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ON CONHARMONIC CURVATURE TENSOR OF SASAKIAN FINSLER STRUCTURES ON TANGENT BUNDLES

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Abstract. The content of this paper is made up of conharmonic curvature tensor$K$ of Sasakian Finsler structures on tangent bundles. In this manner, quasi-conharmonically flat, ξ -conharmonically flat, φ-conharmonically flat Sasakian Finsler structures are studied. Some structure theorems including Einstein Sasakian Finsler manifolds satisfying $R(X^H, Y^H).K = 0$ are clarified.

1. Introduction

Sinha and Yadav, constructed almost contact Finsler structure φ on the total space of a vector bundle in [15], then Hasegawa, Yamauchi and Shimada discussed Sasakian structures on Finsler manifolds in chapter 6 in [9]. In Yaliniz and Caliskan's paper, Sasakian Finsler manifolds and their principal curvature properties are studied [17]. In this study, Sasakian Finsler structures' conharmonic curvature tensor properties are characterized by using the tangent bundle approach.

Conharmonic curvature tensor is defined by Ishii [10] then studied for several manifold structures in differential geometry: such as Riemannian, almost Hermite, Kahler and nearly Kahler manifolds by Mishra [14], Doric et al. [6], Krichenko et al. [12], [13], for K-contact, Sasakian, Kenmotsu and LP- Sasakian manifolds by Khan [11], Dwivedi and Kim [7] , Asghari and Taleshian, Taleshian et al., [4] [16], for Sasakian space forms by De et al. [5] , for N(k)-contact metric manifolds by Ghosh et al. [8], for $C(\lambda)$ manifolds by Akbar and Sarkar [1]. In this paper, conharmonic curvature tensor is studied for Sasakian Finsler structures on tangent bundles. In order to examine this; some characteristics of such kind of structures are given:

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Assume, M be an $m=(2n+1)$-dimensional smooth manifold. TM_x denotes the tangent space to M at x where $x=(x^1,\dots,x^{(2n+1)})\in M$ and $y=y^i\frac{\partial}{\partial x^i}\in TM_x$. TM is notated as the tangent bundle of the manifold M. Thus $u=(x,y)\in TM$.

Suppose $F:TM\to[0,\infty[$ be a Finsler function with the following properties:

(1) F is differentiable of class C^1 on TM and differentiable of class C^∞on $TM_0=TM\backslash\{(x,0)\}$,
(2) $F(x,\lambda y)=|\lambda|F(x,y)$, $(x,y)\in TM$, $\lambda\in\mathbb{R}$,
(3) $g_{ij}(x,y)=\frac{1}{2}[\frac{\partial^2F^2}{\partial y^i\partial y^j}]$ is positive definite on TM_0,
then g is called a Finsler metric tensor with g_{ij} coefficients and $F^m=(M,F)$ is a Finsler manifold [2].

If (x^i,y^i) are the local coordinates of TM, then $\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\}$ denote natural bases of $TTM_{|u|}$ at $u\in TM$. Thus, the complete vector field $X=X^i\frac{\partial}{\partial x^i}+X^j\frac{\partial}{\partial y^j}\in TTM_{|u|}$. Suppose that, $\pi:TM\to M$ is the bundle projection, then $\pi=(TM,\pi,M)$ of rank m is called Finsler tangent bundle. So, the differential map $\pi:TTM_{|u|}\to TM_{\pi(u)}$ generates the vertical distribution VTM of TTM_0 where HTM and VTM are complementary distributions of each other for TTM_0.

