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On some α-admissible contraction mappings on Branciari b-metric spaces

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Abstract

In this paper α-admissible contraction mappings on Branciari b-metric spaces are defined. Conditions for the existence and uniqueness of fixed points for these mappings are discussed and related theorems are proved. Various consequences of these theorems are given and specific examples are presented.

Keywords: Fixed point, Branciari b-metric space, α-admissible contraction mappings, b-comparison functions
2010 MSC: 47H10, 54C60, 54H25, 55M20.

1. Introduction and Preliminaries

In this section we define some basic concepts and notions which are going to be used in the paper.

The concept of b-metric spaces have been introduced by Czerwik [7] and Bakhtin [2].

Definition 1.1. [2, 7] Let X be a nonempty set and let $d : X \times X \to [0, +\infty)$ be a mapping satisfying the following conditions for all $x, y, z \in X$:

(M_b1) $d(x, y) = 0$ if and only if $x = y$;

(M_b2) $d(x, y) = d(y, x)$;

(M_b3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for some real number $s \geq 1$.

Then the mapping d is called a b-metric and the pair (X, d) is called a b-metric space(M_bS) with a constant $s \geq 1$.

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On the other hand, Branciari [3] proposed a generalization of the metric in which he replaced the triangular inequality by a rectangular inequality. This new metric has been referred to by different names such as generalized metric, rectangular metric and Branciari metric. Following the paper by Aydi et.al [1], we will call it Branciari metric.

Definition 1.2. [3] Let X be a nonempty set and let $d : X \times X \to [0, +\infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:

$(BM1)$ $d(x, y) = 0$ if and only if $x = y$;

$(BM2)$ $d(x, y) = d(y, x)$;

$(BM3)$ $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

The map d is called a Branciari metric and the pair (X, d) is called a Branciari metric space (BMS).

Combining the definitions of b-metric and Branciari metric, the so-called Branciari b-metric is defined as follows.

Definition 1.3. [8] Let X be a nonempty set and let $d : X \times X \to [0, +\infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:

(BM_b1) $d(x, y) = 0$ if and only if $x = y$;

(BM_b2) $d(x, y) = d(y, x)$;

(BM_b3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for some real number $s \geq 1$.

The map d is called a Branciari b-metric and the pair (X, d) is called a Branciari b-metric space (BM_bS) with a constant $s \geq 1$.

On a Branciari b-metric space we define and denote an open ball of radius r centered at $x \in X$ as

$$B_r(x, r) = \{y \in X : |d(x, y) < r\}.$$

However, such an open ball is not always an open set.

Let $\mathcal{P}$ be the collection of all subsets $\mathcal{Y}$ of X with the following property: For each $y \in \mathcal{Y}$ there exist $r > 0$ such that $B_r(y) \subseteq \mathcal{Y}$. Then $\mathcal{P}$ defines a topology for the BM_bS (X, d), which is not necessarily Hausdorff.

Convergent sequence, Cauchy sequence, completeness and continuity on Branciari b-metric space are defined as follows.

Definition 1.4. [8] Let (X, d) be a Branciari b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

1. A sequence $\{x_n\} \subset X$ is said to converge to a point $x \in X$ if, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$. The convergence is also represented as follows.

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty.$$

2. A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0$, $p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.
3. (X, d) is said to be a complete Branciari b-metric space if every Cauchy sequence in X converges to some $x \in X$.
4. A mapping $T : X \to X$ on is said to be continuous with respect to the Branciari b-metric d if, for any sequence $\{x_n\} \subset X$ which converges to some $x \in X$, that is $\lim\limits_{n\to\infty} d(x_n, x) = 0$ we have $\lim\limits_{n\to\infty} d(Tx_n, Tx) = 0$.

It should be noted that the limit of a sequence in a BM_bS is not necessarily unique. In addition, a convergent sequence in a BM_bS is not necessarily a Cauchy sequence. Moreover, a Branciari b-metric is not necessarily continuous. The following example illustrates these facts.

Example 1.5. Let $A = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$, $B = \{0, 3\}$ and $X = A \cup B$. Define the function $d(x,y) : X \times X \to [0,\infty)$ such that $d(x,y) = d(y,x)$ in the following way.

