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FIXED POINTS FOR GENERALIZED $(\mathcal{F}, h, \alpha, \mu) - \psi-$CONTRACTIONS IN $b-$METRIC SPACES

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Abstract. In this paper, we defined $(\mathcal{F}, h, \alpha, \mu) - \psi-$contractions using pair of $(\mathcal{F}, h)$ upper class functions for $\alpha-$admissible and $\mu-$subadmissible mappings.We proved some fixed point theorems for this type contractive mappings in $b-$metric spaces. Our results generalize $\alpha-$admissible results in the literature.

1. Introduction

Definition 1. *([9]) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:*

(i) $d(x, y) = 0$ *if and only if* $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, every metric is a b-metric with $s = 1$.

Example 1. *([1]) Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a $b-$metric with $s = 2^{p-1}$.*

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = R$ is the set of real numbers and $d(x, y) = |x - y|$ is usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a $b-$metric on R with $s = 2$.But is not a metric on R.

Definition 2. *([7]) Let $\{x_n\}$ be a sequence in a b-metric space (X, d).*

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(a) $\{x_n\}$ *is called* $b-$*convergent if and only if there is* $x\in X$ *such that* $d(x_n,x)\to 0$ *as* $n\to\infty$.
(b) $\{x_n\}$ *is a* $b-$*Cauchy sequence if and only if* $d(x_n,x_m)\to 0$ *as* $n,m\to\infty$.

A b-metric space is said to be complete if and only if each $b-$*Cauchy sequence in this space is* $b-$*convergent.*

Proposition 1. *([7]) In a* $b-$*metric space* (X,d), *the following assertions hold:*

(p1) A $b-$convergent sequence has a unique limit.
(p2) Each $b-$convergent sequence is $b-$Cauchy.
(p3) In general, a $b-$metric is not continuous.

On the other hand the notion of $\alpha-\psi$-contractive type mapping was introduced by Samet et al. [11],[17]. Also, see ([10],[12],[13-15])

Now we give some definitions that will be used throughout this paper.

A mapping $\psi:[0,\infty)\to[0,\infty)$ is called a comparison function if it is increasing and $\lim_{n\to\infty}\psi^n(t)=0$ for all $t>0$.

Lemma 1. *([5]) Let* $\psi:[0,\infty)\to[0,\infty)$ *is a comparison function then*

(a) *each iterate* ψ^n *of* ψ, $n\geq 1$, *is also a comparison function,*
(b) ψ *is continuous at* $t=0$,
(c) $\psi(t)<t$ *for all* $t>0$.

Definition 3. *([5]) A function* $\psi:[0,\infty)\to[0,\infty)$ *is said to be a (c)-comparison function if*

(c1) ψ *is increasing,*
(c2) *there exists* $k_0\in\mathbb{N}$, $a\in(0,1)$ *and a convergent series of nonnegative terms* $\sum\limits_{k=1}^{\infty}v_k$, *such that* $\psi^{k+1}(t)\leq a\psi^k(t)+v_k$, *for* $k\geq k_0$ *and any* $t\in[0,\infty)$.

Definition 4. *([6]) Let* $s\geq 1$ *be a real number. A function* $\psi:[0,\infty)\to[0,\infty)$ *is said to be a (b)-comparison function if*

(b1) ψ *is monotonically increasing,*
(b1) *there exists* $k_0\in\mathbb{N}$, $a\in(0,1)$ *and a convergent series of nonnegative terms* $\sum\limits_{k=1}^{\infty}v_k$, *such that* $s^{k+1}\psi^{k+1}(t)\leq as^k\psi^k(t)+v_k$, *for* $k\geq k_0$ *and any* $t\in[0,\infty)$.

When $s=1$, (b)-comparison function reduces to (c)-comparison function.

We denote by Ψ_b for the class of (b)-comparison function.

