Iğdır Üniversitesi Fen Bilimleri Enstitüsü Dergisi
Journal of the Institute of Science and Technology

New Travelling Wave Solutions for Time-Space Fractional Liouville and Sine-Gordon Equations

Orkun TAŞBOZAN[1], Ali KURT[1]

ABSTRACT: In this paper, the authors discussed the new wave solutions of time-space fractional Liouville and Sine-Gordon equations by using a reliable analytical method called sub-equation method. The fractional derivatives of considered equations are handled in conformable sense. Conformable derivative which is an easy and applicable type of fractional derivative, satisfies basic properties of known derivative with integer order such as Leibniz rule, quotient rule, chain rule. These properties of conformable derivative superior to other popular definitions on obtaining analytical solutions of fractional equations.

Keywords: Conformable fractional derivative; Sub-Equation method; Liouville equation; Sine-Gordon; Wave Solutions.

Zaman Konum Kesirli Liouville ve Sine-Gordon Denklemlerinin Yeni Dalga Çözümleri

ÖZ: Bu makalede, yazarlar alt denklem yöntemi olarak adlandırılan güvenilir bir yöntem kullanarak zaman-uzay kesirli Lioville ve Sine-Gordon denklemlerinin yeni dalga çözümlerini elde ettiler. Kullanılan denklemlerde mevcut olan kesirli mertebeden türevler, conformable anlamında ele alınmıştır. Kolay, uıygulanabilir olan conformable türevi, bilinen türevin sağladığı Leibniz Kuralı, bölüm kuralı, zincir kuralı gibi kuralları sağlar. Bu özellikler conformable kesirli türeve diğer popüler türevler karşısında bir avantaj sağlamaktadır.

Anahtar Kelimeler: Conformable Kesirli Türev, Liouville Denklemi, Sine Gordon Denklemi, Dalga Çözümü, Alt Denklem Yöntemi.

Cilt/*Volume*: 8, Sayı/*Issue*: 4, Sayfa/*pp*: 295-303, 2018
ISSN: 2146-0574, e-ISSN: 2536-4618 DOI: 10.21597/jist.412948

[1] Orkun TAŞBOZAN (0000-0001-5003-6341), Ali KURT (0000-0002-0617-6037), Mustafa Kemal University, Faculty of Science and Art, Department of Mathematics, Hatay, Turkey
Sorumlu yazar/Corresponding Author: Orkun TAŞBOZAN, otasbozan@mku.edu.tr

Geliş tarihi / *Received:* 05.04.2018
Kabul tarihi / *Accepted:* 26.07.2018

INTRODUCTION

Obtaining the analytical solutions of partial differential equations with arbitrary order is very important in the last decades because of the applications of this subject in many different areas such as physics, engineering, social, biological, chemical sciences and etc. Therefore, many scientists studied to get the analytical solutions of fractional partial differential equations. The popular fractional derivatives such as Riemann-Liouville and Caputo do not satisfy some basic properties. For instance, the Riemann-Liouville derivative of a real constant is not equal to zero but Caputo derivative satisfies this property. Both of these definitions do not satisfy the fractional derivative rule for the product (and quotient) of two functions. Similarly, both definitions do not satisfy the chain rule. Moreover, these definitions do not satisfy $D^{\theta}D^{\phi}(\omega)=D^{\theta+\phi}(\omega)$ generally. Also, Caputo definition is limited because it is assumed that the function must be differentiable (Cenesiz et al., 2017).

Due to these limitations, the scientists studied to state a new type of arbitrary derivative which satisfies all the rules above. Khalil et al. introduced a new definition, called conformable fractional derivative, which is simple, well behaved, applicable, efficient fractional derivative and fractional integral definition (Khalil et al., 2014). This new definition fulfils the basic classical properties which cannot be satisfied by Riemann-Liouville and Caputo definitions.