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 4 & \text{if} \quad x, y \in A, \\ \frac{1}{n} & \text{if} \quad x \in A, y \in B, \\ 2 & \text{if} \quad x, y \in B. \end{cases}$$

Notice that

$$d(\frac{1}{2}, 1) = 4 > d(\frac{1}{2}, 0) + d(0, 1) = \frac{3}{2},$$

so, $d(x,y)$ is not a metric. In addition,

$$d(\frac{1}{2}, 1) = 4 > d(\frac{1}{2}, 0) + d(0, 3) + d(3, 1) = \frac{7}{2},$$

hence, $d(x,y)$ is not a Branciari metric. Moreover,

$$d(\frac{1}{m}, \frac{1}{n}) = 4 > s[d(\frac{1}{n}, 0) + d(0, \frac{1}{m})] = s\frac{m+n}{mn},$$

for $n, m \in \mathbb{N}$ satisfying $\frac{4mn}{m+n} > s$. Therefore, $d(x,y)$ is not a b-metric as well. However, it is Branciari b-metric with $s = 2$. Indeed, then we have

$$d(\frac{1}{m}, \frac{1}{n}) = 4 \leq 2[d(\frac{1}{n}, 0) + d(0, 3) + d(3, \frac{1}{m})] = 2(2 + \frac{m+n}{mn}).$$

Observe also that

$$\lim_{n\to\infty} d(\frac{1}{2n}, 0) = \lim_{n\to\infty} \frac{1}{2n} = 0,$$

and

$$\lim_{n\to\infty} d(\frac{1}{2n}, 3) = \lim_{n\to\infty} \frac{1}{2n} = 0,$$

that is, both 0 and 3 are limits of the sequence $\{\frac{1}{2n}\}$.

Another fact about this metric is that even though the sequence $\{\frac{1}{2n}\}$ is convergent, it is not a Cauchy sequence. Obviously,

$$\lim_{p\to\infty} d(x_n, x_{n+p}) = \lim_{p\to\infty} d(\frac{1}{2n}, \frac{1}{2n+2p}) = \lim_{n\to\infty} 4 = 4.$$

Finally, we note that although the open set $B_1(\frac{1}{3})$ contains 0, that is $B_1(\frac{1}{3}) = \{0, 3, \frac{1}{3}\}$, there is no positive r for which $B_r(0) \subset B_1(\frac{1}{3})$.

Regarding the above facts about Branciari b-metric, we need the following property of Branciari metric space, the proof of which can be found in [10].

Proposition 1.6. *[10] Let $\{x_n\}$ be a Cauchy sequence in a Branciari metric space (X,d) such that $\lim_{n\to\infty} d(x_n, x) = 0$, where $x \in X$. Then $\lim_{n\to\infty} d(x_n, y) = d(x, y)$, for all $y \in X$. In particular, the sequence $\{x_n\}$ does not converge to y if $y \neq x$.*

Remark 1.7. The Proposition 1.6 is valid if we replace Branciari metric space by a Branciari b-metric space.

Berinde [4] and Rus [11] defined and later modified a class of functions called comparison functions. These functions are being used by many authors to replace the usual contractive condition by a more general one. We next define the comparison and (b)-comparison functions.

An increasing function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying $\varphi^n(t) \to 0$, $n \to \infty$ for any $t \in [0, \infty)$ is called a comparison function, (CF) (see e.g. [4],[11])..

A (b)-comparison function, (BCF), (see e.g.[5],[6]) is a function $\varphi_b : [0, +\infty) \to [0, +\infty)$ satisfying the conditions

- (b_1) φ_b is increasing,
- (b_2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} \nu_k$ such that $s^{k+1}\varphi_b^{k+1}(t) \le as^k\varphi_b^k(t) + \nu_k$, for $k \ge k_0$ and any $t \in [0, \infty)$.

for some $s \ge 1$.

In the sequel, we denote the class of comparison functions by Φ and the class of (b)-comparison functions by Φ_b.

Comparison and (b)-comparison functions satisfy the following properties.