Lemma 2. *([4]) If* $\psi:[0,\infty)\to[0,\infty)$ *is a (b)-comparison function then one has the following:*

(i) $\sum\limits_{k=0}^{\infty}s^k\psi^k(t)$ *converges to any* $t\in R^+$,

(ii) *the function* $b_s : [0, \infty) \to [0, \infty)$ *defined by* $b_s(t) = \sum_{k=0}^{\infty} s^k \psi^k(t)$, $t \in R^+$, *increasing and continuous at* 0.

Any (b)-comparison function is a comparison function.

Definition 5. *([17])For any nonempty set* X, *let* $T : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$ *be mappings.* T *is called* α*-admissible if for all* $x, y \in X$,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 6. *([16])Let* $T : X \to X$, $\mu : X \times X \to R^+$. *We say* T *is an* μ*-subadmissible mapping if*

$$x, y \in X, \quad \mu(x, y) \leq 1 \quad \Longrightarrow \quad \mu(Tx, Ty) \leq 1.$$

Bota et. al. in ([8]) gave the definition of $\alpha - \psi -$ contractive mapping of type (b) in $b-$metric space which is a generalization of Definition 9.

Definition 7. *Let* (X, d) *be a* $b-$*metric space and* $T : X \to X$ *be a given mapping.* T *is called an* $\alpha - \psi -$*contractive mapping of type (b), if there exists two functions* $\alpha : X \times X \to [0, \infty)$ *and* $\psi \in \Psi_b$ *such that*

$$\alpha(x, y)\, d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X.$$

Definition 8. *([2],[3])We say that the function* $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is a function of subclass of type I, if* $x \geq 1 \Longrightarrow h(1, y) \leq h(x, y)$ *for all* $y \in \mathbb{R}^+$.

Example 2. *([2],[3])Define* $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *by:*

(a) $h(x, y) = (y + l)^x, l > 1$;
(b) $h(x, y) = (x + l)^y, l > 1$;
(c) $h(x, y) = x^n y$, $n \in \mathbb{N}$;
(d) $h(x, y) = y$;
(e) $h(x, y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y$, $n \in \mathbb{N}$;
(f) $h(x, y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1$, $n \in \mathbb{N}$

for all $x, y \in \mathbb{R}^+$. *Then* h *is a function of subclass of type I.*

Definition 9. *([2],[3]) Let* $h, \mathcal{F} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, *then we say that the pair* $(\mathcal{F}, h)$ *is an upper class of type I, if* h *is a function of subclass of type I and: (i)* $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$, *(ii)* $h(1, y) \leq \mathcal{F}(1, t) \Longrightarrow y \leq t$ *for all* $t, y \in \mathbb{R}^+$.

Example 3. *([2],[3]) Define* $h, \mathcal{F} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *by:*

(a) $h(x, y) = (y + l)^x, l > 1$ *and* $\mathcal{F}(s, t) = st + l$;
(b) $h(x, y) = (x + l)^y, l > 1$ *and* $\mathcal{F}(s, t) = (1 + l)^{st}$;
(c) $h(x, y) = x^m y$, $m \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = st$;
(d) $h(x, y) = y$ *and* $\mathcal{F}(s, t) = t$;
(d) $h(x, y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) y$, $n \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = st$;
(e) $h(x, y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y, l > 1$, $n \in \mathbb{N}$ *and* $\mathcal{F}(s, t) = (1 + l)^{st}$

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type I.

Definition 10. *([2],[3])We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type II, if $x, y \geq 1 \Longrightarrow h(1, 1, z) \leq h(x, y, z)$ for all $z \in \mathbb{R}^+$.*

Example 4. *([2],[3])Define $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $h(x, y, z) = (z + l)^{xy}, l > 1$;
(b) $h(x, y, z) = (xy + l)^z, l > 1$;
(c) $h(x, y, z) = z$;
(d) $h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N}$;
(e) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}$

for all $x, y, z \in \mathbb{R}^+$. Then h is a function of subclass of type II.

