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Weak Solutions for a Coupled System of Partial Pettis Hadamard Fractional Integral Equations

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Abstract

In this paper we investigate the existence of weak solutions under the Pettis integrability assumption for a coupled system of partial integral equations via Hadamard's fractional integral, by applying the technique of measure of weak noncompactness and Mönch's fixed point theorem.

Keywords: Functional integral equation coupled system partial Pettis Hadamard fractional integral measure of weak noncompactness weak solution
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1. Introduction

In this paper $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers, respectively the set of real numbers, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ := [0, \infty)$.

The fractional calculus represents a powerful tool in applied mathematics to study many problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [25, 40]. There has been a significant development in fractional differential and integral equations in recent years; see the monographs of Abbas *et al.* [1, 2], Kilbas *et al.* [26], Miller and Ross [28], and the papers of Abbas *et al.* [3], Darwish *et al.* [16, 17, 18, 19, 20, 21], Vityuk *et al.* [41, 42], and the references therein.

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In [14], Butzer *et al.* investigated properties of the Hadamard fractional integral and derivative. In [15], they obtained the Mellin transform of the Hadamard fractional integral and differential operators, and in [36], Pooseh *et al.* obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [37], and the references therein.

The measure of weak noncompactness was introduced by De Blasi [22]. The strong measure of noncompactness was developed first by Banaŝ and Goebel [6] and subsequently developed and used in many papers; see for example, Akhmerov *et al.* [4], Alvàrez [5], Benchohra *et al.* [10, 12], Guo *et al.* [23], Mönch *et al.* [30, 31], Szufla [38], and the references therein. Recently in [7, 8] Benchohra *et al.* used the measure of weak noncompactness for some classes of fractional differential equations and inclusions, while in [9], a class of hyperbolic differential equations involving the Caputo fractional derivative was considered. Some applications of the measure of weak noncompactness to ordinary differential and integral equations in Banach spaces are reported in [11, 27, 33, 39] and the references therein. Some recent results on coupled systems of operator equations in b-metric spaces are given in [34].

This paper deals with the existence of weak solutions to the following coupled system of Hadamard partial fractional integral equations of the form, for $(x, y) \in J$,

$$\begin{cases} u(x,y) = \mu_1(x,y) + \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{f_1(s,t,u(s,t),v(s,t))}{st\Gamma(r_1)\Gamma(r_2)} dtds, \\ \\ v(x,y) = \mu_2(x,y) + \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{\rho_1-1} \left(\ln \frac{y}{t}\right)^{\rho_2-1} \frac{f_2(s,t,u(s,t),v(s,t))}{st\Gamma(\rho_1)\Gamma(\rho_2)} dtds, \end{cases} \tag{1.1}$$

where $J := [1, a] \times [1, b]$, $a, b > 1$, $r_1, r_2, \rho_1, \rho_2 > 0$, $\mu_1, \mu_2 : J \to E$ and $f_1, f_2 : J \times E \times E \to E$ are given continuous functions, $\Gamma(\cdot)$ is the Euler gamma function and E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X.

The present paper initiates the use of the measure of weak noncompactness and Mönch's fixed point theorem to the coupled system (1.1).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Let $C := C(J, E)$ be the Banach space of continuous functions $u : J \to E$ with the norm

$$\|u\|_C = \sup_{(x,y)\in J} \|u(x, y)\|_E.$$

It is clear that the product space $\mathcal{C} := C \times C$ is a Banach space with the norm

$$\|(u, v)\|_{\mathcal{C}} = \|u\|_C + \|v\|_C.$$

Denote by $L^\infty(J, E)$, the Banach space of essentially bounded measurable functions $u : J \to E$ equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : \|u(x, y)\|_E \le c, \ a.e. \ (x, y) \in J\}.$$

Let $(E, w) = (E, \sigma(E, E^*))$ denote the Banach space E with its weak topology.

Definition 2.1. A Banach space X is called weakly compactly generated (WCG, in short) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.2. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E, w) then $h(u_n) \to h(u)$ in (E, w)).

Definition 2.3. [35] The function $u : J \to E$ is said to be Pettis integrable on J if and only if there is an element $u_j \in E$ corresponding to each $j \subset J$ such that $\phi(u_j) = \int\int_j \phi(u(s,t))dtds$ for all $\phi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_j = \int\int_j u(s,t)dtds$).