Definition 1. Let $g:(0,\infty)\rightarrow\mathbb{R}$ be a function. The $\alpha-th$ order conformable fractional derivative of function g is defined as

$$D_t^{\alpha}g(t)=\lim_{\varepsilon\rightarrow 0}\frac{g(t+\varepsilon t^{1-\alpha})-g(t)}{\varepsilon}$$

for all $t\in(0,1]$, $0<\alpha<1$ (Khalil et al., 2014).

Definition 2. Let $a \geq 0$, $t \geq a$, g be a function defined on $(a,t]$ and $\alpha \in \mathbb{R}$. Then, the $\alpha - th$ order fractional integral of function g is defined as (Khalil et al., 2014),

$$ {}_tI_a^{\alpha} g(t) = \int_a^t \frac{g(x)}{x^{1-\alpha}} dx. $$

This new definition (Khalil et al., 2014; Kurt et al., 2017) satisfies the following properties.

Theorem 1.1. Let $0 < \alpha \leq 1$ and f, g be $\alpha - th$ order differentiable at $t \in (0, \infty)$. Then

a) $T_\alpha(xf + yg) = xT_\alpha(f) + yT_\alpha(g)$, for all $x, y \in \mathbb{R}$.

b) $T_\alpha(t^m) = mt^{m-\alpha}$ for all $m \in \mathbb{R}$.

c) $T_\alpha(\rho) = 0$ for all constant functions $g(t) = \rho$.

d) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f).$

e) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(g) - fT_\alpha(f).}{g^2}$

f) If, in addition to f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

The applicability, effectiveness and convenience of these new type derivative and integral have attracted many researchers. For example, invariant subspace method is considered by Hashemi (Hashemi, 2018) to get the analytical solutions of various conformable differential equations. Kaplan and Ozer used Hirotas method to obtain analytical solutions of (2 + 1)-dimensional nonlinear evolution equation (Kaplan and Ozer, 2018). The modified Kudryashov method is used by Kumar et al. (Kumar et al., 2018) to get analytical solitary wave solutions for the variety of fractional Boussinesq-like equations. First integral method is employed to get analytical solutions for conformable differential equations (Ilie et al., 2018). Analytical solutions of fractional Schrodinger equation are extracted by Hosseini et al. (Hosseini et al., 2017) with the aid of the improved $\tanh(\phi(n)/2)$-expansion method and the exp function method.

MATERIALS AND METHODS

In this section, we give a short description of sub-equation method (Zhang and Zhang, 2011). Consider the following fractional differential equation of the form

$$P\left(u, D_t^{\alpha} u, D_x^{\beta} u, D_t^{2\alpha} u, D_x^{2\beta} u, \ldots\right) = 0 \tag{1}$$

where all fractional derivatives are conformable fractional derivative. Let explain the method step by step as follows.

Step 1: Applying the wave transformation,

$$u(x,t) = U(\xi), \qquad \xi = m\frac{x^{\beta}}{\beta} + k\frac{t^{\alpha}}{\alpha} \tag{2}$$

where m, k are arbitrary constants. Now we can rearrange Equation (1) in form of the following nonlinear ODE by using chain rule (Abdeljawad, 2015):

$$G\left(U, U', U'', \ldots\right) = 0 \tag{3}$$

where prime shows the integer order derivative with respect to ξ.

Step 2. Let us consider that Equation (3) has one solution in the following form

$$U(\xi) = \sum_{i=0}^{N} a_i \varphi^i(\xi), \quad a_N \neq 0 \tag{4}$$

where $a_i \left(0 \leq i \leq N\right)$ are constant coefficients to be determined later. N is an integer which can be calculated by balancing principle terms in Equation (3) and $\varphi(\xi)$ satisfies the ODE in the form

$$\varphi'\left(\xi\right) = \sigma + \left(\varphi\left(\xi\right)\right)^2 \tag{5}$$

where σ is a constant. In the following, we provide some analytical solutions for the Equation (5)

$$\varphi(\xi)=\begin{cases}-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi), & \sigma<0\\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi), & \sigma<0\\ \sqrt{\sigma}\tan(\sqrt{\sigma}\xi), & \sigma<0\\ \sqrt{\sigma}\cot(\sqrt{\sigma}\xi), & \sigma<0\\ -\dfrac{1}{\xi+\varpi},\varpi\ is\ a\ cons., & \sigma=0\end{cases} \tag{6}$$

Step 3. Substituting Equations (4) and (5) into Equation (3) and setting the coefficients of $\varphi^i(\xi)$ to zero, a nonlinear algebraic system respect to $a_i\ (i=0,1,\ldots,N)$ is obtained.