Definition 11. *([2],[3])Let $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then we say that the pair $(\mathcal{F}, h)$ is an upper class of type II, if h is a subclass of type II and: (i) $0 \leq s \leq 1 \Longrightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$, (ii) $h(1, 1, z) \leq \mathcal{F}(s, t) \Longrightarrow z \leq st$ for all $s, t, z \in \mathbb{R}^+$.*

Example 5. *([2],[3]) Define $h \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:*

(a) $h(x, y, z) = (z + l)^{xy}, l > 1, \mathcal{F}(s, t) = st + l$;
(b) $h(x, y, z) = (xy + l)^z, l > 1, \mathcal{F}(s, t) = (1 + l)^{st}$;
(c) $h(x, y, z) = z, F(s, t) = st$;
(d) $h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N}, \mathcal{F}(s, t) = s^p t^p$
(e) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}, \mathcal{F}(s, t) = s^k t^k$

for all $x, y, z, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type II.

2. Main results

Definition 12. *([13])Let (X, d) be a $b-$metric space and $T : X \to X$ be a given mapping. T is called generalized $\alpha - \psi-$contractive mapping of type (I), if there exists two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$ such that for all $x, y \in X$*

$$\alpha (x, y) \, d (Tx, Ty)) \leq \psi(M_s (x, y))$$

where,

$$M_s (x, y) = \max \left\{ d (x, y), d (Tx, x), d (Ty, y), \frac{d (Tx, y) + d (x, Ty)}{2s} \right\}.$$

Theorem 1. *([13])Let (X, d) be a complete $b-$metric space. Suppose that $T : X \to X$ be a generalized $\alpha - \psi-$contractive mapping of type (I) and satisfies:*

(i) *T is $\alpha-$admissible*
(ii) *there exists $x_0 \in X$ such that $\alpha (x_0, Tx_0) \geq 1$*
(iii) *T is continuous.*

Then T has a fixed point.

Definition 13. *([13])Let (X,d) be a $b-$metric space and $T:X\to X$ be a given mapping. T is called generalized $\alpha-\psi-$contractive mapping of type (II), if there exists two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi_b$ such that for all $x,y\in X$*

$$\alpha(x,y)\,d(Tx,Ty))\leq\psi(N_s(x,y))$$

where,

$$N_s(x,y)=\max\left\{d(x,y),\frac{d(Tx,x)+d(Ty,y)}{2s},\frac{d(Tx,y)+d(Ty,x)}{2s}\right\}.$$

Definition 14. *Let (X,d) be a $b-$metric space and $T:X\to X$ be a given mapping. T is called generalized $(\mathcal{F},h,\alpha,\mu)-\psi-$contractive mapping of type (I), if there exists two functions $\alpha,\mu:X\times X\to[0,\infty)$ and $\psi\in\Psi_b$ such that for all $x,y\in X$*

$$h(\alpha(x,y),d(Tx,Ty))\leq\mathcal{F}(\mu(x,y),\psi(M_s(x,y))) \tag{2.1}$$

where,pair $(\mathcal{F},h)$ is an upper class of type I and

$$M_s(x,y)=\max\left\{d(x,y),d(Tx,x),d(Ty,y),\frac{d(Tx,y)+d(x,Ty)}{2s}\right\}.$$

Theorem 2. *Let (X,d) be a complete $b-$metric space. Suppose that $T:X\to X$ be a generalized $(\mathcal{F},h,\alpha,\mu)-\psi-$contractive mapping of type (I) and satisfies:*

(i) *T is $\alpha-$admissible and μ-subadmissible*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq1,\mu(x_0,Tx_0)\leq1$*
(iii) *T is continuous.*

Then T has a fixed point.

Proof. By assumption (ii), there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq1,\mu(x_0,Tx_0)\leq1$. Define the sequence $\{x_n\}$ in X by $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. If $x_n=x_{n+1}$ for some $n\in\mathbb{N}$, then x_n is a fixed point of T.

Assume that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$.

