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THE VORONOVSKAJA TYPE ASYMPTOTIC FORMULA FOR q-DERIVATIVE OF INTEGRAL GENERALIZATION OF q-BERNSTEIN OPERATORS

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Abstract. The Voronovskaja type asymptotic formula for function having q-derivative of the integral generalization Bernstein operators based on q-integer is discussed. The same formula for Stancu type generalization of this operators is mentioned.

1. Introduction

The classical Bernstein-Durrmeyer operators D_n introduced by Durrmeyer [1] is associated with an integrable function f on the interval $[0,1]$ and is defined as

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, \ x \in [0,1], \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$.

These operators have been studied by Derriennic [2] and many others. For the last 30 years, q-calculus has been an active area of research in approximation theory. In 1987, the q-analogues of Bernstein operators was introduced by Lupas [3] and in [4], q-generalization of the operators (1.1) was introduced as

$$D_{n,q}(f;x) = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x)\int_0^1 f(t)p_{n,k}(q;qt)d_qt, \tag{1.2}$$

where $p_{n,k}(q;x) = \binom{n}{k}_q x^k(1-x)_q^{n-k}$.

The rate of convergence of the operators (1.2) was discussed by Zeng et al. [5]. In 2014, Mishra and Patel [6, 7] introduced the generalization due to Stancu

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and proved Voronovskaja type asymptotic formula and various other approximation properties of the q-Durrmeyer-Stancu operators. Here, in this manuscript, we establish Voronovskaja type asymptotic formula for function having q-derivative.

2. Estimation of moments and Asymptotic formula

In the sequel, we shall need the following auxiliary results:

Theorem 1 ([8]). *If m-th ($m > 0, m \in \mathbb{N}$) order moments of operator (1.2) is defined as*

$$D^q_{n,m}(x) = D_{n,q}(t^m, x) = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 p_{n,k}(q;qt) t^m d_q t, \;\; x \in [0,1],$$

then $D^q_{n,0}(x) = 1$ and for $n > m+2$, we have the following recurrence relation,

$$[n+m+2]_q D^q_{n,m+1}(x) = ([m+1]_q + q^{m+1} x [n]_q) D^q_{n,m}(x) + x(1-x) q^{m+1} D_q(D^q_{n,m}(x)).$$

To establish asymptotic formula for functions having q-derivative, it is necessary to compute moments of first to fourth degree. Using above theorem one can have first, second, third and fourth order moments. The first three moments of Lemma 1 was also established in [4].

Lemma 1 ([4, 8]). *For all $x \in [0,1]$, $n = 1, 2, \ldots$ and $0 < q < 1$, we have*

- $D_{n,q}(1, x) = 1;$
- $D_{n,q}(t, x) = \frac{1 + qx[n]_q}{[n+2]_q};$
- $D_{n,q}(t^2, x) = \frac{q^3 x^2 [n]_q ([n]_q - 1) + (1+q)^2 q x [n]_q + 1 + q}{[n+3]_q [n+2]_q};$
- $D_{n,q}(t^3, x) = \frac{q^9 x^3 [n]_q [n-1]_q [n-2]_q + x^2 q^4 [3]^2_q [n]_q [n-1]_q + xq[2]_q [3]^2_q [n]_q + [3]_q [2]_q}{[n+4]_q [n+3]_q [n+2]_q};$
- $D_{n,q}(t^4, x) = \frac{q^{16} x^4 [n]_q [n-1]_q [n-2]_q [n-3]_q + q^9 x^3 [4]^2_q [n]_q [n-1]_q [n-2]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q}$
  $+ \frac{q^4 x^2 [2]_q [3]^2_q (1+q^2)^2 [n]_q [n-1]_q + qx[2]_q [3]_q [4]^2_q [n]_q + [2]_q [3]_q [4]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q}$

