Available online: November 17, 2017

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 67, Number 2, Pages 291–297 (2018)
DOI: 10.1501/Commua1_0000000882
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS
SERIES A1

OSCILLATION OF NONLINEAR FOURTH-ORDER DIFFERENCE EQUATIONS WITH MIDDLE TERM

M. EMRE KAVGACI

Abstract. In this article, we study oscillatory properties of the fourth-order difference equation with middle-term

$$\Delta^4 x_m - a_m \Delta^2 x_{m+1} + b_m f(x_{m+\sigma}) = 0,$$

in case when the corresponding second-order difference equation $\Delta^2 h_m - a_m h_{m+1} = 0$ is nonoscillatory.

1. Introduction

Consider the fourth-order nonlinear difference equation

$$\Delta^4 x_m - a_m \Delta^2 x_{m+1} + b_m f(x_{m+\sigma}) = 0, \tag{1.1}$$

where $\sigma \in \mathbb{N}$ is a deviating argument and $\{a_m\}$, $\{b_m\}$ are real sequences for $m \in \mathbb{N}$. Function $f : \mathbb{R} \to \mathbb{R}$, is continuous such that $uf(u) > 0$ for $u \neq 0$ where $\mathbb{R}$ denotes the set of real numbers.

Throughout the paper we assume

$$a_m a_{m+1} > 0, \quad b_m > 0, \ m \in \mathbb{N}$$

and

$$\sum_{m=1}^{\infty} m|a_m| < \infty. \tag{1.2}$$

By a solution of the equation (1.1), we mean a real sequence $\{x_m\}$ satisfying equation (1.1) for $m \in \mathbb{N}$. A nontrivial solution $\{x_m\}$ of (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and it is oscillatory otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Received by the editors: June 12, 2017, Accepted: September 05, 2017.
2010 *Mathematics Subject Classification.* 39A10, 39A21.
Key words and phrases. Difference equation, oscillatory solution, nonoscillatory solution.

In the recent years, a great importance has been paid to the study of oscillatory behavior of fourth-order differential equations [6, 7] and difference equations [2, 3, 4, 5, 14, 16, 19], see also the monograph [1] and [15].

In the continuous case, the fourth-order differential equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + r(t)f(x(\varphi(t))) = 0$$

can be written as

$$\Big(h^2(t)\Big(\frac{x''(t)}{h(t)}\Big)'\Big)' + h(t)r(t)f(x(t)) = 0$$

$h''(t) + q(t)h(t) = 0$ is nonoscillatory and h is its eventually positive solution, see e.g. [7].

Došlá and Krejcova [11, 12] have investigated a class of fourth-order nonlinear difference equations of the form

$$\Delta\left(a_n\left(\Delta b_n\left(\Delta c_n\left(\Delta x_n\right)^{\gamma}\right)^{\beta}\right)^{\alpha}\right) + d_n x_{n+\tau}^{\lambda} = 0, \tag{1.3}$$

and Jankowski, Schmeidel and Zonenberg [14] have generalized the some results of [11] for neutral equation

$$\Delta\left(a_n\left(\Delta b_n\left(\Delta c_n\left(\Delta\left(x_n + p_n x_{n-\delta}\right)\right)^{\gamma}\right)^{\beta}\right)^{\alpha}\right) + d_n f(x_{n-\tau}) = 0, \tag{1.4}$$

where α, β and γ are the ratios of odd positive integers, integers τ, δ are deviating arguments.

In this paper we investigate oscillatory properties of the equation (1.1). Our approach is based on the transformation of (1.1) to the two-terms equation of the form (1.3) and to application of oscillation results for equation (1.3) stated in [11, 12].

2. Preliminaries

Consider second order linear equation

$$\Delta^2 h_m - a_m h_{m+1} = 0. \tag{2.1}$$

Let (2.1) be nonoscillatory. The following definition is given by Patula [17].

Definition 2.1. *If there exist two linearly independent solutions v and w of* (2.1) *such that $v/w \to 0$, as $n \to \infty$, then v is recessive solution and w is dominant solution of* (2.1)*.*

We remark that the recessive solution always exist and is unique up to a constant factor, see [17, Theorem 1].

Lemma 2.1. *If* (2.1) *is nonoscillatory, there exist a recessive solution h such that*

$$\sum_{m=1}^{\infty} \frac{1}{h_m h_{m+1}} = \infty.$$

Proof. See [17, Theorem 1] and [1, Theorem 6.3.1]. □

Lemma 2.2. *If* $\sum_{m=1}^{\infty} m|a_m| < \infty$, *then* (2.1) *has recessive solution which tends to positive constant.*

Proof. Let $a_m > 0$ for $m \geq 1$. Then the conclusion follows from [10, Theorem 4]. In case $a_m < 0$ for $m \geq 1$, the statement follows from [13, Theorem 4.2]. □

From Lemma 1 and Lemma 2 we have the following Lemma.