Since T is $\alpha-$admissible, then

$$\begin{aligned}\alpha(x_0,x_1)&=\alpha(x_0,Tx_0)\geq1\Longrightarrow\alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\geq1.\\ \mu(x_0,x_1)&=\mu(x_0,Tx_0)\leq1\Longrightarrow\mu(Tx_0,Tx_1)=\mu(x_1,x_2)\leq1.\end{aligned}$$

By induction, we get for all $n\in\mathbb{N}$,

$$\alpha(x_n,x_n+1)\geq1\quad,\quad\mu(x_n,x_n+1)\leq1. \tag{2.2}$$

Using (2.1) and (2.2)

$$\begin{aligned}h(1,d(x_n,x_{n+1}))&=h(1,d(Tx_{n-1},Tx_n))\\&\leq h(\alpha(x_{n-1},x_n),d(Tx_{n-1},Tx_n))\\&\leq\mathcal{F}(\mu(x_{n-1},x_n),\psi(M_s(x_{n-1},x_n)))\\&\leq\mathcal{F}(1,\psi(M_s(x_{n-1},x_n)))\\&\Longrightarrow\end{aligned}$$

$$d\left(x_n, x_{n+1}\right) \leq \psi(M_s\left(x_{n-1}, x_n\right)). \tag{2.3}$$

where

$$\begin{aligned} M_s\left(x_{n-1}, x_n\right) &= \max\left\{ \begin{array}{c} d\left(x_{n-1}, x_n\right), d\left(Tx_{n-1}, x_{n-1}\right), d\left(Tx_n, x_n\right), \\ \frac{d(Tx_{n-1}, x_n) + d(Tx_n, x_{n-1})}{2s} \end{array} \right\}, \\ &= \max\left\{ \begin{array}{c} d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n-1}\right), d\left(x_{n+1}, x_n\right), \\ \frac{d(x_n, x_n) + d(x_{n+1}, x_{n-1})}{2s} \end{array} \right\}, \\ &= \max\left\{ \begin{array}{c} d\left(x_{n-1}, x_n\right), d\left(x_{n+1}, x_n\right), \\ \frac{s[d(x_{n+1}, x_n) + d(x_n, x_{n-1})]}{2s} \end{array} \right\} \\ &\leq \max\left\{d\left(x_{n-1}, x_n\right), d\left(x_{n+1}, x_n\right)\right\}. \end{aligned}$$

If $M_s\left(x_{n-1}, x_n\right) = d\left(x_n, x_{n+1}\right)$, then from (2.3) and definition of ψ,

$$d\left(x_n, x_{n+1}\right) \leq \psi\left(d\left(x_n, x_{n+1}\right)\right) < d\left(x_n, x_{n+1}\right)$$

a contradiction. Thus $M_s\left(x_{n-1}, x_n\right) = d\left(x_{n-1}, x_n\right)$. Hence,

$$d\left(x_n, x_{n+1}\right) \leq \psi\left(d\left(x_{n-1}, x_n\right)\right) < d\left(x_{n-1}, x_n\right)$$

for all $n \geq 1$. If operations are continued in this way,

$$d\left(x_n, x_{n+1}\right) \leq \psi^n\left(d\left(x_0, x_1\right)\right). \tag{2.4}$$

Thus, for all $p \geq 1$,

$$\begin{aligned} d\left(x_n, x_{n+p}\right) &\leq sd\left(x_n, x_{n+1}\right) + s^2 d\left(x_{n+1}, x_{n+2}\right) + ... \\ &\quad + s^{p-1} d\left(x_{n+p-2}, x_{n+p-1}\right) + s^p d\left(x_{n+p-1}, x_{n+p}\right) \\ &\leq s\psi^n\left(d\left(x_0, x_1\right)\right) + s^2\psi^{n+1}\left(d\left(x_0, x_1\right)\right) + ... \\ &\quad + s^{p-1}\psi^{n+p-2}\left(d\left(x_0, x_1\right)\right) + s^p\psi^{n+p-1}\left(d\left(x_0, x_1\right)\right) \\ &= \frac{1}{s^{n-1}}[s^n\psi^n\left(d\left(x_0, x_1\right)\right) + s^{n+1}\psi^{n+1}\left(d\left(x_0, x_1\right)\right) + ... \\ &\quad + s^{p-n-2}\psi^{p-n-2}\left(d\left(x_0, x_1\right)\right) + s^{p+n-1}\psi^{p+n-1}\left(d\left(x_0, x_1\right)\right)]. \end{aligned}$$