Lemma 2. *For all $x \in [0,1]$, $n = 1, 2, \ldots$ and $0 < q < 1$, we have*

- $D_{n,q}\left((t-x)_q, x\right) = \frac{1 - \left(1 + q^{n+1}\right) x}{[n+2]_q};$
- $D_{n,q}\left((t-x)^2_q, x\right) = \frac{qx^2 [2]_q \left([n]^2_q (1-q)^2 q^2 + [n]_q (2q^3 - [3]_q) + [3]_q\right) - x[2]_q ([3]_q + q[n]_q (-1 - q + q^2)) + [2]_q}{[n+3]_q [n+2]_q};$
- $D_{n,q}\left((t-x)^3_q, x\right)$
  $= q^2 x^3 \left\{ \frac{q^7 [n]_q [n-1]_q [n-2]_q}{[n+2]_q [n+3]_q [n+4]_q} - \frac{q^2 [3]_q [n]_q [n-1]_q}{[n+2]_q [n+3]_q} + \frac{[2]_q [n]_q - q[n+2]_q}{[n+2]_q [n+3]_q [n+4]_q} \right\}$
  $+ qx^2 \left\{ \frac{q^3 [3]^2_q [n]_q [n-1]_q}{[n+2]_q [n+3]_q [n+4]_q} - \frac{[2]^2_q [3]_q [n]_q}{[n+2]_q [n+3]_q} + \frac{[2]_q}{[n+2]_q} \right\}$
  $+ x[2]_q [3]_q \left\{ \frac{q[3]_q [n]_q - [n+4]_q}{[n+2]_q [n+3]_q [n+4]_q} \right\} + \frac{[3]_q [2]_q}{[n+2]_q [n+3]_q [n+4]_q};$
- $D_{n,q}\left((t-x)^4_q, x\right)$
  $= q^4 x^4 \left\{ \frac{q^{12} [n]_q [n-1]_q [n-2]_q [n-3]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} - \frac{q^5 [4]_q [n]_q [n-1]_q [n-2]_q}{[n+4]_q [n+3]_q [n+2]_q} + \frac{q\left([5]_q + q^2\right) [n]_q [n-1]_q}{[n+3]_q [n+2]_q} - \frac{[4]_q [n]_q}{[n+2]_q} + q^2 \right\}$
  $+ x^3 q^2 \left\{ \frac{q^7 [4]^2_q [n]_q [n-1]_q [n-2]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} - \frac{q^2 [3]^2_q [4]_q [n]_q [n-1]_q}{[n+4]_q [n+3]_q [n+2]_q} + \frac{\left([5]_q + q^2\right) [2]^2_q [n]_q}{[n+3]_q [n+2]_q} - \frac{q[4]_q}{[n+2]_q} \right\}$
  $+ qx^2 \left\{ \frac{q^3 [2]_q [3]^2_q (1+q^2) [n]_q [n-1]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} - \frac{[2]_q [3]^2_q [4]_q [n]_q}{[n+4]_q [n+3]_q [n+2]_q} + \frac{[2]_q \left([5]_q + q^2\right)}{[n+3]_q [n+2]_q} \right\}$
  $+ x \left\{ \frac{[2]_q [3]_q [4]_q \left(q[4]_q [n]_q - [n+5]_q\right)}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q} \right\} + \frac{[2]_q [3]_q [4]_q}{[n+5]_q [n+4]_q [n+3]_q [n+2]_q}.$

Theorem 2. *Let f be bounded and integrable on the interval $[0,1]$ and (q_n) denote a sequence such that $0<q_n<1$, $q_n\to 1$ and $q_n^n\to c$ as $n\to\infty$, where c is arbitrary constant. Then we have for a point $x\in(0,1)$,*

$$\lim_{n\to\infty}[n]_{q_n}[D_{n,q_n}(f;x)-f(x)]=(1-2x)\lim_{n\to\infty}D_{q_n}f(x)+x(1-x)\lim_{n\to\infty}D^2_{q_n}f(x).$$

Proof: By q-Taylor formula [9] for f, we have

$$f(t)=f(x)+D_{q_n}f(x)(t-x)+\frac{1}{[2]_{q_n}}D^2_{q_n}f(x)(t-x)^2_{q_n}+\theta_{q_n}(x;t)(t-x)^2_{q_n},$$

for $0<q<1$, where

$$\theta_{q_n}(x;t)=\begin{cases}\dfrac{f(t)-f(x)-D_{q_n}f(x)(t-x)-\frac{1}{[2]_{q_n}}D^2_{q_n}f(x)(t-x)^2_{q_n}}{(t-x)^2_{q_n}} & \text{if } x\neq t\\ 0, & \text{if } x=t.\end{cases}\tag{2.1}$$

