Advances in the Theory of Nonlinear Analysis and its Applications **2** (2018) No. 2, 85–87.
https://doi.org/10.31197/atnaa.431135
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Revisiting the Kannan Type Contractions via Interpolation

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Abstract

In the paper we revisited the well-known fixed point theorem of Kannan under the aspect of interpolation. By using the interpolation notion, we propose a new Kannan type contraction to maximize the rate of convergence.

Keywords: Kannan Type Contraction, Interpolation, fixed point, rate of convergence.
2010 MSC: 47H 10, 49T99, 54H25.

1. Introduction

After the distinguished fixed point of Banach, one of the pivotal metric fixed point result was reported by Kannan [1, 2]. A mapping that satisfies Banach contraction inequality is necessarily continuous. In 1968, Kannan [1] introduced a new type of contraction which is an affirmative answer to the natural question below: Whether there is a discontinuous mapping that fulfils certain contractive conditions and posses a fixed point in the frame of complete metric spaces.

Theorem 1.1. *[1] Let (X, d) be a complete metric spaces and $T : X \to X$ be a Kannan contraction mapping, i.e.,*

$$d(Tx, Ty) \leq \lambda [d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point.

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Received April 20, 2018, Accepted: May 18, 2018, Online: June 08, 2018.

2. Main results

We start our results by the generalization of the definition of Kannan type contraction via interpolation notion, as follows.

Definition 2.1. *Let (X, d) be a metric space. We say that the self-mapping $T : X \to X$ is an interpolative Kannan type contraction, if there exist a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda \left[d(x, Tx)\right]^{\alpha} \cdot \left[d(y, Ty)\right]^{1-\alpha}. \tag{2.1}$$

for all $x, y \in X$ with $x \neq Tx$.

Theorem 2.2. *Let (X, d) be a complete metric space and T be an interpolative Kannan type contraction. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in (X, d)$. We shall set a constructive sequence $\{x_n\}$ by $x_{n+1} = T^n(x_0)$ for all positive integer n. Without loss of generality, we assume that $x_n \neq x_{n+1}$ for each nonnegative integer n. Indeed, if there exist a nonnegative integer n_0 such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then, x_{n_0} forms a fixed point. Thus, we have

$$d(x_n, Tx_n) = d(x_n, x_{n+1}) > 0, \text{ for each nonnegative integer } n.$$

Taking $x = x_n$ and $y = x_{n-1}$ in (2.1), we derive that

$$\begin{aligned} d(x_{n+1}, x_n) = &\ d(Tx_n, Tx_{n-1}) \leq \lambda \left[d(x_n, Tx_n)\right]^{\alpha} \cdot \left[d(x_{n-1}, Tx_{n-1})\right]^{1-\alpha} \\ = &\ \lambda \left[d(x_{n-1}, x_n)\right]^{1-\alpha} \cdot \left[d(x_n, x_{n+1})\right]^{\alpha}, \end{aligned} \tag{2.2}$$

which yields that

$$\left[d(x_n, x_{n+1})\right]^{1-\alpha} \leq \lambda \left[d(x_{n-1}, x_n)\right]^{1-\alpha}. \tag{2.3}$$

Thus, we deduce that the sequence $\{d(x_{n-1}, x_n)\}$ is non-increasing and non-negative. As a result, there is a nonnegative constant L such that $\lim_{n\to\infty} d(x_{n-1}, x_n) = L$. We shall indicate that $L > 0$. Indeed, from (2.3), we derive that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \leq \lambda^n d(x_0, x_1). \tag{2.4}$$

Letting $n \to \infty$ in the inequality above, we observe that $L = 0$.

As a next step, we shall show that the sequence $\{x_n\}$ is Cauchy by using a standard arguments based on the triangle inequality. More precisely, we have

$$\begin{aligned} d(x_n, x_{n+r}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+r-1}, x_{n+r}) \\ &\leq \lambda^n d(x_0, x_1) + \cdots + \lambda^{n+r-1} d(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1) \end{aligned} \tag{2.5}$$

Letting $n \to \infty$ in the inequality above, we find that the sequence $\{x_n\}$ is Cauchy. Since (X, d) is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$.

On what follows we shall show that the limit x of the iterative sequence $\{x_n\}$ forms a fixed point for the given self-mapping T. By substituting $x = x_n$ and $y = x$ in (2.1), we find that

$$d(Tx_n, Tx) \leq \lambda \left[d(x_n, Tx_n)\right]^{\alpha} \cdot \left[d(x, Tx)\right]^{1-\alpha}. \tag{2.6}$$

Taking $n \to \infty$ in the inequality above, we derive that $d(x, Tx) = 0$ that is, $Tx = x$.

For the uniqueness, we shall use the method of *Reductio ad Absurdum*. Suppose, on the contrary that T has a two distinct fixed point $x, y \in X$. Thus, from (2.1) we have

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \leq \lambda \left[d(x, Tx)\right]^{\alpha} \cdot \left[d(y, Ty)\right]^{1-\alpha} \\ &\leq \lambda \left[d(x, x)\right]^{\alpha} \cdot \left[d(y, y)\right]^{1-\alpha} = 0, \end{aligned} \tag{2.7}$$

which yields that $d(x, y) = 0$, a contradiction. Hence, the observed fixed point is unique. □

Example 2.3. *Let $X = \{x, y, z, w\}$ be a set endowed with a metric d such that*

$$\begin{aligned} d(x,x) &= d(y,y) = d(z,z) = d(w,w) = 0, \\ d(y,x) &= d(x,y) = 3, \\ d(z,x) &= d(x,z) = 4, \\ d(y,z) &= d(z,y) = \tfrac{3}{2} \\ d(w,x) &= d(x,w) = \tfrac{5}{2} \\ d(w,y) &= d(y,w) = 2 \\ d(w,z) &= d(z,w) = \tfrac{3}{2}. \end{aligned}$$

We define a self-mapping T on X by $T : \begin{pmatrix} x & y & z & w \\ x & w & x & y \end{pmatrix}$. It is clear that T is not Kannan contraction. Indeed, there is no $\lambda \in [0, \frac{1}{2})$ such that the following inequality is fulfilled:

$$d(Tw, Tz) = d(y, x) = 3 \leq \lambda(d(Tw, w) + d(z, Tz)) = 6\lambda.$$

On the other hand, for $\alpha = \frac{1}{8}$ and $\lambda = \frac{9}{10}$, the self-mapping T forms an interpolative Kannan type contraction and x is the desired unique fixed point of T. Notice that in the setting of interpolative Kannan type contraction, the constant lies between 0 and 1 although in the classical version it is restricted with 1/2.

References

[1] R. Kannan, Some results on fixed points. Bull. Calcutta Math. Soc. 60, 71-76 (1968).
[2] R. Kannan, Some results on fixed points. II. Am. Math. Mon. 76, 405-408 (1969.