Lemma 2.3. *If* $\sum_{m=1}^{\infty} m|a_m| < \infty$, *then recessive solution* h *of* (2.1) *provides*

$$\sum_{m=m_0}^{\infty} \frac{1}{h_m h_{m+1}} = \infty, \quad \sum_{m=m_0}^{\infty} h_m = \infty. \tag{2.2}$$

Proof. See [1, Theorem 6.3.8] and [13, Theorem 4.2]. □

Now, we consider equation (1.1) and we write it as a two-terms equation.

Lemma 2.4. *Let the equation* (2.1) *be nonoscillatory and let* h *be its solution such that* $h_m > 0$ *for* $m \geq 1$. *Then, we have for* $m \geq 1$

$$\Delta^4 x_m - a_m \Delta^2 x_{m+1} = \frac{1}{h_{m+1}} \Delta \left[h_m h_{m+1} \Delta \left(\frac{1}{h_m} \Delta^2 x_m \right) \right]. \tag{2.3}$$

Consequently, x *is solution of equation* (1.1) *if and only if it is a solution of equation in the disconjugate form*

$$\Delta \left[h_m h_{m+1} \Delta \left(\frac{1}{h_m} \Delta^2 x_m \right) \right] + b_m h_{m+1} f(x_{m+\sigma}) = 0. \tag{2.4}$$

Proof. Assume that $y_m \equiv h_m u_m$, where $u = (u_m)$ is any sequence. Firstly, we show that

$$h_{m+1}(\Delta^2 y_m - a_m y_{m+1}) = \Delta(h_m h_{m+1} \Delta u_m). \tag{2.5}$$

Using the definition of difference operator, we can easily obtain that

$$\Delta(h_m h_{m+1} \Delta u_m) = h_{m+1}(h_{m+2} \Delta u_{m+1} - h_m \Delta u_m) \tag{2.6}$$

and

$$\Delta^2 y_m = h_{m+2} u_{m+2} - 2h_{m+1} u_{m+1} + h_m u_m. \tag{2.7}$$

From equation (2.1), we can write $a_m h_{m+1} = \Delta^2 h_m$ and

$$a_m y_{m+1} = u_{m+1} \Delta^2 h_m = u_{m+1}(h_{m+2} - 2h_{m+1} + h_m). \tag{2.8}$$

From (2.7) and (2.8)

$$h_{m+1}(\Delta^2 y_m - a_m y_{m+1}) = h_{m+1}(h_{m+2} \Delta u_{m+1} - h_m \Delta u_m). \tag{2.9}$$

Then, right side of equation (2.6) is equal to right side of equation (2.9) and we obtain,

$$\Delta^2 y_m - a_m y_{m+1} = \frac{1}{h_{m+1}} \Delta(h_m h_{m+1} \Delta u_m)$$

where $u_m = \frac{y_m}{h_m}$ and $y_m = \Delta^2 x_m$. □

Remark 2.1. *If h is recessive solution of* (2.1)*, then by Lemma 3,* (2.2) *holds and equation* (2.4) *is said to be in the canonical form.*

Let x be a solution of (2.4) and denote the quasi-differences of x as

$$x^{[1]} = \Delta x_m,\ x^{[2]} = \tfrac{1}{h_m}\Delta x^{[1]},\ x^{[3]} = h_m h_{m+1}\Delta x^{[2]}.$$

Lemma 2.5. *If* (2.2) *holds, then any eventually positive solution $\{x_m\}$ of* (2.4) *is one of the following types:*
type (a): $x_m > 0, x^{[1]} > 0, x^{[2]} > 0, x^{[3]} > 0$ for large m,
type (b): $x_m > 0, x^{[1]} > 0, x^{[2]} < 0, x^{[3]} > 0$ for large m.

Proof. We consider (2.4) as a four-dimensional system

$$\begin{cases} \Delta x_m = y_m \\ \Delta y_m = h_m z_m \\ \Delta z_m = \frac{1}{h_m h_{m+1}} w_m \\ \Delta w_m = -b_m h_{m+1} f(x_{m+\sigma}), \end{cases} \tag{2.10}$$

where

$$(x, y, z, w) = (x, x^{[1]}, x^{[2]}, x^{[3]}).$$

Proceeding by the similar way as in [11], proof of Lemma 2, we obtain the conclusion. The details are omitted here. □

3. Oscillation results

In this section, we give oscillation results for equation (1.1). During this section we assume that equation (2.1) is nonoscillatory and h is a solution of (2.1) such that $h_m > 0$ for $m \geq 1$.