The horizontal distribution $HTM=(N_i^j(x,y))$ of TTM_0 is the non-linear connection on TM where $N_i^j=\frac{\partial N^j}{\partial y^i}$ are obtained via the spray coefficients notated $N^j=\frac{1}{4}g^{jk}(\frac{\partial^2F^2}{\partial y^k\partial x^h}y^h-\frac{\partial F^2}{\partial x^k})$. By using the linear connection $\triangledown$ on VTM, the pair $(HTM,\triangledown)$ is called a Finsler connection on M [3]. So, $X=X^i(\frac{\partial}{\partial x^i}-N_i^j(x,y)\frac{\partial}{\partial y^j})+(N_i^j(x,y)X^i+\widetilde{X}^j)\frac{\partial}{\partial y^j}=X^i\frac{\delta}{\delta x^i}+X^j\frac{\partial}{\partial y^j}$ is obtained. Here, $\frac{\delta}{\delta x^i}$ and $\frac{\partial}{\partial y^j}$ are the bases of $HTM_{|u|}$ and $VTM_{|u|}$, respectively. Besides, their dual bases are dx^i and $\delta y^j=dy^j+N_i^jdx^i$, respectively where $TTM_{|u|}=HTM_{|u|}\oplus VTM_{|u|}$.

The Riemannian metric G with Finsler coefficients, is called the Sasaki-Finsler metric on TM_0 and its distributions are as follows:

$G(X,Y)=G(X^H,Y^H)+G(X^V,Y^V)=G^H(X,Y)+G^V(X,Y)$ for tangent vectors $X,Y\in TTM_{|u|}$ at $u\in TM$ and $X^H,Y^H\in HTM_{|u|}$ and $X^V,Y^V\in VTM_{|u|}$. G^H and $G^V$are Riemannian metrics of type $\begin{pmatrix}0&0\\2&0\end{pmatrix}$ and $\begin{pmatrix}0&0\\0&2\end{pmatrix}$ on hTM and vTM, respectively. Thus, following properties are satisfied:

$G(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})=G(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})=g_{ij}$ and $G(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})=0$. $(\varphi^H,\xi^H,\eta^H,G^H)$ and $(\varphi^V,\xi^V,\eta^V,G^V)$ are $(2n+1)$-dimensional Sasakian Finsler structures on hTM and vTM, respectively and the following relations hold [17]:

$$\varphi^2=-I+\eta^H\otimes\xi^H+\eta^V\otimes\xi^V \tag{1.1}$$

$$\eta^H(\xi^H)=1=\eta^V(\xi^V) \tag{1.2}$$

$$\varphi^H(\xi^H)=0=\varphi^V(\xi^V) \tag{1.3}$$

$$\eta^H\circ\varphi^H=0=\eta^V\circ\varphi^V \tag{1.4}$$

$$\begin{aligned} G(X^H,Y^H) &= G(\varphi^H X^H,\varphi^H Y^H)+\eta^H(X^H)\eta^H(Y^H) \\ G(X^V,Y^V) &= G(\varphi^V X^V,\varphi^V Y^V)+\eta^V(X^V)\eta^V(Y^V) \end{aligned} \tag{1.5}$$

$$G(X^H,\varphi^H Y^H)=-G(\varphi^H X^H,Y^H), G(X^V,\varphi^V Y^V)=-G(\varphi^V X^V,Y^V) \tag{1.6}$$

$$G(X^H,\xi^H)=\eta^H(X^H), G(X^V,\xi^V)=\eta^V(X^V) \tag{1.7}$$

$$G(X^H,\varphi^H Y^H)=d\eta^H(X^H,Y^H), G(X^V,\varphi^V Y^V)=d\eta^H(X^H,Y^H) \tag{1.8}$$

$$\nabla^H_X\xi^H=-\frac{1}{2}\varphi^H X^H, \nabla^V_X\xi^V=-\frac{1}{2}\varphi^V X^V \tag{1.9}$$

$$\begin{aligned} (\nabla^H_X\varphi^H)Y^H &= \frac{1}{2}[G(X^H,Y^H)\xi^H-\eta^H(Y^H)X^H] \\ (\nabla^V_X\varphi^V)Y^V &= \frac{1}{2}[G(X^V,Y^V)\xi^V-\eta^V(Y^V)X^V] \end{aligned} \tag{1.10}$$

