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α-admissible contractions on quasi-metric-like space

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Abstract

In the setting of a complete quasi-metric-like spaces we investigate some fixed point problems via admissible mappings. Contractive condition includes (c)-comparison function. Definition of (α, ψ)-contraction is generalized and continuity of f is replaced with regularity of observed space. Presented results improve and extend several results on quasi-metric-like spaces.

Keywords: quasi-metric-like space, fixed point, α- admissible, (b)-comparison functions,
2010 MSC: 47H10, 54C60, 54H25, 55M20.

1. Introduction and Preliminaries

Among various generalizations of concept of metric, Matthews ([19]) introduced special kind of a partial metric space where the self-distance $d(x, x)$ is not necessarily zero. He studied denotational semantics of dataflow networks and proved generalization of Banach theorem for applications in program verification. On the other hand, Amini-Harandi ([2]) redefined a dislocated metric of Hitzler and Seda ([13]) and introduced metric-like spaces. Combining these two concepts we get quasi-metric-like spaces. The study of partial metric spaces has wide area of application, especially in computer science ([17, 22]). Therefore, we can find many fixed point results in the setting of partial metric spaces ([1, 2, 4], [5], [7, 9], [12], [16], [24, 25], [26, 27]).

In 2012., Samet et al. ([23]) introduced the concept of α-admissible mappings and, one year later, Karapınar et al. ([14]) improved this notion with triangular α-admissible mappings. In that manner, study of ψ-contractions was extended and broadly researched ([3], [11], [14, 15], [23]).

In this paper, we discuss on existence and uniqueness of a fixed point of (α, ψ)-contractive mappings on quasi-metric-like space. Moreover, we generalize some fixed point results regarding (α, ψ)-contractive mappings. Obtained results are discussed, compared and substantiated with several examples.

Let us recall some definitions that will be needed in the sequel.

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Definition 1.1. Let X be a nonempty set. A mapping $d\colon X \times X \to [0, +\infty)$ is said to be a metric-like if for all $x, y, z \in X$, the following conditions are satisfied:

(d_1) $d(x, y) = 0 \Longrightarrow x = y$;

(d_2) $d(x, y) = d(y, x)$;

(d_3) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a metric-like space.

Omitting symmetry property of metric, we get a quasi-metric. If that condition is combined with a notion of metric-like, we get the following definition:

Definition 1.2. Let X be a nonempty set. A mapping $d\colon X \times X \to [0, +\infty)$ is said to be a quasi-metric-like if for all $x, y, z \in X$, the following conditions are satisfied:

(q_1) $d(x, y) = 0 \Longrightarrow x = y$;

(q_2) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a quasi-metric-like space.

Example 1.3. Let $X = [0, \infty)$ and $d : X \times X \mapsto [0, \infty)$ defined with

$$d(x, y) = \max\{x, y\},\ x, y \in X.$$

Then (X, d) is a metric-like space. Obviously, (d_2) holds, so it is not a quasi-metric-like space.

Example 1.4. Let $X = [0, \infty)$ and $d : X \times X \mapsto [0, \infty)$ defined with

$$d(x, y) = \begin{cases} x - y, & \text{if } y \leq x, \\ 1, & \text{otherwise.} \end{cases}$$

Then (X, d) is a quasi-metric-like space.

In order to study fixed point problems on quasi-metric-like spaces, we need to give basic definitions regarding continuity and convergence.

Definition 1.5. Let (X, d) be a quasi-metric-like space and $\{x_n\} \subseteq X$. A sequence $\{x_n\}$ is a Cauchy sequence if both $\lim\limits_{m,n\to\infty, m>n} d(x_n, x_m)$ and $\lim\limits_{m,n\to\infty, m>n} d(x_m, x_n)$ exist and are finite.

Definition 1.6. Let (X, d) be a quasi-metric-like space and $\{x_n\} \subseteq X$. A sequence $\{x_n\}$ is convergent sequence in X if there exists some $x \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x) = \lim\limits_{n\to\infty} d(x, x_n) = d(x, x)$.

If $\{x_n\}$ converges to x, we denote that whit $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$.