Let $P(J,E)$ be the space of all E-valued Pettis integrable functions on J, and $L^1(J,\mathbb{R})$, be the Banach space of Lebesgue integrable functions $u : J \to \mathbb{R}$. Define the class $P_1(J,E)$ by

$$P_1(J,E) = \{u \in P(J,E) : \varphi(u) \in L^1(J,\mathbb{R}) \text{ for every } \varphi \in E^*\}.$$

The space $P_1(J,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi\in E^*,\ \|\varphi\|\le 1} \int_1^a \int_1^b |\varphi(u(x,y))|d\lambda(x,y),$$

where λ stands for the Lebesgue measure on J.

The following result is due to Pettis (see [[35], Theorem 3.4 and Corollary 3.41]).

Proposition 2.4. *[35] If $u \in P_1(J,E)$ and h is a measurable and essentially bounded $E-$valued function, then $uh \in P_1([0,a],E)$.*

For all that follows, the sign "$\int$" denotes the Pettis integral.

Let us recall the definitions of Pettis integral and Hadamard integral of fractional order.

Definition 2.5. [24, 26] The left sided mixed Pettis Hadamard integral of order $q > 0$, for a function $g \in P_1([1,a],E)$, is defined as

$$({}^H I_1^r g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln\frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds.$$

Remark 2.6. Let $g \in P_1([1,a],E)$. For every $\varphi \in E^*$, we have

$$\varphi({}^H I_1^r g)(x) = ({}^H I_1^r \varphi g)(x); for\ a.e.\ x \in [1,a].$$

Definition 2.7. Let $r_1,\ r_2 \ge 0,\ \sigma = (1,1)$ and $r = (r_1, r_2)$. For $w \in P_1(J,E)$, define the left sided mixed Pettis Hadamard partial fractional integral of order r by the expression

$$({}^H I_\sigma^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln\frac{x}{s}\right)^{r_1-1} \left(\ln\frac{y}{t}\right)^{r_2-1} \frac{w(s,t)}{st} dtds.$$

Definition 2.8. [22] Let E be a Banach space, Ω_E the bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0,\infty)$ defined by

$$\beta(X) := \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B \Rightarrow \beta(A) \le \beta(B)$,

(b) $\beta(A) = 0 \Leftrightarrow A$ is relatively weakly compact,

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,

(d) $\beta(\overline{A}^{\omega}) = \beta(A)$, ($\overline{A}^{\omega}$ denotes the weak closure of A),

(e) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(f) $\beta(\lambda A) = |\lambda|\beta(A)$,

(g) $\beta(conv(A)) = \beta(A)$,

(h) $\beta(\cup_{|\lambda|\leq h}\lambda A) = h\beta(A)$.

The next result follows directly from the Hahn-Banach theorem.

Proposition 2.9. *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : J \to E$ let us denote by

$$V(x, y) = \{v(x, y) : v \in V\};\ (x, y) \in J,$$

and

$$V(J) = \{v(x, y) : v \in V,\ (x, y) \in J\}.$$

Lemma 2.10. *[23] Let $H \subset C$ be a bounded and equicontinuous. Then the function $(x, y) \to \beta(H(x, y))$ is continuous on J, and*

$$\beta_C(H) = \max_{(x,y)\in J} \beta(H(x, y)),$$

and

$$\beta\left(\int\int_J u(s,t)dtds\right) \leq \int\int_J \beta(H(s,t))dtds,$$

where $H(s, t) = \{u(s, t) : u \in H,\ (s, t) \in J\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

For our purposes, we will need the following fixed point theorem:

Theorem 2.11. *[32] Let Q be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(J, E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \quad \textit{is relatively weakly compact}, \tag{2.1}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3. Existence Results

Let us start by defining what we mean by a solution of the integral equation (1.1).

Definition 3.1. A pair $(u, v) \in \mathcal{C}$ is said to be a solution of (1.1) if (u, v) satisfies equation (1.1) on J.

Further, we present conditions for the existence of a solution of equation (1.1).