Step 4. Finally, substituting the obtained constants from the nonlinear algebraic system and the solutions (6) into Equation (4) we obtain the analytical solutions for Equation (1).

RESULTS AND DISCUSSION

Wave Solutions For Conformable Fractional Liouville Equation

Consider the following fractional Liouville equation:

$$D_x^{\beta}D_t^{\alpha}u+e^u=0,\ t\geq 0,\quad 0<\alpha,\beta\leq 1 \tag{7}$$

Using the chain rule (Abdeljawad, 2015) with the aid of wave transform $\xi=m\dfrac{x^\beta}{\beta}+k\dfrac{t^\alpha}{\alpha}$ and $u=U(\xi)$ we get

$$mkU_{\xi\xi}+e^u=0. \tag{8}$$

Then making following transformation

$$U=\ln V \tag{9}$$

in the Equation (8) turns into following ordinary differential equation

$$mk(V_{\xi\xi}V-(V_\xi)^2)+V^3=0. \tag{10}$$

Using balancing principle in Equation (10), we obtain $N=2$. Thus we can write

$$V(\xi)=a_0+a_1\varphi+a_2\varphi^2. \tag{11}$$

Subrogating Equation (11) with Equation (5) into Equation (10), we collect the coefficients of power of φ and equate them to zero:

$$
\begin{aligned}
&\varphi^0: \quad a_0^3 - a_1^2 km\sigma^2 + 2a_0 a_2 km\sigma^2 = 0,\\
&\varphi^1: \quad 3a_0^2 a_1 + 2a_0 a_1 km\sigma - 2a_1 a_2 km\sigma^2 = 0,\\
&\varphi^2: \quad 3a_0 a_1^2 + 3a_0^2 a_2 + 8a_0 a_2 km\sigma - 2a_2^2 km\sigma^2 = 0,\\
&\varphi^3: \quad a_1^3 + 6a_0 a_1 a_2 + 2a_0 a_1 km + 2a_1 a_2 km\sigma = 0,\\
&\varphi^4: \quad 3a_1^2 a_2 + 3a_0 a_2^2 + a_1^2 km + 6a_0 a_2 km = 0,\\
&\varphi^5: \quad 3a_1 a_2^2 + 4a_1 a_2 km = 0,\\
&\varphi^6: \quad a_2^3 + 2a_2^2 km = 0.
\end{aligned}
$$

Solving this set of algebraic equations with the help of Mathematica, we get

$$
\begin{aligned}
a_0 &= -2km\sigma,\\
a_1 &= 0,\\
a_2 &= -2km.
\end{aligned}
\tag{12}
$$

Using the equalities (12) along with the wave transform $\xi = m\frac{x^\beta}{\beta} + k\frac{t^\alpha}{\alpha}$ and (9) yields the following travelling wave solutions

$$u_1(x,t) = \ln\left(-2km\sigma + 2km\sigma\tanh\left(\sqrt{-\sigma}\left(k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right)^2\right),$$

$$u_2(x,t) = \ln\left(-2km\sigma + 2km\sigma\coth\left(\sqrt{-\sigma}\left(k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right)^2\right),$$

$$u_3(x,t) = \ln\left(-2km\sigma - 2km\sigma\tan\left(\sqrt{\sigma}\left(k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right)^2\right),$$

$$u_4(x,t) = \ln\left(-2km\sigma - 2km\sigma\cot\left(\sqrt{\sigma}\left(k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}\right)\right)^2\right),$$

$$u_5(x,t) = \ln\left(-\frac{2km}{\left(k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + \omega\right)^2}\right).$$