Definition 1.7. A quasi-metric-like space (X, d) is complete if, for any Cauchy sequence $\{x_n\} \subseteq X$, there exists some $x \in X$ such that

$$\begin{aligned} d(x, x) &= \lim_{n\to\infty} d(x, x_n) \\ &= \lim_{n\to\infty} d(x_n, x) \\ &= \lim_{m,n\to\infty, m>n} d(x_n, x_m) \\ &= \lim_{m,n\to\infty, m>n} d(x_m, x_n). \end{aligned}$$

Definition 1.8. Let (X, d) be a quasi-metric-like space and $\{x_n\} \subseteq X$. A sequence $\{x_n\}$ is a Cauchy sequence if both $\lim_{m,n\to\infty, m>n} d(x_n, x_m)$ and $\lim_{m,n\to\infty, m>n} d(x_m, x_n)$ exist and are finite.

The main difference between metric and quasi-metric like spaces is reflected in topology and properties of a convergence:

- This kind of generalized metric needs not to be continuous.
- Topology of quasi-metric-like space is not necessarily Hausdorff, so the limit of convergent sequence is not always unique.
- There are convergent sequences in quasi-metric-like spaces that are not Cauchy sequences.

Example 1.9. Let $X = \{a, b\}$, $a \neq b$, and $d : X \times X \mapsto [0, \infty)$ defined with $d(x, y) = 1$, $x, y \in X$. Then (X, d) is a metric like space and any constant sequence is convergent with both a and b as limits since

$$d(a, b) = d(b, a) = d(a, a) = d(b, b).$$

Example 1.10. Let $X = \{0, 1, 2\}$ and $d : X \times X \mapsto [0, \infty)$ defined with

x \ y	0	1	2
0	1	1	2
1	2	1	2
2	2	2	2

Thus, (X, d) is a quasi-metric-like space. Observe the sequence $x_{2n} = 1$, $x_{2n-1} = 0$, $n \in \mathbb{N}$. Obviously, $\{x_n\}$ is not a Cauchy sequence, but

$$\lim_{n\to\infty} d(x_n, 2) = \lim_{n\to\infty} d(2, x_n) = d(2, 2),$$

implying that $\lim_{n\to\infty} x_n = 2$.

Definition 1.11. Let (X, d) and (Y, q) be quasi-metric-like spaces. A mapping $f : X \to Y$ is a continuous mapping if, for any $\{x_n\} \subseteq X$,

$$\lim_{n\to\infty} x_n = x^* \in X \Rightarrow \lim_{n\to\infty} f x_n = f x^*,$$

where the limit is taken according to the observed metrics and induced topologies.

Definition 1.12. [23] For some $\alpha : X \times X \to [0, +\infty)$, a mapping $f : X \mapsto X$ is an α-admissible mapping if

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(fx, fy) \geq 1,$$

for any $x, y \in X$.

Very recently, Popescu [21] introduced notions as follows:

Definition 1.13. ([21]) Let $\alpha : X \times X \to [0, \infty)$ be a function. If $f : X \to X$ satisfies the condition

$$(T1)' \;\; \alpha(x, fx) \geq 1 \Rightarrow \alpha(fx, f^2x) \geq 1,$$

for all $x \in X$, then it is called right-α-orbital admissible mapping. If f satisfies the condition

$$(T1)'' \;\; \alpha(fx, x) \geq 1 \Rightarrow \alpha(f^2x, fx) \geq 1,$$

for all $x \in X$, then it is called left-α-orbital admissible mapping. Furthermore, if it is both right-α-orbital admissible and left-α-orbital admissible, then a mapping f is called α-orbital admissible.

Karapinar ([14]) and Popescu ([21])extended notion of α-admissability by defining triangular α-admissability and, respectively, triangular α-orbital admissability.

Class of (b)-comparison functions was introduced by Berinde ([9]) in order to extend some fixed point results integrating comparison functions and c-comparison functions ([8]):

Definition 1.14. [9] Let $s \geq 1$ be a real number. A mapping $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called a (b)-comparison function if the following conditions are fulfilled

(1) ψ is a nondecreasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\psi^{k+1}(t) \leq a s^k \psi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0,\infty)$.