Denoting $S_n = \sum\limits_{k=n}^{\infty} s^k\psi^k\left(d\left(x_0, x_1\right)\right)$, $n \geq 1$, we obtain

$$d\left(x_n, x_{n+p}\right) \leq \frac{1}{s^{n-1}}[S_{n+p-1} - S_{n-1}] \tag{2.5}$$

for $n \geq 1$, $p \geq 1$. From Lemma 2, we conclude that the series $\sum\limits_{k=0}^{\infty} s^k\psi^k\left(d\left(x_0, x_1\right)\right)$ is convergent. Thus, there exists

$$S = \lim_{n\to\infty} S_n \in [0, \infty).$$

Regarding $s \geq 1$ and by (2.5) $\{x_n\}$ is a Cauchy sequence in $b-$metric space (X, d). Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Using continuity of T,

$$x_{n+1} = Tx_n \to Tx^*$$

as $n \to \infty$. By the uniqueness of the limit, we get $x^* = Tx^*$. Hence x^* is a fixed point of T. □

Definition 15. *Let (X, d) be a $b-$metric space and $T : X \to X$ be a given mapping. T is called generalized $(\mathcal{F}, h, \alpha, \mu)-\psi-$contractive mapping of type (II), if there exists two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$ such that for all $x, y \in X$*

$$h(\alpha (x, y), d(Tx, Ty)) \leq \mathcal{F}(\mu (x, y), \psi(N_s (x, y))) \tag{2.6}$$

where,pair $(\mathcal{F}, h)$ is an upper class of type I and

$$N_s (x, y) = \max \left\{ d(x, y), \frac{d(Tx, x) + d(Ty, y)}{2s}, \frac{d(Tx, y) + d(Ty, x)}{2s} \right\}.$$

Theorem 3. *Let (X, d) be a complete $b-$metric space. Suppose that $T : X \to X$ be a generalized $(\mathcal{F}, h, \alpha, \mu)-\psi-$contractive mapping of type (II) and satisfies:*

(i) *T is $\alpha-$admissible,μ-subadmissible*
(ii) *there exists $x_0 \in X$ such that $\alpha (x_0, Tx_0) \geq 1, \mu (x_0, Tx_0) \leq 1$*
(iii) *T is continuous.*

Then T has a fixed point.

Proof. By assumption (ii), there exists $x_0 \in X$ such that $\alpha (x_0, Tx_0) \geq 1, \mu (x_0, Tx_0) \leq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T.

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is $\alpha-$admissible, then

$$\begin{aligned} \alpha (x_0, x_1) &= \alpha (x_0, Tx_0) \geq 1 \Longrightarrow \alpha (Tx_0, Tx_1) = \alpha (x_1, x_2) \geq 1, \\ \mu (x_0, x_1) &= \mu (x_0, Tx_0) \leq 1 \Longrightarrow \mu (Tx_0, Tx_1) = \mu (x_1, x_2) \leq 1. \end{aligned}$$