Lemma 1.8. *(Berinde [4], Rus [11]) Any comparison function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfies the following:*

1. *Every iterate φ^k of φ $k \ge 1$, is also a comparison function;*
2. *φ is continuous at 0;*
3. *$\varphi(t) < t$, for any $t > 0$.*

Lemma 1.9. *[6] A (b)-comparison function $\varphi_b : [0, +\infty) \to [0, +\infty)$ satisfies the following:*

1. *the series $\sum_{k=0}^{\infty} s^k\varphi_b^k(t)$ converges for any $t \in [0, +\infty)$;*
2. *the function $b_s : [0, +\infty) \to [0, +\infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k\varphi_b^k(t)$, $t \in [0, \infty)$ is increasing and continuous at 0.*

Finally, we note that any (b)-comparison function is a comparison function.

We also need to recall the notion of α-admissibility introduced by Samet et al [12] (see also [9]).

Definition 1.10. A mapping $T : X \to X$ is called α-admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \ge 1 \Rightarrow \alpha(Tx, Ty) \ge 1, \tag{1.1}$$

where $\alpha : X \times X \to [0, \infty)$ is a given function.

2. Existence and uniqueness theorems on complete Branciari b-metric spaces

In what follows, we define some classes of α-admissible contractions.

Definition 2.1. Let (X, d) be a Branciari b-metric space with a constant $s \ge 1$ and let $\alpha : X \times X \to [0, \infty)$ and $\varphi_b \in \Phi_b$ be two given functions.

(i) An $\alpha - \varphi_b$ contractive mapping $T : X \to X$ is of type (A) if it is α-admissible and satisfies

$$\alpha(x, y)d(Tx, Ty) \le \varphi_b (M(x, y)), \text{ for all } x, y \in X \tag{2.1}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

(ii) An $\alpha - \varphi_b$ contractive mapping $T : X \to X$ is of type (B) if it is α-admissible and satisfies

$$\alpha(x, y)d(Tx, Ty) \le \varphi_b (N(x, y)), \text{ for all } x, y \in X \tag{2.2}$$

where

$$N(x, y) = \max\{d(x, y), \frac{1}{2s}[d(x, Tx) + d(y, Ty)]\}.$$

Remark 2.2. Clearly, we have $d(x,y) \leq N(x,y) \leq M(x,y)$ for all $x,y \in X$.

We state and prove an existence theorem for fixed point of $\alpha - \varphi_b$ contractive mapping in class (A).

Theorem 2.3. *Let (X,d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is an $\alpha - \varphi_b$ contractive mapping of type (A) satisfying the following conditions.*

(i) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.

(ii) T is continuous.

Then T has a fixed point.

Proof. Regarding the condition (i), we choose $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$ and define the sequence $\{x_n\}$ as

$$x_{n+1} = Tx_n \ \text{ for } \ n \in \mathbb{N}.$$

First, we assume that any two consecutive members of the sequence $\{x_n\}$ are distinct, that is, $x_n \neq x_{n+1}$ for all $n \geq 0$. Otherwise, we would have $x_p = x_{p+1} = Tx_p$ for some $p \in \mathbb{N}$, which means that x_p is a fixed point of T.

Since T is α-admissible, the condition (i) implies

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1, \tag{2.3}$$

or, continuing in this way,

$$\alpha(x_n, x_{n+1}) \geq 1, \ \text{for all } n \in \mathbb{N}. \tag{2.4}$$

In a similar way, starting with

$$\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1, \tag{2.5}$$

we deduce

$$\alpha(x_n, x_{n+2}) \geq 1, \ \text{for all } n \in \mathbb{N}. \tag{2.6}$$

The rest of the proof is done in 4 steps.

Step 1: We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

For $x = x_n$ and $y = x_{n+1}$ with the use of (2.4), the contractive condition (2.1) becomes

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \leq \varphi_b(M(x_{n-1}, x_n)), \end{aligned} \tag{2.8}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \end{aligned}$$

The first possibility, that is $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ for some $n \geq 1$, implies

$$d(x_n, x_{n+1}) \leq \varphi_b(M(x_{n-1}, x_n)) = \varphi_b(d(x_n, x_{n+1})) < d(x_n, x_{n+1}),$$

since $d(x_n, x_{n+1}) > 0$ and $\varphi_b(t) < t$, which is not possible. Hence, for all $n \geq 1$ we must have $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Then the inequality (2.8) becomes

$$d(x_n, x_{n+1}) \leq \varphi_b(M(x_{n-1}, x_n)) \leq \varphi_b(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n), \ \text{for all } n \geq 1. \tag{2.9}$$