The class of (b)-comparison functions will be denoted by Ψ_b. Notice that the notion of (b)-comparison function reduces to the concept of (c)-comparison function if $s = 1$ and therefore includes a set of comparison functions. The following lemma will be used in the proof of our main result.

Lemma 1.15. *[6, 7] Let $s \geq 1$ be a real number. If $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a (b)-comparison function, then*

(1) the series $\sum_{k=0}^{\infty} s^k \psi^k(t)$ converges for any $t \in \mathbb{R}_0^+$;

(2) the function $p_s : [0,\infty) \to [0,\infty)$ defined by

$$p_s(t) = \sum_{k=0}^{\infty} s^k \psi^k(t), \text{ for all } \ t \in [0,\infty),$$

is increasing and continuous at 0.

Remark 1.16. Evidently, if $\psi \in \Psi_b$, then $\psi(t) < t$ for all $t > 0$.

Application of (b)-comparison function is familiar for the setting of b-metric spaces due to the existence of a constant s. Nevertheless, $\Psi_c \subseteq \Psi_b$, thus we may assume $\psi \in \Psi_b$.

2. Main result

In this section we define (α,ψ)-contractions and prove existence and uniqueness of fixed point for this class of mappings under different assumptions. One kind of generalization of (α,ψ)-contractive mappings is given in the sequel with accompanying fixed point results.

Definition 2.1. Let (X,d) be a complete quasi-metric-like space. A self-mapping $f : X \to X$ is called (α,ψ)-contractive mapping if there exist $\psi \in \Psi_b$ and $\alpha : X \times X \to [0,\infty)$ satisfying the following condition:

$$\alpha(x,y)d(fx,fy) \leq \psi(d(x,y)), \, x,y \in X. \tag{2.1}$$

Theorem 2.2. *Let (X,d) be a complete quasi-metric-like space and let $f : X \to X$ be an (α,ψ)-contractive mapping. Suppose also that*

(i) f is α-orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq 1$;
(iii) f is continuous.

Then f has a fixed point x^* in X and $d(x^*, x^*) = 0$.

Proof. Choose x_0 such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq 1$ and define an iterative sequence $\{x_n\}$ in X by $x_{n+1} = fx_n$, $n \in \mathbb{N}_0$. If there is some $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of f. Therefore, suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. α-orbital admissibility of f , from (ii), inductively implies

$$\alpha(x_n, x_{n+1}) \geq 1, \; n \in \mathbb{N}_0,$$

and, analogously,

$$\alpha(x_{n+1}, x_n) \geq 1, \; n \in \mathbb{N}_0.$$

Observe that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(fx_n, fx_{n-1}) \\ &\leq \alpha(x_n, x_{n-1}) d(fx_n, fx_{n-1}) \\ &\leq \psi(d(x_n, x_{n-1})), \end{aligned}$$

leads to

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1})) < d(x_n, x_{n-1}), \; n \in \mathbb{N}, \tag{2.2}$$

and

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) \\ &\leq \psi(d(x_{n-1}, x_n)) \end{aligned}$$

gives

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n), \; n \in \mathbb{N}. \tag{2.3}$$

Continuing in the same manner, after $n-1$ more steps, we get

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \text{ and } d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0)), \quad n \in \mathbb{N}. \tag{2.4}$$

By letting $n \to \infty$, $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = \lim\limits_{n\to\infty} d(x_{n+1}, x_n) = 0$.
Let $n, m \in \mathbb{N}$ such that $m > n$. Then,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} \alpha(x_{i-1}, x_i) d(x_i, x_{i+1}) \\ &= \sum_{i=n}^{m-1} \psi^i(d(x_0, x_1)). \end{aligned}$$

If $n, m \to \infty$, we get that

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Likewise,

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0.$$

Hence, the sequence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there is some $x^* \in X$ such that

$$\lim_{n\to\infty} d(x^*, x_n) = \lim_{n\to\infty} d(x_n, x^*) = d(x^*, x^*) = \lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} d(x_m, x_n) = 0. \tag{2.5}$$

