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NONLINEAR SELF ADJOINTNESS AND EXACT SOLUTION OF FOKAS–OLVER–ROSENAU–QIAO (FORQ) EQUATION

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Abstract. Based on Lie's symmetry approach, conservation laws are constructed for Fokas–Olver–Rosenau–Qiao(FORQ) equation and exact solution is obtained. Nonlocal conservation theorem is used to carry out the analysis of conservation process. Nonlinear self adjointness concept is applied to FORQ equation, it is proved to be strict self adjoint. Characteristic equation and similarity variable help us find exact solution of FORQ equation. Compared with solutions found in previous papers, our solution is new and important, since it is not possible to find exact solution of FORQ equation quite easily.

1. Introduction

In recently past years, more works has been conducted on conservation laws. The existence of conservation laws makes important progress in understanding given in many physical models. The determination of conservation laws, particularly local ones, offers rich knowledge on the mechanism of physical phenomena modeled by nonlinear evolution equations. An effective and impressive way of constructing conservation laws is by means of well known Noether's theorem [1]. This theorem provides explicit formulae for construction conservation laws for Euler-Lagrange differential equations once their Noether symmetries are known. Choosing a proper Lagrangian provides a chance of applying Noether's theorem to related equation. So as to remove this restriction, some methods have been developed in recent years, such as partial Lagrangian, Nonlocal conservation theorem, multiplier approach and so on [2]-[14].

In recent years, there has been an increasing interest in integrable non-evolutionary partial differential equation of the form

$$\left(1-D_x^2\right)u_t = F\left(u, u_x, u_{xx}, u_{xxx}, ...\right), \quad u=u(x,t), \quad D_x = \frac{\partial}{\partial x} \tag{1}$$

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where F is a function of u and its derivatives with respect to x. The most famous example of this type of equations is the Camassa-Holm equation [15, 16].

$$\left(1-D_x^2\right)u_t = 3uu_x - 2u_xu_{xx} - uu_{xxx}. \tag{2}$$

The integrability of Camassa-Holm type equations was shown by inverse scattering transform, infinity hierarchy of local conservation laws, bi-Hamiltonian structure and other remarkable properties of integrable equations [17]. We consider the following form of (1) [18]

$$\begin{aligned}\left(1-\epsilon^2D_x^2\right)u_t &= c_1u^2u_x + \epsilon\left[c_2u^2u_{xx}+c_3uu_x^2\right] + \epsilon^2\left[c_4u^2u_{xxx}+c_5u_xu_{xx}u+c_6u_x^3\right]\\ &+\epsilon^3\left[c_7u^2u_{xxxx}+c_8u_xu_{xxx}u+c_9u_{xx}^2u+c_{10}u_x^2u_{xx}\right]\\ &+\epsilon^4\left[c_{11}u^2u_{xxxxx}+c_{12}u_xu_{xxxx}u+c_{13}u_{xx}u_{xxx}u+c_{14}u_x^2u_{xxx}+c_{15}u_xu_{xx}^2\right]\end{aligned} \tag{3}$$

Here, ϵ and c_i are the complex parameters and $\epsilon \neq 0$. This equation is homogeneous differential polynomials of weight 1. Supposing that weight of u_i is i, weight of ϵ equals -1 and weight of u_t is 1. Particularly, choosing the coefficients in (3) appropriately, we get the (3) in the form of

$$u_t - u_{xxt} + 3u^2u_x - u_x^3 - 4uu_xu_{xx} + 2u_xu_{xx}^2 - u^2u_{xxx} + u_x^2u_{xxx} = 0 \tag{4}$$

which is given as FORQ equation in [19].

In this paper, we concentrate on FORQ equation which was derived by Olver and Rosenau [20], Fuchssteiner [21], and Qiao [22]. Our main motivation in this study is to obtain Lie symmetry generators of FORQ equation with Maple package program. Taking $w = \varphi(x,t,u)$, the construction of nonlinear self adjointness and conservation laws of the FORQ equation is presented. Furthermore, using the similarity variables and reduced equation, exact solution is obtained.