Wave Solutions For Conformable Fractional Sine-Gordon Equation

Let consider the following conformable fractional Sine-Gordon equation:

$$D_t^{2\alpha}u - D_x^{2\beta} + \sin u = 0,\ t \geq 0, \quad 0 < \alpha, \beta \leq 1 \tag{13}$$

Similarly applying the chain rule (Abdeljawad, 2015) and wave transform yield

$$(k^2 - m^2)U_{\xi\xi} + \sin u = 0. \tag{14}$$

Considering following transform

$$V = e^{iU} \tag{15}$$

in Equation (14) yields,

$$2(k^2 - m^2)(V_{\xi\xi}V - (V_\xi)^2) + V^3 - V = 0. \tag{16}$$

Balancing the highest order derivative term $V_{\xi\xi}$ with nonlinear term V^3 in Equation (16), we get $N = 2$, thus we can write

$$V(\xi) = a_0 + a_1\varphi + a_2\varphi^2. \tag{17}$$

Substituting Equation (17) using Equation (5) into Equation (16), arranging the coefficients of φ and equating them to zero led to an algebraic equations set of a_0, a_1, a_2, m as follows:

φ^0 : $-a_0 + a_0^3 - 2a_1^2k^2\sigma^2 + 4a_0a_2k^2\sigma^2 + 2a_1^2m^2\sigma^2 - 4a_0a_2m^2\sigma^2 = 0,$

φ^1 : $-a_1 + 3a_0^2a_1 + 4a_0a_1k^2\sigma - 4a_0a_1m^2\sigma - 4a_1a_2k^2\sigma^2 + 4a_1a_2m^2\sigma^2 = 0,$

φ^2 : $3a_0a_1^2 - a_2 + 3a_0^2a_2 + 16a_0a_2k^2\sigma - 16a_0a_2m^2\sigma - 4a_2^2k^2\sigma^2 + 4a_2^2m^2\sigma^2 = 0,$

φ^3 : $a_1^3 + 6a_0a_1a_2 + 4a_0a_1k^2 - 4a_0a_1m^2 + 4a_1a_2k^2\sigma - 4a_1a_2m^2\sigma = 0,$

φ^4 : $3a_1^2a_2 + 3a_0a_2^2 + 2a_1^2k^2 + 12a_0a_2k^2 - 2a_1^2m^2 - 12a_0a_2m^2 = 0,$

φ^5 : $3a_1a_2^2 + 8a_1a_2k^2 - 8a_1a_2m^2 = 0,$

φ^6 : $a_2^3 + 4a_2^2k^2 - 4a_2^2m^2 = 0.$

Employing computer software Mathematica to obtain the solutions of this system yields the following solution sets.

Set 1:

$$a_0 = 0,$$

$$a_1 = 0,$$

$$a_2 = \frac{1}{\sigma},$$

$$m = \pm\frac{\sqrt{1 + 4k^2\sigma}}{2\sqrt{\sigma}}.$$

Set 2:

$$a_0 = 0,$$
$$a_1 = 0,$$
$$a_2 = -\frac{1}{\sigma},$$
$$m = \pm\frac{\sqrt{-1+4k^2\sigma}}{2\sqrt{\sigma}}.$$

The solutions fractional Sine-Gordon equation (13) corresponding to above solutions sets are

$$u_{1,2}(x,t) = \arccos\left(\frac{1}{2}\left(-\coth\left(\sqrt{-\sigma}\left(\frac{kt^\alpha}{\alpha}\pm\frac{x^\beta\sqrt{1+4k^2\sigma}}{2\beta\sqrt{\sigma}}\right)\right)^2 - \tanh\left(\sqrt{-\sigma}\left(\frac{kt^\alpha}{\alpha}\pm\frac{x^\beta\sqrt{1+4k^2\sigma}}{2\beta\sqrt{\sigma}}\right)\right)^2\right)\right),$$