By induction, we get for all $n \in \mathbb{N}$,

$$\alpha (x_n, x_n + 1) \geq 1 \quad , \quad \mu (x_n, x_n + 1) \leq 1.$$

Using (2.6)

$$\begin{aligned} h(1, d(x_n, x_{n+1})) &= h(1, d(Tx_{n-1}, Tx_n)) \\ &\leq h(\alpha (x_{n-1}, x_n), d(Tx_{n-1}, Tx_n)) \\ &\leq \mathcal{F}(\mu (x_{n-1}, x_n), \psi(N_s (x_{n-1}, x_n))) \\ &\leq \mathcal{F}(1, \psi(N_s (x_{n-1}, x_n))) \\ &\Longrightarrow \end{aligned}$$

$$d(x_n, x_{n+1}) \leq \psi(N_s (x_{n-1}, x_n)) \leq \psi(M_s (x, y)).$$

The rest of proof is evident due to Theorem 2. □

In the following two theorems we are able to remove the continuity condition for the $\alpha - \psi-$ contractive mappings of type (I) and type (II).

Theorem 4. *Let (X,d) be a complete $b-$metric space. Suppose that $T:X\to X$ be a generalized $\alpha-\psi-$contractive mapping of type (I) and satisfies:*

(i) *T is $\alpha-$admissible,*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$,as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n(k)},x\right)\geq 1$, for all k.*

Then T has a fixed point.

Proof. Following the proof of Theorem 2 , we know that the sequence $\{x_n\}$ defined by $x_{n+1}=Tx_n$ for all $n\geq 0$, is Cauchy and converges to some $u\in X$.

We shall show that $Tu=u$. Suppose on the contrary that $d(Tu,u)>0$. From (2.2) and (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n(k)},u\right)\geq 1$ for all k. By (2.1)

$$\begin{aligned} h(1,d\left(x_{n(k)+1},Tu\right)) &= h(1,d\left(Tx_{n(k)},Tu\right)) \\ &\leq h(\alpha\left(x_{n(k)},u\right),d\left(Tx_{n(k)},Tu\right)) \\ &\leq \mathcal{F}(\mu\left(x_{n(k)},u\right),\psi(M_s\left(x_{n(k)},u\right))) \\ &\leq \mathcal{F}(1,\psi(M_s\left(x_{n(k)},u\right))) \\ &\Longrightarrow \end{aligned}$$

$$d\left(x_{n(k)+1},Tu\right)\leq\psi\left(M_s\left(x_{n(k)},u\right)\right), \tag{2.7}$$

where

$$M_s\left(x_{n(k)},u\right)=\max\left\{\begin{array}{c} d\left(x_{n(k)},u\right),d\left(Tx_{n(k)},x_{n(k)}\right),d\left(Tu,u\right), \\ \frac{d\left(Tx_{n(k)},u\right),d\left(Tu,x_{n(k)}\right)}{2s} \end{array}\right\}.$$

As $k\to\infty$, $\lim_{k\to\infty}M_s\left(x_{n(k)},u\right)=d\left(Tu,u\right)$.

In (2.7), as $k\to\infty$

$$d\left(u,Tu\right)\leq\psi\left(d\left(u,Tu\right)\right)<d\left(u,Tu\right)$$

which is a contradiction. Hence, $u=Tu$ and u is a fixed point of T. □

Theorem 5. *Let (X,d) be a complete $b-$metric space. Suppose that $T:X\to X$ be a generalized $\alpha-\psi-$contractive mapping of type (II) and satisfies:*

(i) *T is $\alpha-$admissible,*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$,*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$,as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n(k)},x\right)\geq 1$, for all k.*

then T has a fixed point.

Proof. Following the proof of Theorem 2.5, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, for all $n \geq 0$, is Cauchy and converges to some $u \in X$.