Solution x of (1.1) is called quickly oscillatory, if it is of the form

$$x_m = (-1)^m p_m,\ p_m > 0 \text{ for } m \in \mathbb{N}.$$

The following result can be seen as a necessary condition for existence of quickly oscillatory solutions.

Lemma 3.1. *If σ is even, then equation* (1.1) *has no quickly oscillatory solutions.*

Proof. Let $x_m = (-1)^m p_m$ be a quickly oscillatory solution of (1.1). By Lemma 4, x_m is solution of (2.4) and system (2.10). Then, the proof is the similar way as in [11], proof of Theorem 1 and [14], proof of Theorem 3.1. □

Theorem 3.1. *Let* (1.2) *holds. If $\sum_{i=1}^{\infty} b_i = \infty$, then* (1.1) *is oscillatory.*

Proof. By Lemma 4, we can transform equation (1.1) to equation (2.4). The proof follows from [14], proof of Theorem 4.4. □

Theorem 3.2. *Let* (1.2) *holds and there exist* $\lambda > 0$ *such that*

$$\lim_{u\to\infty} \frac{f(u)}{u^\lambda} > 0. \tag{3.1}$$

Equation (1.1) *with* $\sigma \geq 1$ *is oscillatory if any of the following conditions holds:*
(i) $\lambda < 1, \sum_{m=1}^{\infty} b_m m^\lambda = \infty$;

(ii) $\lambda > 1, \sum_{\mathrm{m=1}}^{\infty} \mathrm{b_m m} = \infty$.

Proof. For the sake of contradiction, let (1.1) have a nonoscillatory solution and let h be recessive solution of (2.4) such that $lim_{m\to\infty} h_m = 1$. Without loss of generality assume $x_m > 0$ for $m \geq 1$. By Lemma 4, x is nonoscillatory solution of (2.4). By Lemma 5, x is type (a) or type (b).

(i) Let x be of type (a) such that $x_m > 0$ for $m \geq 1$. Then, $\lim_{m\to\infty} x_m = \infty$. Consider equation

$$\Delta\left[h_m h_{m+1} \Delta\left(\frac{1}{h_m}\Delta^2 v_m\right)\right] + b_m h_{m+1} \frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}} v^\lambda_{m+\sigma} = 0. \tag{3.2}$$

This equation has a solution $v = x$ of type (a). Using (3.1), we have that there exist $K > 0$ such that $\frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}} \geq K$. We apply to (3.2), lemma in [11, Lemma 4] with $\alpha = \beta = \gamma = 1$ and $\sigma \geq 1$. We have

$$b_m h_{m+1} \frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}} \geq \frac{K}{2} b_m, \text{ for large } m.$$

Thus,

$$\sum_{m=1}^{\infty} b_m h_{m+1} \frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}} m^\lambda = \infty,$$

and by [11, Lemma 4 and Corollary 1], equation (3.2) is oscillatory. This is a contradiction with the fact that (3.2) has a nonoscillatory solution $v = x$.

(ii) Let x be of type (b). Then, there exist $\lim_{m\to\infty} x_m$. Because of the continuity of f there exist $K > 0$ such that

$$\lim_{m\to\infty} \frac{f(x_{m+\sigma})}{x^\lambda_{m+\sigma}} \geq K, \text{ for large } m,$$

and proceeding the similar way as in (i), we get that (3.2) has no nonoscillatory solution of type (b). This completes the proof. □

Theorem 3.3. *Let* (1.2) *holds and there exist* $\lambda > 0$ *such that*

$$\lim_{u\to\infty} \frac{f(u)}{u^\lambda} > 0.$$

Equation (1.1) *with* $\sigma \geq 3$ *is oscillatory if any of the following conditions hold:*

(i) $\lambda > 1$ *and*

$$\sum_{m=m_0}^{\infty} m^2 \sum_{k=m-2}^{\infty} b_k = \infty, \tag{3.3}$$

(ii) $\lambda = 1$ *and*

$$\lim_{m\to\infty} \sup \left(m^3 \sum_{k=m-3}^{\infty} b_k \right) = \infty. \tag{3.4}$$