2. Conservation laws for the FORQ equation

We briefly present notation to be used and recall basic definitions and theorems that appear in [23]-[25].

Consider the k^{th} order system of PDEs of n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$

$$E^\alpha\left(x, u, u_{(1)}, ..., u_{(k)}\right) = 0, \quad \alpha = 1, ..., m. \tag{5}$$

where $u_{(i)}$ is the collection of i^{th}-order partial derivatives and the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha\frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha} + ..., \quad i = 1, ..., n \tag{6}$$

in which the summation convention is used. Suppose $\mathcal{A}$ is the universal space of all differential functions of finite orders, clearly it is a vector space and forms an algebra. *The* Lie-Backlund *generator* is the following vector field operator:

$$\mathbf{X} = \xi^i\frac{\partial}{\partial x_i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A} \tag{7}$$

The operator (7) is an abbreviated form of the infinite formal sum

$$\mathbf{X}=\xi^i\frac{\partial}{\partial x_i}+\eta^\alpha\frac{\partial}{\partial u^\alpha}+\sum_{k\geq 1}\zeta^\alpha_{i_1 i_2\ldots i_k}\frac{\partial}{\partial u^\alpha_{i_1 i_2\ldots i_k}}, \tag{8}$$

where the additional coefficients can be determined from the prolongation formulae

$$\begin{aligned}\zeta^\alpha_i &= D_i\left(\eta^\alpha\right)+\xi^j u^\alpha_{ji}\\ \zeta^\alpha_{i_1\ldots i_k} &= D_{i_1}\ldots D_{i_k}\left(\zeta^\alpha_{i_1\ldots i_{k-1}}\right)+\xi^j u^\alpha_{j i_1\ldots i_k}, \quad k>1.\end{aligned} \tag{9}$$

The Noether operators associated with a Lie–Bäcklund operator $\mathbf{X}$ are

$$N^i=\xi^i+W^\alpha\frac{\delta}{\delta u^\alpha_i}+\sum_{k\geq 1}^{\infty}D_{i_1}\ldots D_{i_k}\frac{\partial}{\partial u^\alpha_{i_1\ldots i_k}}, \qquad i=1,2,...,n$$

in which W^α is the *Lie characteristic function*

$$W^\alpha=\eta^\alpha-\xi^j u^\alpha_j. \tag{10}$$

A conserved vector of (5) is an n-tuple vector $T=\left(T^1,T^2,...,T^n\right)$, $T^j\epsilon\mathcal{A}$, $j=1,...,n$

$$D_i T^i_{|(1)}=0. \tag{11}$$

holds for all solutions of (5).

Then we define the adjoint equation to Eq(5) in the form of

$$E^{\alpha *}(x,u,w,u_{(1)},w_{(1)},...,u_{(k)},w_{(k)})=0, \qquad \alpha=1,...,m$$

with

$$E^{\alpha *}(x,u,w,u_{(1)},w_{(1)},...,u_{(k)},w_{(k)})=\frac{\delta L}{\delta u^\alpha} \tag{12}$$

where L is formal Lagrangian for Eq(5) defined by

$$L=w^\alpha E^\alpha\equiv\sum_{\alpha=1}^{m}w^\alpha E^\alpha. \tag{13}$$

Here, so-called non local variables are $w^\alpha=(w^1,...,w^m)$, their derivatives are $w^\alpha_{(1)},...,w^\alpha_{(k)}$. Here $\frac{\delta}{\delta u}$is the Euler-Lagrange operator and given by