$$\begin{aligned} R(X^H,Y^H)\xi^H &= \frac{1}{4}[\eta^H(Y^H)X^H-\eta^H(X^H)Y^H] \\ R(X^V,Y^V)\xi^V &= \frac{1}{4}[\eta^V(Y^V)X^V-\eta^V(X^V)Y^V] \end{aligned} \tag{1.11}$$

$$\begin{aligned} R(\xi^H,X^H)Y^H &= \frac{1}{4}[G(X^H,Y^H)\xi^H-\eta^H(Y^H)X^H] \\ R(\xi^V,X^V)Y^V &= \frac{1}{4}[G(X^V,Y^H)\xi^V-\eta^V(Y^V)X^V] \end{aligned} \tag{1.12}$$

$$S(X^H,\xi^H)=\frac{n}{2}\eta^H(X^H), S(X^V,\xi^V)=\frac{n}{2}\eta^V(X^V) \tag{1.13}$$

$$S(\xi^H,\xi^H)=\frac{n}{2}, S(\xi^V,\xi^V)=\frac{n}{2} \tag{1.14}$$

$$R(X^H,\xi^H)\xi^H=-\frac{1}{4}\ , R(X^V,\xi^V)\xi^V=-\frac{1}{4} \tag{1.15}$$

For all $X^H, Y^H \in HTM_{|u|}$ and $X^V, Y^V \in VTM_{|u|}$. Additionally, φ denotes the tensor field of type $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, ξ is the structure vector field of type $\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$, η is the 1-form of type $\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$, G is the Sasaki-Finsler metric structure of type $\begin{pmatrix}0 & 0\\ 2 & 2\end{pmatrix}$, ∇ is the Finsler connection with respect to G on TM, R is the Riemann curvature tensor field of type $\begin{pmatrix}1 & 1\\ 3 & 3\end{pmatrix}$, S is the Ricci tensor field of type $\begin{pmatrix}0 & 0\\ 2 & 2\end{pmatrix}$.

As it is seen from above-mentioned preliminaries, Sasakian Finsler structures can be founded either on hTM or vTM. In this paper, hTM is considered on behalf of briefness.

Definition 1.1. For m-dimensional $(hTM, \varphi^H, \xi^H, \eta^H, G^H)$, conharmonic curvature tensor is described as follows:

$$K(X^H, Y^H)Z^H = R(X^H, Y^H)Z^H + \frac{1}{2n-1}[S(Y^H, Z^H)X^H - S(X^H, Z^H)Y^H + G(Y^H, Z^H)QX^H - G(X^H, Z^H)QY^H] \quad (1.16)$$

for $X^H, Y^H, Z^H \in HTM_{|u|}$.

2. QUASI-CONHARMONICALLY FLAT SASAKIAN FINSLER STRUCTURES

Definition 2.1. If $(\varphi^H, \xi^H, \eta^H, G^H)$ is the Sasakian Finsler metric structure on hTM, then hTM is quasi-conharmonically flat when the below relation is verified:

$$G(K(X^H, Y^H)Z^H, \varphi W^H) = 0 \quad (2.1)$$

for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$.

For a Sasakian Finsler manifold, because of the scalar curvature tensor $r = 0$, with the help of (1.13), it is possible to get the below relation:

$$S(Y^H, Z^H) = -\frac{1}{4}G(Y^H, Z^H) + (\frac{n}{2} + \frac{1}{4})\eta^H(Y^H)\eta^H(Z^H) \quad (2.2)$$

for $Y^H, Z^H \in HTM_{|u|}$ and that means hTM is the η-Einstein.