Therefore, the sequence $\{d(x_{n-1}, x_n)\}$ is decreasing ,that is,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \text{ for all } n \geq 1. \tag{2.10}$$

Repeated application of (2.9) yields,

$$d(x_{n+1}, x_n) \leq \varphi_b^n(d(x_0, x_1)), \text{ for all } n \geq 1. \tag{2.11}$$

Taking limit as $n \to \infty$ in(2.11) and using the statement (1) of Lemma 1.9, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Step 2: At this step we prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.12}$$

Let $x = x_{n-1}$ and $x = x_{n+1}$ in (2.1) and take into account (2.6). This gives

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \alpha(x_{n-1}, x_{n+1}) d(Tx_{n-1}, Tx_{n+1}) \leq \varphi_b(M(x_{n-1}, x_{n+1})), \end{aligned} \tag{2.13}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\} \\ &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\} \end{aligned} \tag{2.14}$$

Regarding (2.10), $M(x_{n-1}, x_{n+1})$ can be either $d(x_{n-1}, x_{n+1})$ or $d(x_{n-1}, x_n)$.
Define $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Thus, from (2.13) we have

$$\begin{aligned} a_n &= d(x_n, x_{n+2}) \leq \varphi_b(M(x_{n-1}, x_{n+1})) \\ &= \varphi_b(\max\{a_{n-1}, b_{n-1}\}) < \max\{a_{n-1}, b_{n-1}\}, \text{ for all } n \geq 1 \end{aligned} \tag{2.15}$$

On the other hand, by (2.10) we also have

$$b_n \leq b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\}.$$

As a result, we get

$$\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\} \text{ for all } n \geq 1,$$

that is, the sequence $\{\max\{a_n, b_n\}\}$ is non increasing and hence, it converges to some $l \geq 0$. If $l > 0$, due to (2.7) we have

$$l = \lim_{n\to\infty} \max\{a_n, b_n\} = \max\{\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\} = \lim_{n\to\infty} a_n$$

Now, we let $n \to \infty$ in (2.15), so that we conclude

$$l = \lim_{n\to\infty} a_n < \lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\} = l,$$

which is a contradiction and hence, $l = 0$. Then, we conclude

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0,$$

that is, (2.12) is proved.

Step 3: We shall prove that for all $n \neq m$,

$$x_n \neq x_m. \tag{2.16}$$

Assume that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. We already have $d(x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, hence, without loss of generality we may take $m > n+1$. Consider now

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, Tx_n) = d(x_m, Tx_m) \\ &= d(Tx_{m-1}, Tx_m) \leq \alpha(x_{m-1}, x_m) d(Tx_{m-1}, Tx_m) \\ &\leq \varphi_b(M(x_{m-1}, x_m)), \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} M(x_{m-1}, x_m) &= \max\{d(x_{m-1}, x_m), d(x_{m-1}, Tx_{m-1}), d(x_m, Tx_m)\} \\ &= \max\{d(x_{m-1}, x_m), d(x_{m-1}, x_m), d(x_m, x_{m+1})\} \\ &= \max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\} = d(x_{m-1}, x_m), \end{aligned} \tag{2.18}$$

because of (2.10). Then we have,

$$d(x_m, Tx_m) \leq \varphi_b(d(x_{m-1}, x_m)),$$

for all $m \in \mathbb{N}$. Hence,

$$d(x_m, Tx_m) \leq \varphi_b(d(x_{m-1}, x_m)) \leq \varphi_b^2(d(x_{m-2}, x_{m-1})) \leq \cdots \leq \varphi_b^{m-n}(d(x_n, x_{n+1})), \tag{2.19}$$

Combining (2.17) and (2.19) we get

$$d(x_n, x_{n+1}) = d(x_m, Tx_m) \leq \varphi_b^{m-n}(d(x_n, x_{n+1})). \tag{2.20}$$

Since every iterate of a comparison function is also a comparison function, then

$$\varphi_b^{m-n}(d(x_n, x_{n+1})) < d(x_n, x_{n+1}),$$

thus, the inequality (2.20) yields

$$d(x_n, x_{n+1}) \leq \varphi_b^{m-n}(d(x_n, x_{n+1})) < d(x_n, x_{n+1}), \tag{2.21}$$

which is not possible. Therefore, our initial assumption is incorrect and we should have $x_n \neq x_m$ for all $m \neq n$.