We shall show that $Tu = u$. Suppose on the contrary that $d(Tu, u) > 0$. From (2.2) and (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (2.6),

$$\begin{aligned} h(1, d(x_{n(k)+1}, Tu)) &= h(1, d(Tx_{n(k)}, Tu)) \\ &\leq h(\alpha(x_{n(k)}, u), d(Tx_{n(k)}, Tu)) \\ &\leq \mathcal{F}(\mu(x_{n(k)}, u), \psi(N_s(x_{n(k)}, u))) \\ &\leq \mathcal{F}(1, \psi(N_s(x_{n(k)}, u))) \\ &\Longrightarrow \end{aligned}$$

$$d(x_{n(k)+1}, Tu) \leq \psi(N_s(x_{n(k)}, u)) \tag{2.8}$$

where

$$N_s(x_{n(k)}, u) = \max\left\{ \begin{array}{c} d(x_{n(k)}, u), \frac{d(Tx_{n(k)}, x_{n(k)}) + d(Tu, u)}{2s}, \\ \frac{d(Tx_{n(k)}, u), d(Tu, x_{n(k)})}{2s} \end{array} \right\}.$$

As $k \to \infty$, $\lim_{k\to\infty} N_s(x_{n(k)}, u) = \frac{d(Tu,u)}{2s}$, for $s \geq 1$.

In (2.8), as $k \to \infty$

$$d(u, Tu) \leq \psi\left(\frac{d(Tu, u)}{2s}\right) < \frac{d(Tu, u)}{2s}$$

which is a contradiction. Hence, $u = Tu$ and u is a fixed point of T. □

Example 6. *Let $X = (0, \infty)$ endowed with $b-$metric*

$$d : X \times X \to R^+, \quad d(x, y) = (x - y)^2$$

with constant $s = 2$. (X, d) is a complete $b-$metric space. Let the functions $T : X \to X$, $\alpha : X \times X \to [0, \infty)$ and $\eta : X \times X \to [0, \infty)$ be defined by

$$\begin{aligned} T(x) &= \begin{cases} \frac{x+1}{4}, & x \in (0, 1] \\ 2x, & x > 1 \end{cases}, \\ \alpha(x, y) &= \begin{cases} 1, x \in (0, 1] \\ 0, otherwise \end{cases}, \\ \eta(x, y) &= \begin{cases} \frac{1}{2}, x \in (0, 1] \\ 1, otherwise \end{cases}. \end{aligned}$$

Clearly, T is $\alpha-$admissible, continuous and $\eta-$subadmissible. Let h,$\mathcal{F}$: $R^+ \times R^+ \to R$ be defined by;

$h(x, y) = (y + l)^x, l > 1$ and $\mathcal{F}(s, t) = st + l$. $(\mathcal{F}, h, \alpha, \eta) - \psi-$contraction of type (I) is satisfied with $\psi(t) = \frac{t}{2}$, for all $t \geq 0$.

Let $x, y \in X$ if $\alpha(x, y) \geq 1$ and $\eta(x, y) \geq 1$, then $x, y \in (0, 1]$ .Thus

$$\begin{aligned} h(\alpha(x,y)\,d(Tx,Ty)) &= h\left(1, \left(\frac{x+1}{4} - \frac{y+1}{4}\right)^2\right) = \frac{1}{16}(x-y)^2 + l \\ &\leq \frac{1}{2}.\frac{1}{2}(x-y)^2 + l = \mathcal{F}(\eta(x,y), \psi(d(x,y))) \\ &\leq \mathcal{F}(\eta(x,y), \psi(M_s(x,y)). \end{aligned}$$

Then all conditions of Theorem 5 are satisfied. $\frac{1}{3}$ is fixed point of T.

Corollary 1. *Let (X, d) be a complete $b-$metric space and $T : X \to X$ be continuous mapping. Suppose that there exists a function $\psi \in \Psi_b$ such that*

$$d(Tx, Ty) \leq \psi(M_s(x, y))$$

for all $x, y \in X$, then T has a fixed point.

Similarly, be taken $\alpha(x, y) = 1$ in Theorem 4, the following result is obtained.

Corollary 2. *Let (X, d) be a complete $b-$metric space and $T : X \to X$ be continuous mapping. Suppose that there exists a function $\psi \in \Psi_b$ such that*

$$d(Tx, Ty) \leq \psi(N_s(x, y))$$

for all $x, y \in X$, then T has a fixed point.

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