Theorem 3.2. *Assume that the following hypotheses hold:*

(H_1) *For a.e. $(x, y) \in J$, the functions $u \to f_i(x, y, u, v)$, $v \to f_i(x, y, u, v)$, $i = 1, 2$, are weakly sequentially continuous,*

(H_2) *For a.e. $u, v \in E$, the functions $(x, y) \to f_i(x, y, u, v)$; $i = 1, 2$ are Pettis integrable a.e. on J,*

(H_3) *There exist functions $P_i \in C(J,[0,\infty))$; $i=1,2$ such that for all $\varphi \in E^*$, we have*

$$|\varphi(f_i(x,y,u,v))| \leq \frac{P_i(x,y)\|\varphi\|}{1+\|\varphi\|+\|u\|_E+\|v\|_E}, for\ a.e.\ (x,y)\in J, and\ u,v\in E,$$

(H_4) *For each bounded set $B\subset E$ and for each $(x,y)\in J$, we have*

$$\beta(f_i(x,y,B)) \leq P_i(x,y)\beta(B);\ i=1,2.$$

If

$$L := \frac{P_1^*(\ln a)^{r_1}(\ln b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} + \frac{P_2^*(\ln a)^{\rho_1}(\ln b)^{\rho_2}}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)} < 1, \tag{3.1}$$

where $P_i^* = \|P_i\|_{L^\infty}$; $i=1,2$, then the coupled system (1.1) has at least one solution defined on J.

Proof. Define the operators $N_i : C\to C$; $i=1,2$ by

$$\begin{aligned}(N_1u)(x,y) &= \mu_1(x,y)\\ &+\int_1^x\int_1^y \left(\ln\frac{x}{s}\right)^{r_1-1}\left(\ln\frac{y}{t}\right)^{r_2-1}\frac{f_1(s,t,u(s,t),v(s,t))}{s\,t\Gamma(r_1)\Gamma(r_2)}dtds,\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(N_2v)(x,y) &= \mu_2(x,y)\\ &+\int_1^x\int_1^y \left(\ln\frac{x}{s}\right)^{\rho_1-1}\left(\ln\frac{y}{t}\right)^{\rho_2-1}\frac{f_2(s,t,u(s,t),v(s,t))}{s\,t\Gamma(\rho_1)\Gamma(\rho_2)}dtds.\end{aligned} \tag{3.3}$$

Consider the continuous operator $N:\mathcal{C}\to\mathcal{C}$ defined by

$$(N(u,v))(x,y) = ((N_1u)(x,y),(N_2v)(x,y)). \tag{3.4}$$

First notice that, the hypothesis (H_2) implies that

$$\forall u,v\in C,\ f(\cdot,\cdot,u(\cdot,\cdot),v(\cdot,\cdot))\in P(J,E).$$

From (H_3) we have that for all $(x,y)\in J$, the functions

$$\left(\ln\frac{x}{s}\right)^{r_1-1}\left(\ln\frac{y}{t}\right)^{r_2-1}\frac{f_1(s,t,u(s,t),v(s,t))}{st}$$

and

$$\left(\ln\frac{x}{s}\right)^{\rho_1-1}\left(\ln\frac{y}{t}\right)^{\rho_2-1}\frac{f_2(s,t,u(s,t),v(s,t))}{st}$$

are Pettis integrable and thus, the operator N makes sense.

Let $R,R_i>0$; $i=1,2$ be such that

$$R_1 > \|\mu_1\|_C + \frac{P_1^*(\ln a)^{r_1}(\ln b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)},\ R_2 > \|\mu_2\|_C + \frac{P_2^*(\ln a)^{\rho_1}(\ln b)^{\rho_2}}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)},$$

$$\text{and } R = R_1+R_2,$$

and consider the set

$$\begin{aligned} Q &= \{(u,v) \in \mathcal{C} : \|(u,v)\|_{\mathcal{C}} \leq R \text{ and } \|(u,v)(x_1,y_1) - (u,v)(x_2,y_2)\|_E \\ &\leq \|\mu_1(x_1,y_1) - \mu_1(x_2,y_2)\|_E + \|\mu_2(x_1,y_1) - \mu_2(x_2,y_2)\|_E \\ &\quad + \frac{P_1^*}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &\quad \times [2(\ln y_2)^{r_2}(\ln x_2 - \ln x_1)^{r_1} + 2(\ln x_2)^{r_1}(\ln y_2 - \ln y_1)^{r_2} \\ &\quad + (\ln x_1)^{r_1}(\ln y_1)^{r_2} - (\ln x_2)^{r_1}(\ln y_2)^{r_2} \\ &\quad - 2(\ln x_2 - \ln x_1)^{r_1}(\ln y_2 - \ln y_1)^{r_2}] \\ &\quad + \frac{P_2^*}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)} \\ &\quad \times [2(\ln y_2)^{\rho_2}(\ln x_2 - \ln x_1)^{\rho_1} + 2(\ln x_2)^{\rho_1}(\ln y_2 - \ln y_1)^{\rho_2} \\ &\quad\quad + (\ln x_1)^{\rho_1}(\ln y_1)^{\rho_2} - (\ln x_2)^{\rho_1}(\ln y_2)^{\rho_2} \\ &\quad - 2(\ln x_2 - \ln x_1)^{\rho_1}(\ln y_2 - \ln y_1)^{\rho_2}]\}. \end{aligned}$$

Clearly, the subset Q is closed, convex and equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 2.11. The proof will be given in several steps.

