilde{R}_{mjl}{}^{t}F_i{}^m) \\ &= F_k{}^m(\nabla_m\widetilde{R}_{ijl}{}^{t}+\Gamma^s_{mi}\widetilde{R}_{sjl}{}^{t}+\Gamma^s_{mj}\widetilde{R}_{isl}{}^{t}+\Gamma^s_{ml}\widetilde{R}_{ijs}{}^{t}-\Gamma^t_{ms}\widetilde{R}_{ijl}{}^{m}) \\ &\quad -F_i{}^m(\nabla_k\widetilde{R}_{mjl}{}^{t}+\Gamma^s_{km}\widetilde{R}_{sjl}{}^{t}+\Gamma^s_{kj}\widetilde{R}_{msl}{}^{t}+\Gamma^s_{kl}\widetilde{R}_{mjs}{}^{t}-\Gamma^t_{ks}\widetilde{R}_{mjl}{}^{s}) \\ &= (\nabla_m\widetilde{R}_{ijl}{}^{t})F_k{}^m-(\nabla_k\widetilde{R}_{mjl}{}^{t})F_i{}^m\end{aligned} \tag{4.13}$$

from which, by (4.5), we find

$$\begin{aligned}(\phi_F\widetilde{R})_{kijl}{}^{t} &= (\phi_F R)_{kijl}{}^{t}+[(\nabla_k\mathcal{A}_{jm})\,F_l{}^m-(\nabla_m\mathcal{A}_{jl})\,F_k{}^m]\delta_i^t \\ &\quad +[(\nabla_m\mathcal{A}_{il})\,F_k{}^m-(\nabla_k\mathcal{A}_{im})\,F_l{}^m]\delta_j{}^t \\ &\quad +[(\nabla_m\mathcal{A}_j^t)F_k{}^m-(\nabla_k\mathcal{A}_j^m)F_m{}^t]g_{il} \\ &\quad +[(\nabla_k\mathcal{A}_i^m)F_m{}^t-(\nabla_m\mathcal{A}_i^t)F_k{}^m]g_{jl}.\end{aligned}$$

In a locally decomposable metallic Riemann manifold (M,g,F), since the Riemann curvature tensor R is a $\phi-$tensor [1], considering Lemma 4.2, the last relation becomes $\phi_F\widetilde{R}=0$. Also, from the equation (2.8), we can say that $\phi_J\widetilde{R}=0$, where J is the product structure associated with the metallic structure F. Thus we obtain the following theorem:

Theorem 4.3. *Let (M,g,F) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). The curvature tensor $\widetilde{R}$ of the connection (3.4) is a $\phi-$tensor (or decomposable tensor).*

References

[1] Gezer A., Karaman C., On metallic Riemannian structures. *Turk J Math*, 39, (2015), 954-962.

[2] Gezer A., Karaman C., On golden semi-symmetric metric F-connecitons, *Turk J Math*, DOI: 10.3906/mat-1510-77.

[3] Hayden H. A. , Sub-spaces of a space with torsion. *Proc. London Math. Soc.* S2-34 (1932), 27-50.

[4] Hretcanu C. ,Crasmareanu M., Metallic structures on Riemannian manifolds. *Rev Un Mat Argentina* 2013; 54: 15-27.

[5] Crasmareanu M. ,Hretcanu C. E. , Golden differential geometry. *Chaos Solitons Fractals* 38 (2008), no. 5, 1229–1238.

[6] de Spinadel VW., The metallic means family and multifractal spectra. *Nonlinear Anal Ser B* 1999; 36: 721-745.

[7] de Spinadel VW., The family of metallic means. *Vis Math* 1999; 1: 3.

[8] Pusic N., On some connections on locally product Riemannian manifolds-part II. *Novi Sad J. Math.* 41 (2011), no. 2, 41-56.

[9] Pusic N., On some connections on locally product Riemannian manifolds-part I. *Novi Sad J. Math.* 41 (2011), no. 2, 29-40.

[10] Prvanovic M., Locally decomposable Riemannian manifold endowed with some semi-symmetric $F-$connection. *Bull. Cl. Sci. Math. Nat. Sci. Math.* No. 22 (1997), 45–56.

[11] Prvanovic M., Some special product semi-symmetric and some special holomorphically semi-symmetric $F-$connections. *Publ. Inst. Math.* (Beograd) (N.S.) 35(49) (1984), 139-152.

[12] Prvanovic M., Product semi-symmetric connections of the locally decomposable Riemannian spaces. *Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur.* (N.S.) 10 (1979), 17-27.

[13] Salimov A., Tensor operators and their applications. Mathematics Research Developments Series. Nova Science Publishers, Inc., New York, 2013. xii+186 pp.

[14] Tachibana S., Analytic tensor and its generalization. *Tohoku Math. J.* 12 (1960), 208-221.

[15] Yano K., On semi-symmetric metric connection. *Rev. Roumaine Math. Pures Appl.* 15 (1970), 1579-1586.

[16] Yano K., M. Ako, On certain operators associated with tensor fields. *Kodai Math. Sem. Rep.* 20 (1968), 414-436.

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