We know that for n large enough

$$\lim_{t\to x}\theta_{q_n}(x;t)=0.\tag{2.2}$$

That is for any $\epsilon>0$, there exists a $\delta>0$ such that

$$|\theta_{q_n}(x;t)|\le\epsilon;\tag{2.3}$$

for $|t-x|<\delta$ and n sufficiently large. Using (2.1), we can write

$$D_{n,q_n}(f;x)-f(x)=D_{q_n}f(x)D_{n,q_n}((t-x)_{q_n};x)+\frac{D^2_{q_n}f(x)}{[2]_{q_n}}D_{n,q_n}((t-x)^2_{q_n};x)+E^{q_n}_n(x),$$

where

$$E^q_n(x)=[n+1]_{q_n}\sum_{k=0}^{n}q^{-k}p_{n,k}(q_n;x)\int_0^1\theta_{q_n}(x;t)p_{n,k}(q_n;q_nt)\,(t-x)^2_{q_n}\,d_{q_n}t.$$

By Lemma 2, we have

$$\lim_{n\to\infty}[n]_{q_n}D_{n,q_n}((t-x)_{q_n};x)=(1-2x)\text{ and }\lim_{n\to\infty}[n]_{q_n}D_{n,q_n}((t-x)^2_{q_n};x)=2x(1-x).$$

In order to complete the proof of the theorem, it is sufficient to show that $\lim_{n\to\infty}[n]_{q_n}E^{q_n}_n(x)=0$. We proceed as follows: Let

$$P^{q_n}_{n,1}(x)=[n]_{q_n}[n+1]_{q_n}\sum_{k=0}^{n}{q_n}^{-k}p_{n,k}(q_n;x)\int_0^1\theta_{q_n}(x;t)p_{n,k}(q_n;q_nt)\,(t-x)^2_{q_n}\,\chi_x(t)d_{q_n}t$$

and

$$P^{q_n}_{n,2}(x)=$$
$$[n]_{q_n}[n+1]_{q_n}\sum_{k=0}^{n}{q_n}^{-k}p_{n,k}(q_n;x)\int_0^1\theta_{q_n}(x;t)p_{n,k}(q_n;q_nt)\,(t-x)^2_{q_n}\,(1-\chi_x(t))\,d_{q_n}t,$$

so that

$$[n]_{q_n} E_n^{q_n}(x) \leq P_{n,1}^{q_n}(x) + P_{n,2}^{q_n}(x),$$

where $\chi_x(t)$ is the characteristic function of the interval $\{t : |t-x| < \delta\}$.

It follows from (2.3) that

$$P_{n,1}^{q_n}(x) = 2\epsilon x(1-x) \text{ as } n \to \infty.$$

If $|t-x| \geq \delta$, then $|\theta_{q_n}(x;t)| \leq \frac{M}{\delta^2}(t-x)^2$, where $M > 0$ is a constant. Since

$$\begin{aligned}(t-x)^2 &= \left(t - q_n^2 x + q_n^2 x - x\right)\left(t - q_n^3 x + q_n^3 x - x\right) \\ &= \left(t - q_n^2 x\right)\left(t - q_n^3 x\right) + x(q_n^3 - 1)\left(t - q_n^2 x\right) + x(q_n^2 - 1)\left(t - q_n^2 x\right) \\ &\quad + x^2(q_n^2 - 1)(q_n^2 - q_n^3) + x^2(q_n^2 - 1)(q_n^3 - 1),\end{aligned}$$

we have

$$\begin{aligned}|P_{n,2}^{q_n}(x)| \leq \frac{M}{\delta^2} &\left\{[n]_{q_n} D_{n,q_n}((t-x)_{q_n}^4; x) + x(2 - q_n^2 - q_n^3)[n]_{q_n} D_{n,q_n}((t-x)_{q_n}^3; x)\right. \\ &\left. + x^2(q_n^2 - 1)^2 [n]_{q_n} D_{n,q_n}((t-x)_{q_n}^2; x)\right\}.\end{aligned}$$

Using Lemma 2, we have

$$D_{n,q_n}((t-x)_{q_n}^4; x) \leq \frac{C_1}{[n]_{q_n}^3},\ D_{n,q_n}((t-x)_{q_n}^3; x) \leq \frac{C_2}{[n]_{q_n}^2} \text{ and } D_{n,q_n}((t-x)_{q_n}^2; x) \leq \frac{C_3}{[n]_{q_n}},$$

and the desired result is obtained.