Theorem 2.2. *For a Sasakian Finsler manifold $(hTM, \varphi^H, \xi^H, \eta^H, G^H)$ necessary and sufficient condition to be quasi-conharmonically flat is: the below relation holds:*

$$\begin{aligned} R(X^H, Y^H)Z^H = & -\frac{1}{2(2n-1)}[G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H] \\ & + \frac{(2n+1)}{4(2n-1)}[\eta^H(Y^H)\eta^H(Z^H)X^H - \eta^H(X^H)\eta^H(Z^H)Y^H] \\ & + \frac{(2n+1)}{4(2n-1)}[G(Y^H, Z^H)\eta^H(X^H)\xi^H - G(X^H, Z^H)\eta^H(Y^H)\xi^H] \end{aligned} \quad (2.3)$$

for $X^H, Y^H, Z^H \in HTM_{|u|}$.

Proof. Due to the Sasakian Finsler manifold is quasi-conharmonically flat with dimension $(2n+1)$, using (1.16) and (2.1), it is known that $r = 0$ and taking $W^H = \varphi W^H$, the equality is herein below:

$$\begin{aligned} G(R(X^H, Y^H)Z^H, \varphi^2 W^H) = & \frac{1}{2(2n-1)}[G(Y^H, Z^H)G(\varphi X^H, \varphi W^H) \\ & - G(X^H, Z^H)G(\varphi Y^H, \varphi W^H)] \\ & + (\frac{1}{4} + \frac{n}{2})[G(\varphi Y^H, \varphi W^H)\eta^H(X^H)\eta^H(Z^H) - G(\varphi X^H, \varphi W^H)\eta^H(Y^H)\eta^H(Z^H)] \end{aligned} \quad (2.4)$$

for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$.

By using (1.16) and (2.1) in (2.4), it is possible to get (2.3). □

3. ξ-CONHARMONICALLY FLAT SASAKIAN FINSLER STRUCTURES

Definition 3.1. Assume that, $(\varphi^H, \xi^H, \eta^H, G^H)$ is the Sasakian Finsler metric structure on hTM, then it is called ξ-conharmonically flat if the following relation holds:

$$K(X^H, Y^H)\xi^H = 0 \tag{3.1}$$

for $X^H, Y^H \in HTM_{|u|}$.

Theorem 3.2. *For a Sasakian Finsler manifold $(hTM, \varphi^H, \xi^H, \eta^H, G^H)$ necessary and sufficient condition to be ξ-conharmonically flat is: hTM is an η-Einstein manifold.*

Proof. For a $(2n+1)$-dimensional $(n > 1)$ Sasakian Finsler manifold hTM, (1.13) holds. By using this in (2.4), it is possible to get

$$S(X^H, W^H) = -\frac{1}{4}G(X^H, W^H) + (\frac{n}{2} + \frac{1}{4})\eta^H(X^H)\eta^H(W^H) \tag{3.2}$$

for $X^H, W^H \in HTM_{|u|}$. Namely, the Sasakian Finsler manifold is η-Einstein and vice versa. □

4. φ-CONHARMONICALLY FLAT SASAKIAN FINSLER STRUCTURES

Definition 4.1. Let $(hTM, \varphi^H, \xi^H, \eta^H, G^H)$ be a Sasakian Finsler manifold, then hTM is said to be φ-conharmonically flat when the below equality is satisfied:

$$G(K(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) = 0 \tag{4.1}$$

for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$.

Theorem 4.2. *For an m-dimensional Sasakian Finsler manifold necessary and sufficient condition to be φ -conharmonically flat is: following relation holds:*

$$G(R(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) = -\frac{1}{2(2n-1)}\{G(\varphi Y^H, \varphi Z^H) G(\varphi X^H, \varphi W^H) - G(\varphi X^H, \varphi Z^H)G(\varphi Y^H, \varphi W^H)\}. \tag{4.2}$$