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}+\sum_{k\geq 1}^{\infty}\left(-1\right)^k D_{i_1}\ldots D_{i_k}\frac{\partial}{\partial u^\alpha_{i_1\ldots i_k}}, \qquad \alpha=1,...,m. \tag{14}$$

so that

$$\frac{\delta L}{\delta u^\alpha}=\frac{\delta(w^\alpha E^\alpha)}{\delta u^\alpha}=\frac{\partial(w^\alpha E^\alpha)}{\partial u^\alpha}-D_i\left(\frac{\partial(w^\alpha E^\alpha)}{\partial u^\alpha_i}\right)+D_i D_k\left(\frac{\partial(w^\alpha E^\alpha)}{\partial u^\alpha_{ik}}\right)-...$$

Definition 1. *The differential equation (5) is said to be nonlinearly self-adjoint if there exists a function*

$$w^{\alpha} = \varphi^{\alpha}(x,t,u) \neq 0 \tag{15}$$

such that it satisfy

$$E^{\alpha *}(x,u,\varphi(x,u),...,u_{(k)},\varphi_{(k)}) = \lambda_{\alpha}^{\beta} E^{\alpha}(x,u,...u_{(k)}), \ \alpha = 1,..,m, \tag{16}$$

for some undetermined coefficient $\lambda = \lambda_{\alpha}^{\beta}(x,t,u)$. If we take $w = \varphi(u)$ in (16) the equation (5) is called quasi self-adjoint. If we take $w = u$, we say that the equation (5) is strictly self-adjoint.

Theorem 2 ([23])**.** *Every Lie point, Lie-Bäcklund and non-local symmetry of equation (5) gives a conservation law for the considered equation. The conserved vector components are*

$$\begin{aligned} T^{i} &= \xi^{i} L + W^{\alpha}\left[\tfrac{\partial L}{\partial u_i} - D_j(\tfrac{\partial L}{\partial u_{ij}}) + D_j D_k(\tfrac{\partial L}{\partial u_{ijk}})\right] \\ &\quad + D_j\left(W^{\alpha}\right)\left[\tfrac{\partial L}{\partial u_{ij}} - D_k(\tfrac{\partial L}{\partial u_{ijk}})\right] + D_j D_k\left(W^{\alpha}\right)\frac{\partial L}{\partial u_{ijk}} \end{aligned} \tag{17}$$

and ξ^{i}, η^{α} are the coefficient functions of the associated generator (7) .

The conserved vectors obtained from (17) involve the arbitrary solutions w^{α} of the adjoint equation (12) and hence one obtains an infinite number of conservation laws for (5) by choosing w^{α}.

Now we use the nonlocal conservation theorem method given by Ibragimov. We consider the following sub-algebra with infinitesimal generators of symmetries given by,

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_3 = t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}. \tag{18}$$

Then the corresponding formal Lagrangian of Eq(4) is given by

$$L = \left(u_t - u_{xxt} + 3u^2 u_x - u_x^3 - 4uu_x u_{xx} + 2u_x u_{xx}^2 - u^2 u_{xxx} + u_x^2 u_{xxx}\right) w(x,t). \tag{19}$$

The adjoint equation for Eq(4) is

$$\begin{aligned} E^{*}(t,x,u,w,...,w_{xxx}) &= \frac{\delta}{\delta u}\big[w(x,t) \\ &\quad \times\left(u_t - u_{xxt} + 3u^2 u_x - u_x^3 - 4uu_x u_{xx}\right. \\ &\quad \left. + 2u_x u_{xx}^2 - u^2 u_{xxx} + u_x^2 u_{xxx}\right)\big] \\ &= -w_t + w_{txx} - 3w_x u^2 + w_x u_x^2 + w_{xxx} u^2 - w_{xxx} u_x^2 \\ &\quad + 2w_x u u_{xx} - 2w_{xx} u_x u_{xx} + 2w_{xx} u u_x \end{aligned} \tag{20}$$

where w is the adjoint variable.