Step 1. N maps Q into itself.
Let $u, v \in Q$, $(x,y) \in J$ and assume that $(N_i u)(x,y) \neq 0;\ i = 1,2$. Then there exists $\phi_i \in E^*;\ i = 1,2$ such that $\|(N_i u)(x,y)\|_E = \phi_i((Nu)(x,y))$. Thus

$$\begin{aligned} &\|(N_1 u)(x,y)\|_E \\ &= \phi_1\left(\mu_1(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \right. \\ &\quad \left. \times \frac{f_1(s,t,u(s,t),v(s,t))}{st} dtds\right) \\ &= \phi_1(\mu_1(x,y)) + \phi_1\left(\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \right. \\ &\quad \left. \times \frac{f_1(s,t,u(s,t),v(s,t))}{st} dtds\right) \\ &\leq \|\mu_1(x,y)\|_E + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{P_1(s,t)}{st} dtds \\ &\leq \|\mu_1\|_C + \frac{P_1^*(\ln a)^{r_1}(\ln b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &\leq R_1. \end{aligned}$$

Also, we get

$$\begin{aligned}
&\|(N_2v)(x,y)\|_E \\
&= \phi_2\left(\mu_2(x,y) + \frac{1}{\Gamma(\rho_1)\Gamma(\rho_2)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\rho_1-1}\left(\ln\frac{y}{t}\right)^{\rho_2-1}\right.\\
&\quad\left.\times\frac{f_2(s,t,u(s,t),v(s,t))}{st}dtds\right)\\
&= \phi_2(\mu_2(x,y)) + \phi_2\left(\frac{1}{\Gamma(\rho_1)\Gamma(\rho_2)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\rho_1-1}\left(\ln\frac{y}{t}\right)^{\rho_2-1}\right.\\
&\quad\left.\times\frac{f_2(s,t,u(s,t),v(s,t))}{st}dtds\right)\\
&\leq \|\mu_2(x,y)\|_E + \frac{1}{\Gamma(\rho_1)\Gamma(\rho_2)}\int_1^x\int_1^y\left(\ln\frac{x}{s}\right)^{\rho_1-1}\left(\ln\frac{y}{t}\right)^{\rho_2-1}\frac{P_2(s,t)}{st}dtds\\
&\leq \|\mu_2\|_C + \frac{P_2^*(\ln a)^{\rho_1}(\ln b)^{\rho_2}}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)}\\
&\leq R_2.
\end{aligned}$$

Thus,

$$\|(N(u,v))(x,y)\|_E \leq R_1 + R_2 = R.$$

Next, let $(x_1,y_1),(x_2,y_2)\in J$ be such that $x_1<x_2$ and $y_1<y_2$, and let $u,v\in Q$, with $(N_1u)(x_1,y_1)-(N_1u)(x_2,y_2)\neq 0$ and $(N_2v)(x_1,y_1)-(N_2v)(x_2,y_2)\neq 0$. Then there exists $\phi_i\in E^*;\ i=1,2$ with $\|\varphi_i\|=1$ such that

$$\|(N_1u)(x_1,y_1)-(N_1u)(x_2,y_2)\|_E = \phi_1((N_1u)(x_1,y_1)-(N_1u)(x_2,y_2))$$

and

$$\|(N_2v)(x_1,y_1)-(N_2v)(x_2,y_2)\|_E = \phi_1((N_2v)(x_1,y_1)-(N_2v)(x_2,y_2)).$$