Step 4: At this step we will prove that $\{x_n\}$ is a Cauchy sequence, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \text{ for all } k \in \mathbb{N}. \tag{2.22}$$

The cases $k = 1$ and $k = 2$ are proved, respectively in (2.7) and (2.12). Assume that $k \geq 3$. We have two cases:

Case 1: Suppose that $k = 2m+1$ where $m \geq 1$. Regarding Step 3, we have $x_l \neq x_s$ for all $l \neq s$, so that we can apply the condition BM_b3 in Definition 1.3, together with (2.11) implies

$$\begin{aligned} d(x_n, x_{n+k}) &= d(x_n, x_{n+2m+1}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ &+ s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\ &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\ &+ s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6})] + \ldots + s^{m+1}[d(x_{n+2m}, x_{n+2m+1})] \\ &\vdots \\ &\leq s[\varphi_b^n(d(x_0, x_1)) + \varphi_b^{n+1}(d(x_0, x_1))] + s^2[\varphi_b^{n+2}(d(x_0, x_1)) + \varphi_b^{n+3}(d(x_0, x_1))] \\ &+ s^3[\varphi^{n+4}(d(x_0, x_1) + \varphi^{n+5}(d(x_0, x_1))] + \ldots + s^m[\varphi^{n+2m}(d(x_0, x_1))] \\ &\leq s\varphi_b^n(d(x_0, x_1)) + s^2\varphi_b^{n+1}(d(x_0, x_1)) + s^3\varphi_b^{n+2}(d(x_0, x_1)) \\ &+ s^4\varphi_b^{n+3}(d(x_0, x_1)) + s^5\varphi^{n+4}(d(x_0, x_1) + \ldots + s^{2m+1}\varphi^{n+2m}(d(x_0, x_1)) \\ &= \frac{1}{s^{n-1}}\left[s^n\varphi_b^n(d(x_0, x_1)) + s^{n+1}\varphi_b^{n+1}(d(x_0, x_1) + s^{n+2}\varphi_b^{n+2}(d(x_0, x_1)))\right. \\ &+ \left. \ldots + s^{n+2m}\varphi_b^{n+2m}(d(x_0, x_1))\right]. \end{aligned}$$

Define

$$\mathcal{S}_n = \sum_{p=0}^{n} s^p \varphi_b^p(d(x_0,x_1)) \text{ for } n \geq 1. \tag{2.23}$$

Then, the inequality above becomes

$$d(x_n, x_{n+2m+1}) \leq \frac{1}{s^{n-1}} \left[\mathcal{S}_{n+2m} - \mathcal{S}_{n-1}\right],\ n \geq 1, m \geq 1.$$

By the initial assumption, $x_0 \neq x_1$ and by the Lemma 1.9, we observe that the series $\sum_{p=0}^{\infty} s^p \varphi_b^p(d(x_0,x_1))$ converges to some $\mathcal{S} \geq 0$. Thus,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = \lim_{n\to\infty} d(x_n, x_{n+2m+1}) = 0. \tag{2.24}$$

Case 2. Suppose that $k = 2m$ where $m \geq 2$. We use again the condition BM_b3 in Definition 1.3, together with (2.11) so that,