Since f is continuous,

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fx^*.$$

□

Example 2.3. Let $X = \{0, 1, 2\}$ and $d : X \times X \mapsto [0, \infty)$ defined with

x \ y	0	1	2
0	0	1	2
1	1	1	2
2	2	3	4

Then (X, d) is a quasi-metric-like space. Define a mapping $f : X \mapsto X$ with

$$f : \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

Let $\alpha : X \times X \mapsto [0, \infty)$ such that

$$\alpha(x, y) = \begin{cases} 0, & x = 1 \, or \, y = 1 \\ 1, & \text{otherwise} \end{cases},$$

and $\psi(t) = \frac{t}{2}$, $t \geq 0$. The mapping f is then (α, ψ)-contractive mapping, but it is not a contraction due to $x = y = 1$. Furthermore, all requirements of Theorem 2.2 are fulfilled, thus f has a unique fixed point in X.

Remark 2.4. Observe that in Example 2.3 f is α-admissible. The same would hold if $f(1) = 2$ and $f(2) = 1$, and it still would not be a contraction. But in case $f(1) = 0$ and $f(2) = 1$, we would get a contractive mapping on a quasi-metric-like space. Obviously, $f(0)$ stays 0, due to Theorem 2.2 because $d(0, 0) = 0$.

Omitting continuity condition in Theorem 2.2 is possible if we introduce notion of α-regularity as presented in [21].

Definition 2.5. ([21]) Quasi-metric-like space (X, d) is α-regular for some $\alpha : X \times X \mapsto [0, \infty)$, if for every sequence $\{x_n\} \subseteq X$ such that $\alpha(x_n, x_{n+1}) \geq 1 (\alpha(x_{n+1}, x_n) \geq 1)$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x \in X$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1 (\alpha(x, x_{n_k}) \geq 1)$, for all $k \in \mathbb{N}$.

Theorem 2.6. *Let (X, d) be a complete quasi-metric-like space and let $f : X \to X$ be an (α, ψ)-contractive mapping. If*

(i) f is α-orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq 1$;
(iii) X is α-regular.

Then f has a fixed point x^* in X and $d(x^*, x^*) = 0$.

Proof. Similarly as in the proof of Theorem 2.2, we define an iterative sequence $\{x_n\}$ which converges to a point $x^* \in X$ such that (2.5) holds. Hence, there exists some subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ and $\alpha(x^*, x_{n_k}) \geq 1$, $k \in \mathbb{N}$. Thus,

$$\begin{aligned} d(x_{n_k+1}, fx^*) &\leq \alpha x_{n_k}, x^*) d(x_{n_k+1}, fx^*) \\ &\leq \psi(d(x_{n_k}, x^*)) \\ &\leq d(x_{n_k}, x^*) \end{aligned}$$

along with

$$d(fx^*, x_{n_k+1}) \leq d(x^*, x_{n_k}), \; k \in \mathbb{N},$$

and (2.5) lead to the conclusion $\lim_{k\to\infty} d(x_{n_k+1}, fx^*) = \lim_{k\to\infty} d(fx^*, x_{n_k+1}) = 0$.
On the other hand, triangle inequality

$$d(x^*, fx^*) \leq d(x^*, x_{n_k+1}) + d(x_{n_k+1}, fx^*), \; k \in \mathbb{N},$$

when $k \to \infty$, implies $d(x^*, fx^*) = 0$, so $fx^* = x^*$. □

Through the following example we will consider uniqueness of a fixed point of a (α,ψ)-contractive mapping on a complete quasi-metric-like space.

Example 2.7. Let (X,d) be the quasi-metric-like space defined in Example 2.3. Also we will use α and ψ defined therein.
If $f : X \mapsto X$ is defined with

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix},$$

then f is α-admissible mapping. Additionally, f is (α,ψ)-contractive mapping. On the other hand, f has two fixed points.

The counterexample indicates, along with previously made comment, that uniqueness of fixed point is related to the absence of the indiscernibility of identicals characteristic for quasi-metric. We notice that we need to add an additional condition to guarantee the uniqueness.