If we take $w = \varphi(t, x, u)$ and necessary derivatives:

$$\begin{aligned}
w &= \varphi(t,x,u),\\
w_t &= \varphi_u u_t + \varphi_t,\\
w_x &= \varphi_u u_x + \varphi_x,\\
w_{xx} &= \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{ux} u_x + \varphi_{xx},\\
w_{xxx} &= \varphi_{xxx} + 3\varphi_{xxu} u_x + 3\varphi_{xuu} u_x^2 + 3\varphi_{xu} u_{xx} + \varphi_{uuu} u_x^3 + 3\varphi_{uu} u_x u_{xx} + \varphi_u u_{xxx},\\
w_{xxt} &= \varphi_{uu} u_t u_{xx} + \varphi_{ut} u_{xx} + \varphi_u u_{xxt} + \varphi_{tuu} u_x^2 + 2\varphi_{uu} u_x u_{xt} + \varphi_{uuu} u_t u_x^2\\
&\quad +2\varphi_{uux} u_x u_t + 2\varphi_{uxt} u_x + 2\varphi_{ux} u_{xt} + \varphi_{xxu} u_t + \varphi_{xxt},
\end{aligned}$$

with the self-adjointness condition (16), Eq(20) as follows:

$$\begin{aligned}
E^*(t,x,u,w,...,w_{xxx}) &= -\varphi_u u_t - \varphi_t - 3(\varphi_u u_x + \varphi_x)u^2 + \varphi_u u_x^3 + u_x^2\varphi_x +\\
&\quad (u^2 - u_x^2)(\varphi_{xxx} + 3\varphi_{xxu} u_x + 3\varphi_{xuu} u_x^2 + 3\varphi_{xu} u_{xx} + \varphi_{uuu} u_x^3\\
&\quad +3\varphi_{uu} u_x u_{xx} + \varphi_u u_{xxx})\\
&\quad +2(\varphi_u u_x + \varphi_x)u_{xx}u - 2(\varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{ux} u_x + \varphi_{xx})u_x u_{xx}\\
&\quad +2(\varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{ux} u_x + \varphi_{xx})u_x u\\
&= \lambda\left(u_t - u_{xxt} + 3u^2 u_x - u_x^3 - 4uu_x u_{xx} + 2u_x u_{xx}^2 - u^2 u_{xxx} + u_x^2 u_{xxx}\right)
\end{aligned} \tag{21}$$

The comparison of the coefficients of all derivatives yields $\varphi = C_1 u + C_2$ where C_1, C_2 are constants. Therefore we can take two different values of w, namely $w = 1$ and $w = u$.

The conserved components of Eq(4), associated with a Lie symmetry, can be obtained from (17) as follows:

$$\begin{aligned}
T^t &= \xi^t L + W\left[\frac{\partial L}{\partial u_t} + D_x^2\left(\frac{\partial L}{\partial u_{xxt}}\right)\right] + D_x(W)\left[-D_x\left(\frac{\partial L}{\partial u_{xxt}}\right)\right] + D_x^2(W)\left[\frac{\partial L}{\partial u_{xxt}}\right]\\
T^x &= \xi^x L + W\left[\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right) + D_x^2\left(\frac{\partial L}{\partial u_{xxx}}\right) + D_{xt}^2\left(\frac{\partial L}{\partial u_{xxt}}\right)\right]\\
&\quad +D_x(W)\left[\frac{\partial L}{\partial u_{xx}} - D_x\left(\frac{\partial L}{\partial u_{xxx}}\right) - D_t\left(\frac{\partial L}{\partial u_{xxt}}\right)\right]\\
&\quad +D_t(W)\left[-D_x\left(\frac{\partial L}{\partial u_{xxt}}\right)\right] + D_x^2(W)\left(\frac{\partial L}{\partial u_{xxx}}\right) + D_{xt}^2(W)\left(\frac{\partial L}{\partial u_{xxt}}\right).
\end{aligned} \tag{22}$$

where $W = \eta - u_x\xi^x - u_t\xi^t$ is Lie characteristic function.

Now, we will find conservation laws of Eq(4) with the help of formulae (22).

i) Firstly, we will construct conservation laws with $\mathbf{w = 1}$.