Proof. For a $(2n+1)$-dimensional hTM with the help of (1.16), it is possible to have the below relation:

$$S(\varphi X^H, \varphi W^H) = G(Q(\varphi X^H), \varphi W^H). \tag{4.3}$$

In consequence of this relation, the equality herein below can be written:

$$\begin{aligned} G(K(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) &= G(R(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) \\ -\frac{1}{2n-1}\{S(\varphi Y^H, \varphi Z^H)G(\varphi X^H, \varphi W^H) &- S(\varphi X^H, \varphi Z^H)G(\varphi Y^H, \varphi W^H) \\ +G(\varphi Y^H, \varphi Z^H)S(\varphi X^H, \varphi W^H) &- G(\varphi X^H, \varphi Z^H)S(\varphi Y^H, \varphi W^H)\}. \end{aligned} \tag{4.4}$$

Owing to the fact that $\{E_i^H\}$ is orthonormal basis of $HTM_{|u|}$, $\{\varphi E_i^H\}$ is orthonormal basis either. By taking summation over $i = 1, 2, \ldots, (2n+1)$ in (4.4) and changing $X^H = W^H = E_i^H$, it takes the following form:

$$\begin{aligned} G(K(\varphi E_i^H, \varphi Y^H)\varphi Z^H, \varphi E_i^H) &= G(R(\varphi E_i^H, \varphi Y^H)\varphi Z^H, \varphi E_i^H) \\ -\frac{1}{2n-1}\{S(\varphi Y^H, \varphi Z^H)G(\varphi E_i^H, \varphi E_i^H) &- S(\varphi E_i^H, \varphi Z^H)G(\varphi Y^H, \varphi E_i^H) \\ +G(\varphi Y^H, \varphi Z^H)S(\varphi E_i^H, \varphi E_i^H) &- G(\varphi E_i^H, \varphi Z^H)S(\varphi Y^H, \varphi E_i^H)\} \end{aligned} \tag{4.5}$$

for $Y^H \in HTM_{|u|}$. Due to HTM is φ-conharmonically flat, (4.1) holds and by virtue of (4.5),

$$S(\varphi Y^H, \varphi Z^H) = (r - \frac{1}{4})G(\varphi Y^H, \varphi Z^H) \tag{4.6}$$

the above equality holds for $Y^H, Z^H \in HTM_{|u|}$. Also, if we take summation over $i = 1, 2, \ldots, 2n+1$ in (4.6) and changing $Y^H = Z^H = E_i^H$, it is obtained that $r = 0$. Using (4.1) in (4.4),we have (4.2). □

Theorem 4.3. *For a $(2n+1)$-dimensional $(n > 1)$ Sasakian Finsler manifold hTM, following items are equal to each other:*

(1) *hTM is conharmonically flat.*
(2) *hTM is φ-conharmonically flat.*
(3) *The below relation holds:*

$$\begin{aligned} G(R(X^H, Y^H)Z^H, W^H) = {} & \frac{1}{2(2n-1)}[G(Y^H, Z^H)G(X^H, W^H) \\ & -G(X^H, Z^H)G(Y^H, W^H)] \\ & -\frac{(2n+1)}{4(2n-1)}[-G(X^H, W^H)\eta^H(Y^H)\eta^H(Z^H) \\ & +G(Y^H, W^H)\eta^H(X^H)\eta^H(Z^H) \\ -G(Y^H, Z^H)\eta^H(X^H)\eta^H(W^H) & + G(X^H, Z^H)\eta^H(Y^H)\eta^H(W^H)] \end{aligned} \tag{4.7}$$

for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$.

Proof. $1 \Rightarrow 2$: Due to the Sasakian Finsler manifold hTM is conharmonically flat, $K(X^H, Y^H)Z^H = 0$ for $X^H, Y^H, Z^H \in HTM_{|u|}$. Therefore (4.1) holds, namely $G(K(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) = 0$. Then manifold is φ-conharmonically flat.