Theorem 2.8. *In addition to Theorem 2.2 (Theorem 2.6) assume that, if $x^* \in X$ is a fixed point obtained as a limit of determined iterative sequence, for all $y \in X$, either $\alpha(x^*,y) \geq 1$ or $\alpha(y,x^*) \geq 1$, then x^* is a unique fixed point of f.*

Proof. Suppose that $z \in X$ is such that $fz = z$.
If, without loss of generality, $\alpha(x^*,z) \geq 1$, then

$$\begin{aligned} d(x^*,z) &= d(fx^*,fz) \\ &\leq \alpha(x^*,z)d(fx^*,fz) \\ &\leq \psi(d(x^*,z)), \end{aligned}$$

If $d(x^*,z) \neq 0$, then $\psi(d(x^*,z)) < d(x^*,z)$ which leads to a contradiction with presented inequality. Therefore, $z = x^*$ and it is a unique fixed point of f. □

Remark 2.9. On several papers studying (α,ψ)-contractions, uniqueness is obtained by adding the condition:

(U) For all $x,y \in \mathrm{Fix}(f)$, either $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$.

where $\mathrm{Fix}(f)$ denotes the set of all fixed points of f. But if we know elements of this set, than we assume knowing its cardinality.
Otherwise, if we assume $\alpha(x,y) \geq 1$, $x,y \in X$, than we lose any impact of α-admissability and we get just ψ-contraction.

Definition 2.10. Let (X,d) be a complete quasi-metric-like space. A mapping $f : X \to X$ is called generalized (α,ψ)-contractive mapping if there exist two functions $\psi \in \Psi_b$ and $\alpha : X \times X \to [0,\infty)$ satisfying the following condition:

$$\alpha(x,y)d(fx,fy) \leq \psi(M(x,y)) \tag{2.6}$$

for all $x,y \in X$, where

$$M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{(x,fy)+d(y,fx)}{2}\right\}. \tag{2.7}$$

Theorem 2.11. *Let (X,d) be a complete quasi-metric-like space and let $f : X \to X$ be a generalized (α,ψ)-contractive mapping. Assume that*

(i) f is α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$ and $\alpha(fx_0,x_0) \geq 1$;

(*iii*) f *is continuous.*

Then f *has a fixed point* x^* *in* X *and* $d(x^*, x^*) = 0$.

Proof. Analogously to the proof of Theorem 2.2, there exists an iterative sequence $x_{n+1} = fx_n$, $n \in \mathbb{N}_0$, where $x_0 \in X$ is chosen with respect to (ii), such that

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ and } \alpha(x_{n+1}, x_n) \geq 1, \text{ for all } n \in \mathbb{N}_0, \tag{2.8}$$

assuming $x_n \neq x_{n+1}$, $n \in \mathbb{N}_0$,, since otherwise we would directly obtain fixed point of f.
Therfore,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) \\ &\leq \psi(M(x_{n-1}, x_n)), \end{aligned}$$

for all $n \in \mathbb{N}$ and

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_n, fx_n), d(x_{n-1}, fx_{n-1}), \tfrac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\right\} \\ &\leq \max\left\{d(x_{n-1}, x_n), d(x_n, fx_n), d(x_{n-1}, fx_{n-1}), \tfrac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\} \\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}. \end{aligned}$$

Since the equality $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ do not hold due to previous assumption $x_n \neq x_{n+1}$, it follows $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, $n \in \mathbb{N}$.
Thus,

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n), \quad \text{for all } n \in \mathbb{N},$$

and

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)), \ n \in \mathbb{N}. \tag{2.9}$$

Analogously, by letting $x = x_n$ and $y = x_{n-1}$ in (2.6), it follows

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \alpha(x_n, x_{n-1}) d(fx_n, fx_{n-1}) \\ &\leq \psi(M(x_n, x_{n-1})), \end{aligned} \tag{2.10}$$

where,

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\left\{d(x_n, x_{n-1}), d(x_n, fx_n), d(x_{n-1}, fx_{n-1}), \tfrac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2}\right\} \\ &\leq \max\left\{d(x_n, x_{n-1}), d(x_n, fx_n), d(x_{n-1}, fx_{n-1}), \tfrac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\} \\ &= \max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right\}. \end{aligned}$$

If $M(x_n, x_{n-1}) = d(x_{n-1}, x_n)$, then, by (2.9) and (2.10),

$$d(x_{n+1}, x_n) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^n(d(x_0, x_1)). \tag{2.11}$$