$$\begin{array}{rcl}
d(x_n, x_{n+k}) & = & d(x_n, x_{n+2m}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\
& \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\
& + & s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\
& \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})] \\
& + & \ldots + s^{m-1}\left[d(x_{n+2m-4}, x_{n+2m-3}) + d(x_{n+2m-3}, x_{n+2m-2})\right. \\
& + & \left. d(x_{n+2m-2}, x_{n+2m})\right] \\
& \vdots & \\
& \leq & s[\varphi_b^n(d(x_0,x_1)) + \varphi_b^{n+1}(d(x_0,x_1))] + s^2[\varphi_b^{n+2}(d(x_0,x_1)) + \varphi_b^{n+3}(d(x_0,x_1))] \\
& + & \ldots + s^{m-1}[\varphi^{n+2m-4}(d(x_0,x_1) + \varphi^{n+2m-3}(d(x_0,x_1))] \\
& + & s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\
& \leq & s\varphi_b^n(d(x_0,x_1)) + s^2\varphi_b^{n+1}(d(x_0,x_1)) + s^3\varphi_b^{n+2}(d(x_0,x_1)) \\
& + & \ldots + s^{2m-3}\varphi_b^{n+2m-4}(d(x_0,x_1)) + s^{2m-2}\varphi^{n+2m-3}(d(x_0,x_1)) \\
& + & s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\
& = & \dfrac{1}{s^{n-1}}\left[s^n\varphi_b^n(d(x_0,x_1)) + s^{n+1}\varphi_b^{n+1}(d(x_0,x_1) + s^{n+2}\varphi_b^{n+2}(d(x_0,x_1)))\right. \\
& + & \left. \ldots + +s^{n+2m-3}\varphi_b^{n+2m-3}(d(x_0,x_1))\right] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\
& = & \displaystyle\sum_{p=n}^{n+2m-3} s^p\varphi_b^p(d(x_0,x_1)) + s^{m-1} d(x_{n+2m-2}, x_{n+2m}).
\end{array}$$

Using the notation in (2.23), we rewrite the inequality above as

$$d(x_n, x_{n+k}) = \frac{1}{s^{n-1}}\left[\mathcal{S}_{n+2m-3} - \mathcal{S}_{n-1}\right] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}). \tag{2.25}$$

From (2.12) we have $\lim_{n\to\infty} s^{m-1} d(x_{n+2m-2}, x_{n+2m}) = 0$, and using the Lemma 1.9 we get

$$\begin{array}{rcl}
\lim\limits_{n\to\infty} d(x_n, x_{n+k}) & = & \lim\limits_{n\to\infty} d(x_n, x_{n+2m}) \\
& \leq & \lim\limits_{n\to\infty}\left[\dfrac{1}{s^{n-1}}(\mathcal{S}_{n+2m-3} - \mathcal{S}_{n-1}) + s^{2m-1} d(x_{n+2m-2}, x_{n+2m})\right] = 0.
\end{array} \tag{2.26}$$

Therefore, for any $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0,$$

that is, the sequence $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete Branciari b-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{2.27}$$

By the condition (ii) of the hypothesis, T is continuous. Then, from (2.27) we have

$$\lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) = 0,$$

that is, the sequence $\{x_n\}$ converges to Tu as well. But then, the Proposition 1.6 implies that $Tu = u$, that is, u is a fixed point of T. □

The Theorem 2.3 provides the existence of a fixed point. To have uniqueness we impose an additional requirement.

(U) For every pair x and y of fixed points of T, $\alpha(x, y) \geq 1$.

Theorem 2.4. *If we add the condition (U) to the statement of Theorem 2.3, the fixed point of the mapping is unique.*

Proof. The existence of a fixed point is proved in Theorem 2.3. Assume that the map T has two fixed points, say $x, y \in X$, such that $x \neq y$. The condition (U) implies that $\alpha(x, y) \geq 1$. If $d(x, y) > 0$ then the contractive condition (2.1) with the fixed points x and y yields

$$d(x, y) = \alpha(x, y)d(Tx, Ty) \leq \varphi_b(M(x, y)),$$

where,

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(Ty, y)\} = d(x, y).$$

Since $\varphi_b(t) < t$ for $t > 0$, we have

$$d(x, y) \leq \varphi_b(d(x, y)) < d(x, y),$$

which is not possible. Therefore, $d(x, y) = 0$, or, $x = y$ which completes the proof of the uniqueness. □

The strong condition on continuity of the map T can be replaced by a weaker condition called α-regularity of the space. This condition reads as follows.

(RG) A Branciari b-metric space (X, d) is called α-regular if for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n, x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

If we replace the continuity condition of the mapping T by the α-regularity of the space (X, d) we have the following result.

Theorem 2.5. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is an $\alpha - \varphi_b$ contractive mapping of type (A) and that the following conditions hold.*

(i) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.*

(ii) *(X, d) is α-regular, that is (RG) holds on (X, d).*

Then T has a fixed point. If, in addition the condition (U) holds on X, the fixed point is unique.