$2 \Rightarrow 3$: If the Sasakian Finsler manifold hTM is φ -conharmonically,(4.1) holds. By using (1.11) and (1.12) in (4.1),

$$\begin{aligned} G(R(\varphi^2 X^H, \varphi^2 Y^H)\varphi^2 Z^H, \varphi^2 W^H) &= G(R(X^H, Y^H)Z^H, W^H) \\ +\frac{1}{4}\{-G(X^H, W^H)\eta^H(Y^H)\eta^H(Z^H) &+ G(Y^H, W^H)\eta^H(X^H)\eta^H(Z^H) \\ -G(Y^H, Z^H)\eta^H(X^H)\eta^H(W^H) &+ G(X^H, Z^H)\eta^H(Y^H)\eta^H(W^H)\} \end{aligned} \tag{4.8}$$

the above relation can be calculated for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$. Changing X^H, Y^H, Z^H, W^H with $\varphi X^H, \varphi Y^H, \varphi Z^H, \varphi W^H$ respectively, following relation is obtained:

$$\begin{aligned} G(R(\varphi X^H, \varphi Y^H)\varphi Z^H, \varphi W^H) = -\frac{1}{2(2n-1)}\{G(Y^H, Z^H)G(X^H, W^H) \\ -G(X^H, Z^H)G(Y^H, W^H) - G(X^H, W^H)\eta^H(Y^H)\eta^H(Z^H) \\ +G(Y^H, W^H)\eta^H(X^H)\eta^H(Z^H) \\ -G(Y^H, Z^H)\eta^H(X^H)\eta^H(W^H) + G(X^H, Z^H)\eta^H(Y^H)\eta^H(W^H)\}. \end{aligned} \tag{4.9}$$

With the help of (4.8) and (4.9),

$$\begin{aligned} G(R(X^H, Y^H)Z^H, W^H) = -\frac{1}{2(2n-1)}[G(Y^H, Z^H)G(X^H, W^H) \\ -G(X^H, Z^H)G(Y^H, W^H)] \\ -\frac{(2n+1)}{4(2n-1)}[-G(X^H, W^H)\eta^H(Y^H)\eta^H(Z^H) + G(Y^H, W^H)\eta^H(X^H)\eta^H(Z^H) \\ -G(Y^H, Z^H)\eta^H(X^H)\eta^H(W^H) + G(X^H, Z^H)\eta^H(Y^H)\eta^H(W^H)] \end{aligned} \tag{4.10}$$

is obtained for $X^H, Y^H, Z^H, W^H \in HTM_{|u|}$. Thereby (4.7) holds.

$3 \Rightarrow 1$: Accept that (4.7) holds for Sasakian Finsler manifold hTM . By taking summation over $i = 1, 2, \ldots, 2n+1$ in (4.7) and taking $X^H = W^H = E_i^H$, the below relation is satisfied:

$$S(Y^H, Z^H) = -\frac{1}{4}G(Y^H, Z^H) + (\frac{n}{2} + \frac{1}{4})\eta^H(Y^H)\eta^H(Z^H) \tag{4.11}$$

for $Y^H, Z^H \in HTM_{|u|}$. If (4.11) and (4.7) are used in (1.16), $K(X^H, Y^H)Z^H = 0$ is obtained. Namely the Sasakian Finsler manifold hTM is conharmonically flat. □

5. EINSTEIN SASAKIAN FINSLER STRUCTURES SATISFYING $R(X^H, Y^H).K = 0$

Theorem 5.1. *Let hTM be a $(2n+1)$-dimensional conharmonically flat Einstein Sasakian Finsler manifold and the relation $R(X^H, Y^H).K = 0$ is satisfied, then it is locally isometric to $S^m(1)$.*