If $M(x_n, x_{n-1}) = d(x_n, x_{n+1})$, then by

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n+1})).$$

along with (2.9), it follows

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n+1})) < \psi^{n+1}\left(d(x_0, x_1)\right).$$

In the last case, $M(x_n, x_{n-1}) = d(x_n, x_{n-1})$, so

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1})). \tag{2.12}$$

If we denote $\max\{d(x_0,x_1),d(x_1,x_0)\}$ with ω, we get $d(x_{n+1},x_n)\le\psi^n(\omega)$ and $d(x_n,x_{n+1})\le\psi^n(\omega)$, for any $n\in\mathbb{N}$, thus

$$\lim_{n\to\infty} d(x_{n+1},x_n)=\lim_{n\to\infty} d(x_n,x_{n+1})=0.$$

If $n,m\in\mathbb{N}$, $m>n$,

$$\begin{aligned} d(x_n,x_m) &\le \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \\ &\le \sum_{i=n}^{m-1} \psi^i(\omega). \end{aligned}$$

Hence, $\lim\limits_{n,m\to\infty} d(x_n,x_m)=0$ and $\lim\limits_{n,m\to\infty} d(x_m,x_n)=0$. Since, X is a complete space, there exists $x^*\in X$ such that $\lim\limits_{n\to\infty} x_n=x^*$ and

$$\lim_{n\to\infty} d(x^*,x_n)=\lim_{n\to\infty} d(x_n,x^*)=d(x^*,x^*)=0. \tag{2.13}$$

Then $x^*=\lim\limits_{n\to\infty} x_n=\lim\limits_{n\to\infty} fx_{n-1}=fx^*$, because f is continuous, and x^* is a fixed point of f. □

Theorem 2.12. *Let (X,d) be a complete quasi-metric-like space and let $f:X\to X$ be a generalized (α,ψ)-contractive mapping. Assume that*

(i) f is α-orbital admissible;
(ii) there exists $x_0\in X$ such that $\alpha(x_0,fx_0)\ge 1$ and $\alpha(fx_0,x_0)\ge 1$;
(iii) X is α-regular.

Then f has a fixed point x^* in X and $d(x^*,x^*)=0$.

Proof. As in the proof of Theorem 2.11, there is an iterative sequence therein defined such that $\lim\limits_{n\to\infty} x^n=x^*$. Also, $\alpha(x_n,x_{n+1})\ge 1$ and $\alpha(x_{n+1},x_n)\ge 1$, $n\in\mathbb{N}_0$, therefore, there exists some subsequence $\{x_{n_k}\}\subseteq\{x_n\}$ such that $\alpha(x_{n_k},x^*)\ge 1$ and $\alpha(x^*,x_{n_k})\ge 1$.
For arbitrary $\varepsilon>0$, choose $n_{k_0}\in\mathbb{N}$ such that $d(x^*,x_n),d(x_n,x^*),d(x_n,x_m),d(x_m,x_n)<\frac{\varepsilon}{2}$ for any $m>n\ge n_{k_0}$.
Accordingly, for any $k\ge k_0$,

$$\begin{aligned} d(x^*,fx^*) &\le d(x^*,x_{n_k+1})+d(x_{n_k+1},fx^*) \\ &\le \frac{\varepsilon}{2}+\alpha x_{n_k},x^*)d(x_{n_k+1},fx^*) \\ &\le \frac{\varepsilon}{2}+\psi(M(x_{n_k},x^*)), \end{aligned}$$

where

$$\begin{aligned} \psi(M(x_{n_k},x^*)) &= max\left\{d(x_{n_k},x^*),d(x_{n_k},x_{n_k+1}),d(x^*,fx^*),\frac{d(x_{n_k},fx^*)+d(x^*,x_{n_k+1})}{2}\right\} \\ &\le \max\left\{\frac{\varepsilon}{2},d(x^*,fx^*),\frac{d(x_{n_k},x^*)+d(x^*,fx^*)+\varepsilon/2}{2}\right\} \\ &\le \frac{\varepsilon+d(x^*,fx^*)}{2}. \end{aligned}$$

Hence,

$$\begin{aligned} d(x^*,fx^*) &\le \varepsilon+\frac{d(x^*,fx^*)}{2} \\ &\le 2\varepsilon. \end{aligned}$$

Since $\varepsilon>0$ was arbitrary, $d(x^*,fx^*)=0$, so x^* is a fixed point of f. □

Uniqueness issue could be solve as for Theorem 2.2 or Theorem 2.6, respectively, but with stronger assumptions.