Corollary 1. *Let f be bounded and integrable on the interval $[0,1]$ and (q_n) denote a sequence such that $0 < q_n < 1$, $q_n \to 1$ and $q_n^n \to c$ as $n \to \infty$, where c is arbitrary constant. Suppose that the first and second derivatives $f'(x)$ and $f''(x)$ exist at a point $x \in (0,1)$. Then, we have, for a point $x \in (0,1)$*

$$\lim_{n\to\infty} [n]_{q_n} [D_{n,q_n}(f;x) - f(x)] = (1-2x)f'(x) + x(1-x)f''(x).$$

3. Asymptotic formula for the Durrmeyer-Stancu Operators

In the year 1968, Stancu [10] generalized Bernstein operators and discussed its approximation properties. After that many researchers gave Stancu type generalization of several operators on finite and infinite intervals. We refer the readers to [11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the references there in. As mention in the introduction, Stancu generalization of q-Durrmeyer operators (1.2) was discussed by Mishra and Patel [6], which is defined as follows:

$$D_{n,q}^{\alpha,\beta} = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 f\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right) p_{n,k}(q;qt) d_q t, \tag{3.1}$$

where $0 \leq \alpha \leq \beta$ and $p_{n,k}(q;x)$ as same as defined in (1.2). We shall need the following lemmas for proving our results.

Lemma 3 ([7]). *We have* $D_{n,q}^{\alpha,\beta}(1;x)=1,\ D_{n,q}^{\alpha,\beta}(t;x)=\frac{[n]_q+\alpha[n+2]_q+qx[n]_q^2}{[n+2]_q([n]_q+\beta)}$,

$$D_{n,q}^{\alpha,\beta}(t^2;x)=\frac{q^3[n]_q^3([n]_q-1)x^2+\left(\left(q(1+q)^2+2\alpha q^4\right)[n]_q^3+2\alpha q[3]_q[n]_q^2\right)x}{([n]_q+\beta)^2[n+2]_q[n+3]_q} + \frac{\alpha^2}{([n]_q+\beta)^2}+\frac{(1+q+2\alpha q^3)[n]_q^2+2\alpha[3]_q[n]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q}.$$

Lemma 4 ([7]). *We have*

$$D_{n,q}^{\alpha,\beta}(t-x,x)=\left(\frac{q[n]_q^2}{[n+2]_q([n]_q+\beta)}-1\right)x+\frac{[n]_q+\alpha[n+2]_q}{[n+2]_q([n]_q+\beta)},$$

$$D_{n,q}^{\alpha,\beta}((t-x)^2,x)=\frac{q^4[n]_q^4-q^3[n]_q^3-2q[n]_q^2[n+3]_q([n]_q+\beta)+[n+2]_q[n+3]_q([n]_q+\beta)^2}{([n]_q+\beta)^2[n+2]_q[n+3]_q}x^2 + \frac{q(1+q)^2[n]_q^3+2q\alpha[n]_q^2[n+3]_q-(2[n]_q+2\alpha[n+2]_q)[n+3]_q([n]_q+\beta)}{([n]_q+\beta)^2[n+2]_q[n+3]_q}x + \frac{(1+q)[n]_q^2+2\alpha[n]_q[n+3]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q}.$$

Remark 1 ([7]). *For all* $m\in\mathbf{N}\cup\{0\},\ 0\le\alpha\le\beta$, *we have the following recursive relation for the images of the monomials* t^m *under* $D_{n,q}^{\alpha,\beta}$ *in terms of* $D_{n,q},\ j=0,1,2,\dots,m$, *as*

$$D_{n,q}^{\alpha,\beta}(t^m;x)=\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\alpha^{m-j}}{([n]_q+\beta)^m}D_{n,q}(t^j,x).$$

Now, let us compute the moments and central moments of order 3 and 4 for the operators (3.1) in the following manner:

$$D_{n,q}^{\alpha,\beta}(t^3;x)=\frac{q^9[n]_q^4[n-1]_q[n-2]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}x^3+\frac{q^4[n]_q^3[n-1]_q\left([3]_q^2[n]_q+\alpha[n+4]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}x^2 + \frac{q[n]_q^2\left([2]_q[3]_q^2[n]_q^2+\alpha[2]_q^2[n]_q[n+4]_q+\alpha^2[n+4]_q[n+3]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}x + \frac{[n]_q^3[3]_q[2]_q+\alpha[2]_q[n]_q^2[n+4]_q+\left(\alpha^2[n]_q+\alpha^3[n+2]_q\right)[n+4]_q[n+3]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q}.$$