Proof. (i) $\lambda > 1$, by [12, Corollary 2-(i)] equation (1.1) with $\sigma \geq 3$ has no solution of type (a) or type (b) if

$$\sum_{m=m_0}^{\infty} m^2 \sum_{k=m-2}^{\infty} b_k = \infty.$$

(ii) $\lambda = 1$, by [12, Corollary 2-(ii)] equation (3.4) implies

$$\lim_{m\to\infty} \sup \left(m \sum_{m=m_0}^{\infty} b_k k^2 \right) > 1.$$

This completes the proof. □

Acknowledgement. The author emits his sincere thanks to Prof. Zuzana Došlá for her kind interest, encouragements, valuable suggestions and comments. The paper has been prepared during the study of the author in Department of Mathematics and Statistics, Faculty of Science, Masaryk University.

References

[1] R.P. Agarwal, *Difference equations and inequalities.* Marcel Dekker, Second Edition, New York, 2000.

[2] R.P. Agarwal, S.R. Grace, P.J.Y. Wong, *Oscillatory behavior of fourth order nonlinear difference equations*, New Zealand J. Math., 36 (2007), 101-111.

[3] R.P. Agarwal, S.R. Grace, J.V. Manojlović, *On the oscillatory properties of certain fourth order nonlinear difference equations*, J. Math. Anal. Appl., 322 (2006), 930-956.

[4] R.P. Agarwal, J.V. Manojlović, *Asymptotic behavior of positive solutions of fourth order nonlinear difference equations*, Ukrainian Math. J. 60 (2008), 6-28.

[5] R.P. Agarwal, J.V. Manojlović, *Asymptotic behavior of nonoscillatory solutions of fourth order nonlinear difference equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 16 (2009), 155-174.

[6] M. Bartusek, Z. Došlá, *Asymptotic problems for fourth-order nonlinear differential equations*, Bound. Value Probl. (2013), 2013:89, 15 pp.

[7] M. Bartusek, M. Cecchi, Z. Dosla, M. Marini, *Fourth-order differential equation with deviating argument*, Abstr. Appl. Anal. (2012), Art. ID 185242, 17 pp.

[8] M. Cecchi, Z. Došlá, M. Marini, *Limit behavior for quasilinear difference equations*, Proceeding of the Sixth International Conference on Difference Equation, (2004), 383-390.

[9] M. Cecchi, Z. Došlá, M. Marini, I. Vrkoč, *Asymptotic properties for half-linear difference equation*, Math. Bohem., 131 (2006), No. 4, 347-363

[10] M. Cecchi , Z. Došlá, M. Marini, *Positive decreasing solutions of quasi-linear difference equations*, Comput. Math. Appl. 42 (2001), no. 10-11, 1401-1410.

[11] Z. Došlá, J. Krejčová, *Oscillation of a class of the fourth-order nonlinear difference equations*, Adv. Difference Equ. (2012), 2012:99, 14 pp.

[12] Z. Došlá, J. Krejčová, *Asymptotic and oscillatory properties of the fourth-order nonlinear difference equations*, Appl. Math. Comput. 249 (2014), 164-173.

[13] J. W. Hooker, W. T. Patula, *A second-order nonlinear difference equation: oscillation and asymptotic behavior*, J. Math. Anal. Appl. 91 (1983), no. 1, 9-29.

[14] R. Jankowski, E. Schmeidel, J. Zonenberg, *Oscillatory properties of solutions of the fourth order difference equations with quasidifferences*, Opuscula Math. 34 (2014), no. 4, 789-797.

[15] W. G. Kelley, A. C. Peterson, *Difference equations: An introduction with applications, 2nd ed.,* Academic Press, Boston, 2001.

[16] M. Migda, A. Musielak, E. Schmeidel, *On a class of fourth-order nonlinear difference equations*, Adv. Difference Equ. (2004), no. 1, 23-36.

[17] W. T. Patula, *Growth and oscillation properties of second order linear difference equations*, SIAM J. Math. Anal. 10 (1979), no. 1, 55-61.

[18] T. Peil, A. Peterson, *Asymptotic behavior of solutions of a two term difference equation*, Rocky Mountain J. Math. 24 (1994), no. 1, 233-252.

[19] E. Schmeidel, *Oscillation and nonoscillation theorems for fourth order difference equations*, Rocky Mountain J. Math. 33 (2003), no. 3, 1083-1094.

Current address: Ankara University, Faculty of Sciences, Dept. of Mathematics, Ankara, TURKEY

E-mail address: ekavgaci@ankara.edu.tr

ORCID Address: http://orcid.org/0000-0002-8605-4346