Proof. Starting with the element $x_0 \in X$ satisfying the condition (i), we construct the sequence of successive iterations $\{x_n\}$ as $x_n = Tx_{n-1}$, for $n \in \mathbb{N}$.

The convergence of this sequence can be shown exactly as in the proof of Theorem 2.3.

Let u be the limit of $\{x_n\}$, that is,

$$\lim_{n\to\infty} d(x_n, u) = 0.$$

We will show that u is a fixed point of T. For the sequence $\{x_n\}$ which converges to u we have from (2.4) that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$. Then, the α-regularity condition (RG) implies that

$$\alpha(x_n, u) \geq 1, \text{ for all } n \in \mathbb{N}_0.$$

The contractive inequality (2.1) with x_n and u becomes

$$d(Tx_n, Tu) \leq \alpha(x_n, u)d(Tx_n, Tu) \leq \varphi_b(M(x_n, u)), \tag{2.28}$$

where

$$M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}.$$

If $M(x_n, u) > 0$, then (2.28) implies

$$\begin{array}{rcl} d(Tx_n, Tu) & \leq & \alpha(x_n, u)d(Tx_n, Tu) \leq \varphi_b(M(x_n, u)) \\ & < & M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}, \end{array} \tag{2.29}$$

whereupon, by letting $n \to \infty$ and regarding the Proposition 1.6, we obtain

$$d(u, Tu) = \lim_{n\to\infty} d(x_{n+1}, Tu) < \lim_{n\to\infty} \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\} = d(u, Tu), \tag{2.30}$$

which is a contradiction. Then we should have $M(x_n, u) = 0$, that is $d(u, Tu) = 0$, hence, u is a fixed point of T.

The proof of uniqueness is identical to the proof of Theorem 2.4. □

We present next some immediate consequences of the main results given in Theorems 2.3, 2.4 and 2.5. First, we observe that regarding the Remark 2.2, the existence and uniqueness of a fixed point of the contraction mappings of type (B) is easily concluded.

Theorem 2.6. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is an $\alpha - \varphi_b$ contractive mapping of type (B) satisfying the following:*

(i) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.
(ii) Either T is continuous or (X, d) satisfies (RG).

Then T has a fixed point.
If, in addition the condition (U) holds on X, the fixed point is unique.

Another result follows from the Remark 2.2.

Theorem 2.7. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $\alpha(x, y) : X \times X \to [0, \infty)$ is a given mapping and that $T : X \to X$ is an α-admissible continuous mapping satisfying the conditions:*

(i) $\alpha(x, y)d(Tx, Ty) \leq \varphi_b(d(x, y))$, for all $x, y \in X$.
(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.
(iii) Either T is continuous or (X, d) satisfies (RG).

Then T has a fixed point. If, in addition the condition (U) holds on X, the fixed point is unique.

Taking $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.3, we obtain the following corollary the proof of which is also obvious.

Corollary 2.8. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is a continuous mapping satisfying*

$$d(Tx, Ty) \leq \varphi_b(M(x, y)), \tag{2.31}$$

for all $x, y \in X$, where $\varphi_b \in \Psi_b$.

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then T has a unique fixed point.

Corollary 2.9. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $T : X \to X$ is a continuous mapping satisfying*

$$d(Tx, Ty) \leq \varphi_b (d(x, y)), \text{ for all } x, y \in X. \tag{2.32}$$

Then T has a unique fixed point.

The following result is obtained by choosing a particular (b)-comparison function as $\varphi_b(t) = \dfrac{k}{s}t$ with $0 < k < 1$.

Corollary 2.10. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$. Suppose that $\alpha : X \times X \to [0, \infty)$ is a given function and $T : X \to X$ is an α-admissible mapping satisfying the following.*

(i)

$$\alpha(x, y)d(Tx, Ty) \leq \frac{k}{s}M(x, y), \tag{2.33}$$

for all $x, y \in X$ and some $0 < k < 1$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

(ii) $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$ *for some* $x_0 \in X$.

(iii) Either T is continuous or (X, d) satisfies (RG). Then T has a fixed point in X. If, in addition, the condition (U) holds on X, the fixed point is unique.