Also,

$$D_{n,q}^{\alpha,\beta}(t^4;x)=\frac{q^{16}[n]_q^5[n-1]_q[n-2]_q[n-3]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}x^4+\frac{q^9[n]_q^4[n-1]_q[n-2]_q\left([4]_q^2[n]_q+\alpha[n+5]_q\right)}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}x^3 + q^4[n]_q^3[n-1]_q\left\{\frac{[2]_q[3]_q^2(1+q^2)^2[n]_q^2+\alpha[3]_q^2[n]_q[n+5]_q+\alpha^2[n+4]_q[n+5]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}\right\}x^2 + \frac{q[n]_q^2\left([2]_q[3]_q[4]_q^2[n]_q^3+[2]_q[3]_q^2\alpha[n]_q^2[n+5]_q+[2]_q^2\alpha^2[n]_q[n+4]_q[n+5]_q+\alpha^3[n+3]_q[n+4]_q[n+5]_q\right)}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}x + \frac{[4]_q[3]_q[2]_q[n]^4+\alpha[3]_q[2]_q[n]_q^3[n+5]_q+\alpha^2[2]_q[n]_q^2[n+4]_q[n+5]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q}+\frac{\alpha^3[n]_q+\alpha^4[n+2]_q}{([n]_q+\beta)^4[n+2]_q}.$$

Now, using the identity $(t-x)_q^3 = t^3 - [3]_q x t^2 + q[2]_q x^2 t - q^3 x^3$ and linear properties of the operators $D_{n,q}^{\alpha,\beta}$, we get

$$\begin{aligned}
D_{n,q}^{\alpha,\beta}\left((t-x)_q^3;x\right) &= q^2 \Bigg[\frac{q^7[n]_q^4[n-1]_q[n-2]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} - \frac{q^2[3]_q[n]_q^3[n-1]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q} + \frac{[2]_q[n]_q^2}{[n+2]_q\left([n]_q+\beta\right)} - q \Bigg] x^3 \\
&+ q \Bigg[\frac{q^3[n]_q^3[n-1]_q\left([n]_q[3]_q^2+\alpha[n+4]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} - \frac{[3]_q\left(\left([2]_q^2+2\alpha q^3\right)[n]_q^3+2\alpha[3]_q[n]_q^2\right)}{([n]_q+\beta)^2[n+2]_q[n+3]_q} + \frac{[2]_q([n]_q+\alpha[n+2]_q)}{[n+2]_q\left([n]_q+\beta\right)} \Bigg] x^2 \\
&+ \Bigg[\frac{q[n]_q^2\left([2]_q[3]_q^2[n]_q^2+[2]_q^2\alpha[n]_q[n+4]_q+\alpha^2[n+4]_q[n+3]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} - \frac{[3]_q\alpha^2}{([n]_q+\beta)^2} - \frac{(1+q+2\alpha q^3)[3]_q[n]_q^2+2\alpha[3]_q^2[n]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q} \Bigg] x \\
&+ \frac{[n]_q^3[3]_q[2]_q+\alpha[2]_q[n]_q^2[n+4]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} + \frac{[n]_q\alpha^2+\alpha^3[n+2]_q}{[n+2]_q([n]_q+\beta)^3}.
\end{aligned}$$

Finally, using identity $(t-x)_q^4 = t^4 - [4]_q x t^3 + q\left([5]_q+q^2\right)x^2t^2 - q^3x^3[4]_q t + q^6x^4$, we have