Theorem 2.13. *In addition to conditions of Theorem 2.11 (Theorem 2.12) assume that, if $x^* \in X$ is a fixed point obtained as a limit of determined iterative sequence, for all $y \in X$, $\alpha(x^*, y) \geq 1$ or $\alpha(y, x^*) \geq 1$, then x^* is a unique fixed point of f.*

Proof. If $fy = y$, without loss of generality, assume that $d(y, x^*) \geq d(x^*, y)$, then

$$\begin{aligned} d(y, x^*) &\leq \alpha(y, x^*)d(y, x^*) \\ &\leq \psi\left(M(y, x^*)\right) \\ &\leq \max \psi\left(d(y, x^*)\right), \psi\left(\frac{d(y, x^*) + d(x^*, y)}{2}\right) \\ &= \psi\left(d(y, x^*)\right). \end{aligned}$$

Thus, $y = x^*$. On contrary, we would get $d(y, x^*) < d(y, x^*)$. □

Similar result for (α, ψ)-contraction could be formulated on metric-like space endowed with a partial ordering. Thus as a consequence we get Corollary 3.8 and Corollary 3.9 of [11], as well as results of Ran and Reurings regarding contractions on partially ordered metric spaces.

Definition 2.14. Let $(X, \preceq)$ be a partially ordered set. The mapping $f : X \to X$ is nondecreasing with respect to $\preceq$ if for all $x, y \in X$

$$x \preceq y \Longrightarrow fx \preceq fy.$$

Analogously we would define nonincreasing mapping with respect to $\preceq$.

Definition 2.15. Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subseteq X$ is said to be nondecreasing (respectively nonincreasing) with respect to $\preceq$ if $x_n \preceq x_{n+1}$, $n \in \mathbb{N}$ (respectively $x_{n+1} \preceq x_n$, $n \in \mathbb{N}$).

Definition 2.16. Let (X, d) be a metric-like space with a partial ordering $\preceq$. The space $(X, \preceq, d)$ is regular with respect to $\preceq$ if for every nondecreasing (respectively, nonincreasing) sequence $\{x_n\} \subseteq X$ such that $\lim\limits_{n\to\infty} x_n = x \in X$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ (respectively, $x \preceq x_{n_k}$) for all $k \in \mathbb{N}$.

We have the following result.

Corollary 2.17. *Let $(X, \preceq)$ be a partially ordered set (which does not contain an infinite totally unordered subset) and (X, d) be a complete metric-like space. Let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi_b$, such that*

$$d(fx, fy) \leq \psi(d(x, y)),\ x, y \in X, x \preceq y. \tag{2.14}$$

Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$ or $fx_0 \preceq x_0$;

(ii) f is continuous or

$(ii)'$ $(X, \preceq, d)$ is regular.

Then f has a fixed point $x^* \in X$ with $d(x^*, x^*) = 0$.
Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, than f has a unique fixed point.

Proof. Choose $x_0 \in X$ as described in (i) and, without loss of generality, assume that $x_0 \preceq fx_0$. If $x_n = fx_{n-1}$, $n \in \mathbb{N}_0$, then $x_n \preceq x_{n+1}$, $n \in \mathbb{N}_0$. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preceq y \text{ or } x \succeq y, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to obtain that f is α-admissible mapping. Moreover, it is (α, ψ)-contractive mapping, so the existence of fixed point follows from Theorem 2.2 or Theorem 2.6, respectively.
If $fx = x$ and $fy = y$, observe z such that $x \preceq z$ and $y \preceq z$. Then, $x \preceq f^n z$ and $y \preceq f^n z$, $n \in \mathbb{N}$, so

$$\begin{aligned} d(x,y) &\preceq d(x, f^n z) + d(f^n z, y) \\ &\preceq \psi^n(d(x,z)) + \psi^n(d(z,y)), \end{aligned}$$

and $x = y$ that guarantees uniqueness of a fixed point. □

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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