Then

$$\begin{aligned}
&\|(N_1u)(x_2,y_2)-(N_1u)(x_1,y_1)\|_E\\
&= \phi_1((N_1u)(x_2,y_2)-(N_1u)(x_1,y_1))\\
&\leq \|\mu_1(x_1,y_1)-\mu_1(x_2,y_2)\|_E\\
&\quad+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_1}\int_1^{y_1}\left[\left|\ln\tfrac{x_2}{s}\right|^{r_1-1}\left|\ln\tfrac{y_2}{t}\right|^{r_2-1}-\left|\ln\tfrac{x_1}{s}\right|^{r_1-1}\left|\ln\tfrac{y_1}{t}\right|^{r_2-1}\right]\\
&\quad\times\tfrac{|\phi_1(f_1(s,t,u(s,t),v(s,t)))|}{st}dtds\\
&\quad+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\ln\tfrac{x_2}{s}\right|^{r_1-1}\left|\ln\tfrac{y_2}{t}\right|^{r_2-1}\tfrac{|\phi_1(f_1(s,t,u(s,t),v(s,t)))|}{st}dtds\\
&\quad+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_1}\int_{y_1}^{y_2}\left|\ln\tfrac{x_2}{s}\right|^{r_1-1}\left|\ln\tfrac{y_2}{t}\right|^{r_2-1}\tfrac{|\phi_1(f_1(s,t,u(s,t),v(s,t)))|}{st}dtds\\
&\quad+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_1^{y_1}\left|\ln\tfrac{x_2}{s}\right|^{r_1-1}\left|\ln\tfrac{y_2}{t}\right|^{r_2-1}\tfrac{|\phi_1(f(s,t,u(s,t),v(s,t)))|}{st}dtds.
\end{aligned}$$

This gives

$$\begin{aligned}
&\|(N_1u)(x_2,y_2)-(N_1u)(x_1,y_1)\|_E \leq \|\mu_1(x_1,y_1)-\mu_1(x_2,y_2)\|_E\\
&\quad+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_1}\int_1^{y_1}\left[\left|\ln\tfrac{x_2}{s}\right|^{r_1-1}\left|\ln\tfrac{y_2}{t}\right|^{r_2-1}\right.\\
&\quad\left.-\left|\ln\tfrac{x_1}{s}\right|^{r_1-1}\left|\ln\tfrac{y_1}{t}\right|^{r_2-1}\right]\tfrac{P_1^*}{st}dtds\\
&\quad+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\ln\tfrac{x_2}{s}\right|^{r_1-1}\left|\ln\tfrac{y_2}{t}\right|^{r_2-1}\tfrac{P_1^*}{st}dtds\\
&\quad+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_1}\int_{y_1}^{y_2}\left|\ln\tfrac{x_2}{s}\right|^{r_1-1}\left|\ln\tfrac{y_2}{t}\right|^{r_2-1}\tfrac{P_1^*}{st}dtds\\
&\quad+\tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_1^{y_1}\left|\ln\tfrac{x_2}{s}\right|^{r_1-1}\left|\ln\tfrac{y_2}{t}\right|^{r_2-1}\tfrac{P_1^*}{st}dtds\\
&\leq \|\mu_1(x_1,y_1)-\mu_1(x_2,y_2)\|_E\\
&\quad+\tfrac{P_1^*}{\Gamma(1+r_1)\Gamma(1+r_2)}[2(\ln y_2)^{r_2}(\ln x_2-\ln x_1)^{r_1}+2(\ln x_2)^{r_1}(\ln y_2-\ln y_1)^{r_2}\\
&\quad+(\ln x_1)^{r_1}(\ln y_1)^{r_2}-(\ln x_2)^{r_1}(\ln y_2)^{r_2}\\
&\quad-2(\ln x_2-\ln x_1)^{r_1}(\ln y_2-\ln y_1)^{r_2}].
\end{aligned}$$

Also,

$$\begin{aligned}&\|(N_2v)(x_2,y_2)-(N_2v)(x_1,y_1)\|_E \leq \|\mu_2(x_1,y_1)-\mu_2(x_2,y_2)\|_E\\&+\tfrac{P_2^*}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)}[2(\ln y_2)^{\rho_2}(\ln x_2-\ln x_1)^{\rho_1}+2(\ln x_2)^{\rho_1}(\ln y_2-\ln y_1)^{\rho_2}\\&+(\ln x_1)^{\rho_1}(\ln y_1)^{\rho_2}-(\ln x_2)^{\rho_1}(\ln y_2)^{\rho_2}\\&-2(\ln x_2-\ln x_1)^{\rho_1}(\ln y_2-\ln y_1)^{\rho_2}].\end{aligned}$$