$$u_{3,4}(x,t) = \arccos\left(\frac{1}{2}\left(\cot\left(\sqrt{\sigma}\left(\frac{kt^\alpha}{\alpha}\pm\frac{x^\beta\sqrt{1+4k^2\sigma}}{2\beta\sqrt{\sigma}}\right)\right)^2 + \tan\left(\sqrt{\sigma}\left(\frac{kt^\alpha}{\alpha}\pm\frac{x^\beta\sqrt{1+4k^2\sigma}}{2\beta\sqrt{\sigma}}\right)\right)^2\right)\right),$$

$$u_{5,6}(x,t) = \arccos\left(\frac{1}{2}\left(\coth\left(\sqrt{-\sigma}\left(\frac{kt^\alpha}{\alpha}\pm\frac{x^\beta\sqrt{-1+4k^2\sigma}}{2\beta\sqrt{\sigma}}\right)\right)^2 + \tanh\left(\sqrt{-\sigma}\left(\frac{kt^\alpha}{\alpha}\pm\frac{x^\beta\sqrt{-1+4k^2\sigma}}{2\beta\sqrt{\sigma}}\right)\right)^2\right)\right),$$

$$u_{7,8}(x,t) = \arccos\left(\frac{1}{2}\left(-\cot\left(\sqrt{\sigma}\left(\frac{kt^\alpha}{\alpha}\pm\frac{x^\beta\sqrt{-1+4k^2\sigma}}{2\beta\sqrt{\sigma}}\right)\right)^2 - \tan\left(\sqrt{\sigma}\left(\frac{kt^\alpha}{\alpha}\pm\frac{x^\beta\sqrt{-1+4k^2\sigma}}{2\beta\sqrt{\sigma}}\right)\right)^2\right)\right).$$

To best of our knowledge all the obtained solutions are firstly seen in the literature. The solutions show that the method is applicable, reliable and accurate for conformable differential equations.

REFERENCES

Abdeljawad T, 2015. On conformable fractional calculus. Journal of computational and Applied Mathematics, 279:57-66.

Cenesiz Y, Tasbozan O, Kurt A, 2017. Functional Variable Method for conformable fractional modifed KdV-ZK equation and Maccari system. Tbilisi Mathematical Journal, 10: 117-125.

Hashemi MS, 2018. Invariant subspaces admitted by fractional differential equations with conformable derivatives. Chaos, Solitons and Fractals, 107: 161-169.

Hosseini K, Manafian J, Samadani F, Foroutan M, Mirzazade, M, Zhou Q, 2017. Resonant Optical Solitons with Perturbation Terms and Fractional Temporal Evolution Using Improved tanh $(\varnothing(n)/2)$ - Expansion Method and Exp Function Approach. Optik-International Journal for Light and Electron Optics, 158:933-939.

Ilie M, Biazar J, Ayati Z, 2018. The first integral method for solving some conformable fractional differential equations. Optical and Quantum Electronics, 50: 55.

Kaplan M, Ozer MN, 2018. Multiple-soliton solutions and analytical solutions to a nonlinear evolution equation. Optical and Quantum Electronics, 50:2.

Khalil R, Horani A, Yousef A, Sababheh M, 2014. A new definition of fractional derivative. Journal of Computational and Applied Mathematics, 264: 65-70.

Kumar D, Darvishi MT, Joardar AK, 2018. Modified Kudryashov method and its application to the fractional version of the variety of Boussinesq-like equations in shallow water. Optical and 2 Quantum Electronics, 50: 128.

Kurt A, Tasbozan O, Baleanu D, 2017. New solutions for conformable fractional Nizhnik– Novikov–Veselov system via G'/G expansion method and homotopy analysis methods. Optical and Quantum Electronics, 49: 333.

Zhang S, Zhang HQ, 2011. Fractional sub-equation method and its applications to nonlinear fractional PDEs. Physics Letters A, 375: 1069-