Case 1:

We consider $X_1 = \frac{\partial}{\partial x}$ with $W = -u_x$, corresponding conserved vectors are

$$\begin{aligned} T_1^t &= -u_x \\ T_1^x &= u_t. \end{aligned} \tag{23}$$

Case 2:

If we use $X_2 = \frac{\partial}{\partial t}$ with $W = -u_t$, we obtain conserved vectors

$$\begin{aligned} T_2^t &= 3u^2 u_x - u_x^3 - 4uu_x u_{xx} + 2u_x u_{xx}^2 - u^2 u_{xxx} + u_x^2 u_{xxx} \\ T_2^x &= -3u^2 u_t + u_t u_x^2 + 2uu_t u_{xx} + 2uu_x u_{xt} - 2u_x u_{xt} u_{xx} + u^2 u_{xxt} - u_x^2 u_{xxt}. \end{aligned} \tag{24}$$

Obtained conservation laws in case1,2 satisfy $D_t\left(T_2^t\right) + D_x\left(T_2^x\right) = 0$, so these vectors are trivial.

Case 3:

For the Lie-point symmetry generator

$$X_3 = t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u},$$

we have

$$W = -\tfrac{1}{2}u - tu_t.$$

If we use (22) , obtained conserved vectors are

$$\begin{aligned} T_3^t &= 3tu^2 u_x - tu_x^3 - 4tuu_x u_{xx} + 2tu_x u_{xx}^2 - tu^2 u_{xxx} + tu_x^2 u_{xxx} - \frac{1}{2}u + \frac{1}{2}u_{xx} \\ T_3^x &= -\frac{3}{2}u^3 + \frac{3}{2}uu_x^2 + \frac{3}{2}u^2 u_{xx} - 3tu_t u^2 + tu_t u_x^2 + 2tu_t uu_{xx} - \frac{3}{2}u_x^2 u_{xx} \\ &\quad + 2tuu_x u_{xt} - 2tu_x u_{xx} u_{xt} + tu^2 u_{xxt} - tu_x^2 u_{xxt} + \frac{3}{2}u_{xt} + tu_{xtt}. \end{aligned} \tag{25}$$

Divergence condition can be expressed for these conservation laws as follows:

$$\begin{aligned} D_t\left(T_3^t\right) + D_x\left(T_3^x\right) &= \frac{3}{2}u_{xxt} + tu_{xxtt} \\ &= D_x\left(\frac{3}{2}u_{xt} + tu_{xtt}\right). \end{aligned} \tag{26}$$

In Eq(26) , since there are some terms leftover, we should find modified conservation laws to satisfy divergence condition. Therefore, modified conservation laws are

$$\begin{aligned} \widetilde{T_3^t} &= T_3^t \\ \widetilde{T_3^x} &= T_3^x - tu_{xtt} - \frac{3}{2}u_{xt}. \end{aligned} \tag{27}$$

ii) We will find conservation laws with $\mathbf{w} = \mathbf{u}$.

Case 4:

According to generator $X_1 = \frac{\partial}{\partial x}$, we get trivial conserved vectors

$$\begin{array}{rcl} T_1^t & = & -uu_x \\ T_1^x & = & uu_t. \end{array} \tag{28}$$

Case 5:

Using the symmetry generator $X_2 = \frac{\partial}{\partial t}$, we obtain following trivial conserved vectors

$$\begin{array}{rcl} T_2^t & = & 3u^3u_x - uu_x^3 - 4u^2u_xu_{xx} + 2uu_xu_{xx}^2 - u^3u_{xxx} + uu_x^2u_{xxx} + u_tu_{xx} - uu_{xxt} \\ T_2^x & = & -3u^3u_t + uu_tu_x^2 + 3u^2u_tu_{xx} - u_tu_x^2u_{xx} + u^2u_xu_{xt} \\ & & -2uu_xu_{xt}u_{xx} + u_x^3u_{xt} - u_xu_{tt} + u^3u_{xxt} - uu_x^2u_{xxt} + uu_{xtt}. \end{array} \tag{29}$$