As a final consequence, we give the following corollary.

Corollary 2.11. *Let (X, d) be a complete Branciari b-metric space with a constant $s \geq 1$ and $T : X \to X$ be a continuous mapping. Suppose that for some constants $a, b, c \geq 0$ and $0 < k < 1$ with $a + b + c \leq \dfrac{k}{s}$ the inequality*

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty), \tag{2.34}$$

holds for all $x, y \in X$. Then T has a unique fixed point.

Proof. Observe that for all $x, y \in X$

$$ad(x, y) + bd(x, Tx) + cd(y, Ty) \leq \frac{k}{s}M(x, y)$$

where $0 < k < 1$. Then the proof follows from Corollary 2.10. □

We give an example to illustrate the theoretical results presented above.

Example 2.12. Suppose that $X = A \cup B$ where $A = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}\right\}$ and $B = [1, 4]$. Define the mapping $d : X \times X \to [0, \infty)$ with $d(x, y) = d(y, x)$ as follows.

For $x, y \in B$, or $x \in A$ and $y \in B$, $d(x, y) = |x - y|$ and

$$d(\tfrac{1}{2}, \tfrac{1}{4}) = d(\tfrac{1}{6}, \tfrac{1}{8}) = 0.2$$

$$d(\tfrac{1}{2}, \tfrac{1}{6}) = d(\tfrac{1}{4}, \tfrac{1}{6}) = d(\tfrac{1}{4}, \tfrac{1}{8}) = 0.1$$

$$d(\tfrac{1}{2}, \tfrac{1}{8}) = 1$$

This mapping is a Branciari b-metric with $s = 2$. Let $T : X \to X$ be defined as

$$Tx = \begin{cases} \frac{x}{4} & \text{if} \quad x \in B, \\ \\ \frac{1}{6} & \text{if} \quad x \in A. \end{cases}$$

Then, the mapping T satisfies the condition

$$d(Tx, Ty) \leq \varphi_b\left(d(x, y)\right),$$

for all $x, y \in X$ where $\varphi_b(t) = \frac{t}{4}$ is a (b)-comparison function. Hence, by Corollary 2.11, T has a unique fixed point which is $x = \frac{1}{6}$.

3. Concluding Remarks

The main contributions of this study to Fixed point theory are the existence-uniqueness results given in Theorems 2.3, 2.4 and 2.5. These theorems provides existence and uniqueness conditions for a large class of contractive mappings on Branciari b-metric spaces. By taking $s = 1$ and/or $\alpha(x, y) = 1$ in all the theorems and corollaries, various existing results on Branciari b-metric and Branciari metric spaces can be obtained.

On the other hand, it should be mentioned that by choosing the function α in the definition of α-admissible mappings in a particular way, it is possible to obtain existence and uniqueness results for maps defined on partially ordered metric spaces.

Define a partial ordering $\preceq$ on a Branciari b-metric space (X, d). Let $T : X \to X$ be an increasing mapping. Then, we can easily proof the following fixed point theorem.

Theorem 3.1. *Let $(X, d, \preceq)$ be a complete Branciari b-metric space with a constant $s \geq 1$ on which a partial ordering $\preceq$ is defined. Suppose that $T : X \to X$ is an increasing mapping satisfying the following:*

(i)

$$d(Tx, Ty) \leq \varphi_b(M(x, y)),$$

for all x, y in X with $x \preceq y$ and some (b)-comparison function φ_b where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

(ii) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$.

(iii) Either T is continuous or, for any increasing sequence $\{x_n\} \in X$ which converges to x we have $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a fixed point.If, in addition any two fixed points of T are comparable, that is, $x \preceq y$ or $y \preceq x$, then the fixed point of T is unique.

Proof. Observe that all the conditions of Theorems 2.3, 2.4 and 2.5 hold if we choose the function α as

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x \preceq y \text{ or } y \preceq x \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

Then, the mapping T has a unique fixed point. □

In addition, all the consequent results of Theorems 2.3, 2.4 and 2.5 can be written on Branciari b-metric spaces with a partial ordering can be proved in a similar way.

4. Competing Interests

The authors declare that they have no competing interests.

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