Proof. Due to Sasakian Finsler manifold hTM is Einstein, with the help of (1.16)

$$K(X^H, Y^H)Z^H = R(X^H, Y^H)Z^H + \frac{2\lambda}{2n-1}[G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H] \tag{5.1}$$

holds for $X^H, Y^H, Z^H \in HTM_{|u|}$ and $\lambda \in \mathbb{R}$. Then the below equality is satisfied:

$$\eta^H(K(X^H, Y^H)Z^H) = [\frac{2\lambda}{2n-1} - \frac{1}{4}][G(X^H, Z^H)\eta^H(Y^H) - G(Y^H, Z^H)\eta^H(X^H)] \tag{5.2}$$

for $X^H, Y^H, Z^H \in HTM_{|u|}$. Taking $X^H = \xi^H$ in (5.2),

$$\eta^H(K(\xi^H, Y^H)Z^H) = [\frac{2\lambda}{2n-1} - \frac{1}{4}][\eta^H(Y^H)\eta^H(Z^H) - G(Y^H, Z^H)] \tag{5.3}$$

is obtained. Changing $Z^H = \xi^H$ in (5.2), it is possible to get

$$\eta^H(K(X^H, Y^H)\xi^H) = 0 \tag{5.4}$$

for $X^H, Y^H \in HTM_{|u|}$. If $R(X^H, Y^H)$ is considered as the derivation of the tensor algebra at each point of the Sasakian Finsler manifold hTM for X^H and Y^H, following relation holds for conharmonic curvature tensor:

$$\begin{aligned}[R(X^H, Y^H)K](U^H, V^H)W^H &= R(X^H, Y^H)[K(U^H, V^H)W^H \\ -K(R(X^H, Y^H)U^H, V^H)W^H &- K(U^H, R(X^H, Y^H)V^H)W^H \\ &-K(U^H, V^H)R(X^H, Y^H)W^H.\end{aligned} \tag{5.5}$$

Owing to $R(X^H, Y^H).K = 0$, by taking $X^H = \xi^H$ in the last equality,

$$\begin{aligned}G([R(\xi^H, Y^H)K](U^H, V^H)W^H, \xi^H) &= -G(K(R(\xi^H, Y^H)U^H, V^H)W^H, \xi^H) \\ -G(K(U^H, R(\xi^H, Y^H)V^H)W^H, \xi^H) &- G(K(U^H, V^H)R(\xi^H, Y^H)W^H, \xi^H)\end{aligned} \tag{5.6}$$

the above relation holds for the tangent vector fields that are orthogonal to ξ^H. By using (1.11) and (1.12) in (5.6),

$$\begin{aligned}0 = \frac{1}{4}\{&G(Y^H, K(U^H, V^H)W^H) - \eta^H(Y^H)\eta^H(K(U^H, V^H)W^H) \\ &-G(Y^H, U^H)\eta^H(K(\xi^H, V^H)W^H) + \eta^H(U^H)\eta^H(K(Y^H, V^H)W^H) \\ &-G(Y^H, V^H)\eta^H(K(U^H, \xi^H)W^H) + \eta^H(V^H)\eta^H(K(U^H, Y^H)W^H) \\ &-G(Y^H, W^H)\eta^H(K(U^H, V^H)\xi^H) + \eta^H(W^H)\eta^H(K(U^H, V^H)Y^H)\}\end{aligned} \tag{5.7}$$

is obtained. With the help of (5.2), it is possible to get

$$\begin{aligned}G(K(U^H, V^H)W^H, Y^H) = [\frac{2\lambda}{2n-1} - \frac{1}{4}][&G(Y^H, V^H)G(U^H, W^H) \\ &-G(Y^H, U^H)G(V^H, W^H)]\end{aligned} \tag{5.8}$$

for $U^H, V^H, W^H, Y^H \in HTM_{|u|}$. By using (5.1) in (5.8), the below relation is obtained:

$$G(R(U^H, V^H)W^H, Y^H) = \frac{1}{4}[G(Y^H, U^H)G(V^H, W^H) - G(Y^H, V^H)G(U^H, W^H)] \tag{5.9}$$

for $U^H, V^H, W^H, Y^H \in HTM_{|u|}$. □

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