Case 6:

Finally, using the following Lie-point symmery generator

$$X_3 = t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}, \tag{30}$$

conserved vectors are

$$\begin{array}{rcl} T_3^t & = & 3tu^3u_x - tuu_x^3 - 4tu^2u_xu_{xx} + 2tuu_xu_{xx}^2 - tu^3u_{xxx} + tuu_x^2u_{xxx} \\ & & -\frac{1}{2}u^2 + uu_{xx} + tu_tu_{xx} - \frac{1}{2}u_x^2 - tu_xu_{xt} \\ T_3^x & = & -2tuu_xu_{xx}u_{xt} + u^2u_x^2 + 2u^3u_{xx} + 2uu_{xt} - 2u_xu_t + tu_{xt}u_x^3 - tu_xu_{tt} \\ & & +tu^3u_{xxt} + tuu_tu_x^2 + 3tu_tu^2u_{xx} - tu_tu_x^2u_{xx} + tu^2u_xu_{xt} - tuu_x^2u_{xxt} \\ & & -\frac{3}{2}u^4 - 2uu_x^2u_{xx} + tuu_{xtt} + \frac{1}{2}u_x^4 - 3tu_tu^3. \end{array} \tag{31}$$

If we look at divergence condition, we get

$$\begin{array}{rcl} D_t\left(T_3^t\right) + D_x\left(T_3^x\right) & = & D_t\left(tuu_{xxt} - u_x^2 - tu_xu_{xt}\right) \\ & & +D_x\left(uu_{xt}\right). \end{array} \tag{32}$$

Again, there are some terms leftover. To satisfy divergence condition modified conservation laws are obtained as follows

$$\begin{array}{rcl} \widetilde{T_3^t} & = & T_3^t - tuu_{xxt} + u_x^2 + tu_xu_{xt} \\ \widetilde{T_3^x} & = & T_3^x - uu_{xt}. \end{array} \tag{33}$$

3. Exact Solution

Now we can find the exact solution of Eq(4) with Lie-point symmetry generator

$$X_3 = t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}.$$

Using this generator, we find characteristic equation

$$\frac{dt}{t} = -\frac{2du}{u},\ dx = 0.$$

Since similarity variables are $c_1 = ut^{\frac{1}{2}}$ and $c_2 = x$, then, solution $f(c_2) = c_1$ implies,

$$u = t^{-1/2} f(x) \tag{34}$$

where f is arbitrary function of x. Through (34), reduced ODE reads

$$-\frac{f}{2} + \frac{f^{''}}{2} + 3f^2 f' - \left(f^{'}\right)^3 - 4ff^{'}f'' + 2f^{'}\left(f^{''}\right)^2 - f^2 f^{'''} + f^{'''}\left(f^{'}\right)^2 = 0 \tag{35}$$

where $f' = \frac{df}{dx}$. If we solve the above ODE, we obtain

$$f(x) = ae^x + be^{-x} \tag{36}$$

where a and b are arbitrary constants. Therefore exact solution of the Eq(4) is

$$u(x,t) = t^{-1/2}\left(ae^x + ce^{-x}\right). \tag{37}$$

4. Conclusion

It is well known that Lie symmetry analysis is widely used in finding conservation laws and reduction of given PDE's, ODE's. In this paper, we have examined FORQ equation, by obtaining new families of conservation laws and exact solution. Nonlocal conservation theorem was employed to construct conservation laws, while reduced ODE was being employed to obtain exact solution. The concept of self adjoint and quasi self adjoint equations were introduced by Ibragimov in [25]. With the same idea, taking nonlocal variable $w = \varphi(x, t, u)$, the self adjointness of FORQ equation was investigated. We have expressed each generator with corresponding conservation law in detail. We hope that obtained conservation laws and exact solution could further assist in understanding and identifying FORQ equation in previous and future works.

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