$$\begin{aligned}
D_{n,q}^{\alpha,\beta}\left((t-x)_q^4;x\right) &= q^4 \Bigg[\frac{q^{12}[n]_q^5[n-1]_q[n-2]_q[n-3]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q} - \frac{q^5[4]_q[n]_q^4[n-1]_q[n-2]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} \\
&+ \frac{q\left([5]_q+q^2\right)[n]_q^3[n-1]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q} - \frac{[4]_q[n]_q^2}{[n+2]_q\left([n]_q+\beta\right)} + q^2 \Bigg] x^4 \\
&+ q^2 \Bigg[\frac{q^7[n]_q^4[n-1]_q[n-2]_q\left([4]_q^2[n]_q+\alpha[n+5]_q\right)}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q} - \frac{q^2[4]_q[n]_q^3[n-1]_q\left([3]_q^2[n]_q+\alpha[n+4]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} \\
&+ \frac{\left([5]_q+q^2\right)\left(\left([2]_q^2+2\alpha q^3\right)[n]_q^3+2\alpha[3]_q[n]_q^2\right)}{([n]_q+\beta)^2[n+2]_q[n+3]_q} - \frac{q[4]_q\left([n]_q+\alpha[n+2]_q\right)}{[n+2]_q\left([n]_q+\beta\right)} \Bigg] x^3 \\
&+ q \Bigg[q^3[n]_q^3[n-1]_q \left\{ \frac{[2]_q[3]_q^2(1+q^2)^2[n]_q^2+\alpha[3]_q^2[n]_q[n+5]_q+\alpha^2[n+4]_q[n+5]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q} \right\} \\
&- \frac{[4]_q[n]_q^2\left([2]_q[3]_q^2[n]_q^2+[2]_q\alpha[n]_q[n+4]_q+\alpha^2[n+4]_q[n+3]_q\right)}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} \\
&+ \frac{\alpha^2\left([5]_q+q^2\right)}{([n]_q+\beta)^2} + \frac{\left([5]_q+q^2\right)(1+q+2\alpha q^3)[n]_q^2+2\alpha[3]_q[n]_q}{([n]_q+\beta)^2[n+2]_q[n+3]_q} \Bigg] x^2 \\
&+ \Bigg[\frac{q[n]_q^2\left([2]_q[3]_q[4]_q^2[n]_q^3+\alpha[2]_q[3]_q^2[n]_q^2[n+5]_q+\alpha^2[2]_q^2[n]_q[n+4]_q[n+5]_q+\alpha^3[n+3]_q[n+4]_q[n+5]_q\right)}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q} \\
&- \frac{[n]_q^3[4]_q[3]_q[2]_q}{([n]_q+\beta)^3[n+4]_q[n+3]_q[n+2]_q} - \frac{\alpha[4]_q[2]_q[n]_q^2}{([n]_q+\beta)^3[n+3]_q[n+2]_q} - \frac{[4]_q[n]_q\alpha^2+\alpha^3[4]_q[n+2]_q}{[n+2]_q([n]_q+\beta)^3} \Bigg] x \\
&+ \frac{[4]_q[3]_q[2]_q[n]^4+\alpha[3]_q[2]_q[n]_q^3[n+5]_q+\alpha^2[2]_q[n]_q^2[n+4]_q[n+5]_q}{([n]_q+\beta)^4[n+5]_q[n+4]_q[n+3]_q[n+2]_q} + \frac{\alpha^3[n]_q+\alpha^4[n+2]_q}{([n]_q+\beta)^4[n+2]_q}.
\end{aligned}$$

Theorem 3. *Let f be bounded and integrable on the interval $[0,1]$ and let (q_n) denote a sequence such that $0<q_n<1$, $q_n\to 1$ and $q_n^n\to c$ as $n\to\infty$, where c is arbitrary constant. Then, we have, for a point $x\in(0,1)$*

$$\lim_{n\to\infty}[n]_{q_n}[D_{n,q_n}^{\alpha,\beta}(f;x)-f(x)] = (1+\alpha-(2+\beta)x)\lim_{n\to\infty}D_{q_n}f(x)+x(1-x)\lim_{n\to\infty}D_{q_n}^2f(x).$$

The proof of the above lemma follows along the lines of the proof of Theorem 2, using Lemma 4 and Remark 1; thus, we omit the details.

Corollary 2 ([6]). *Let f be bounded and integrable on the interval $[0,1]$ and let (q_n) denote a sequence such that $0 < q_n < 1$, $q_n \to 1$ and $q_n^n \to c$ as $n \to \infty$, where c is arbitrary constant. Suppose that the first and second derivatives $f'(x)$ and $f''(x)$ exist at a point $x \in (0,1)$. Then, we have, for a point $x \in (0,1)$,*

$$\lim_{n\to\infty}[n]_{q_n}[D_{n,q_n}^{\alpha,\beta}(f;x) - f(x)] = (1+\alpha-(2+\beta)x)f'(x) + x(1-x)f''(x).$$

Remark 2. *Theorem 2 and Theorem 3, give asymptotic formula for q-Durrmeyer operators and q-Durrmeyer-Stancu operators respectively. If f has first and second derivatives, then $\lim_{n\to\infty} D_{q_n}f(x) = f'(x)$ and $\lim_{n\to\infty} D_{q_n}^2 f(x) = f''(x)$. We obtain the results of Mishra and Patel* [6, Theorem 5]*, which are mentioned in Corollary 2. So our results are more general than the existing ones.*

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