Thus,

$$\begin{aligned}&\|(N(u,v))(x_2,y_2)-(N(u,v))(x_1,y_1)\|_E\\&\leq \|\mu_1(x_1,y_1)-\mu_1(x_2,y_2)\|_E+\|\mu_2(x_1,y_1)-\mu_2(x_2,y_2)\|_E\\&+\tfrac{P_1^*}{\Gamma(1+r_1)\Gamma(1+r_2)}[2(\ln y_2)^{r_2}(\ln x_2-\ln x_1)^{r_1}+2(\ln x_2)^{r_1}(\ln y_2-\ln y_1)^{r_2}\\&+(\ln x_1)^{r_1}(\ln y_1)^{r_2}-(\ln x_2)^{r_1}(\ln y_2)^{r_2}\\&-2(\ln x_2-\ln x_1)^{r_1}(\ln y_2-\ln y_1)^{r_2}]\\&+\tfrac{P_2^*}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)}[2(\ln y_2)^{\rho_2}(\ln x_2-\ln x_1)^{\rho_1}+2(\ln x_2)^{\rho_1}(\ln y_2-\ln y_1)^{\rho_2}\\&+(\ln x_1)^{\rho_1}(\ln y_1)^{\rho_2}-(\ln x_2)^{\rho_1}(\ln y_2)^{\rho_2}\\&-2(\ln x_2-\ln x_1)^{\rho_1}(\ln y_2-\ln y_1)^{\rho_2}].\end{aligned}$$

Hence $N(Q)\subset Q$.

Step 2. N is weakly-sequentially continuous.
Let (u_n,v_n) be a sequence in Q and let $(u_n(x,y))\to u(x,y)$ and $(v_n(x,y))\to v(x,y)$ in (E,ω) for each $(x,y)\in J$. Fix $(x,y)\in J$. Since $f_i;\ i=1,2$ satisfy the assumption (H_1), then for each $i\in\{1,2\}$ the function $f_i(x,y,u_n(x,y),\ v_n(x,y))$ converges weakly uniformly to $f_i(x,y,u(x,y),v(x,y))$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies that for each $(x,y)\in J$, the sequence $(N_1u_n)(x,y)$ converges weakly uniformly to $(N_1u)(x,y)$ in (E,ω), and $(N_2v_n)(x,y)$ converges weakly uniformly to $(N_2v)(x,y)$ in (E,ω). So $N(u_n)\to N(u)$. Then $N:Q\to Q$ is weakly-sequentially continuous.

Step 3. The implication (2.1) holds.
Let V be a subset of Q such that $\overline{V}=\overline{\mathrm{conv}}(N(V)\cup\{0\})$. Obviously

$$V(x,y)\subset\overline{\mathrm{conv}}(NV)(x,y))\cup\{0\}),\ (x,y)\in J.$$

Further, as V is bounded and equicontinuous, by Lemma 3 in [13] the function $(x,y)\to v(x,y)=\beta(V(x,y))$ is continuous on J. Since the functions $\mu_i;\ i=1,2$ are continuous on J, the set $\{\mu(x,y);\ (x,y)\in J\}\subset E$ is compact. From (H_3), Lemma 2.10 and the properties of the measure β, for any $(x,y)\in J$, we have

$$\begin{aligned}v(x,y)&\leq\beta((NV)(x,y)\cup\{0\})\\&\leq\beta((NV)(x,y))\\&\leq\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left|\ln\frac{x}{s}\right|^{r_1-1}\left|\ln\frac{y}{t}\right|^{r_2-1}\frac{P_1(s,t)\beta(V(s,t))}{st}dtds\\&+\frac{1}{\Gamma(\rho_1)\Gamma(\rho_2)}\int_1^x\int_1^y\left|\ln\frac{x}{s}\right|^{\rho_1-1}\left|\ln\frac{y}{t}\right|^{\rho_2-1}\frac{P_2(s,t)\beta(V(s,t))}{st}dtds\\&\leq\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left|\ln\frac{x}{s}\right|^{r_1-1}\left|\ln\frac{y}{t}\right|^{r_2-1}\frac{P_1(s,t)v(s,t)}{st}dtds\\&+\frac{1}{\Gamma(\rho_1)\Gamma(\rho_2)}\int_1^x\int_1^y\left|\ln\frac{x}{s}\right|^{\rho_1-1}\left|\ln\frac{y}{t}\right|^{\rho_2-1}\frac{P_2(s,t)v(s,t)}{st}dtds\\&\leq\frac{\|v\|_{\mathcal{C}}}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left|\ln\frac{x}{s}\right|^{r_1-1}\left|\ln\frac{y}{t}\right|^{r_2-1}\frac{P_1(s,t)}{st}dtds\\&+\frac{\|v\|_{\mathcal{C}}}{\Gamma(\rho_1)\Gamma(\rho_2)}\int_1^x\int_1^y\left|\ln\frac{x}{s}\right|^{\rho_1-1}\left|\ln\frac{y}{t}\right|^{\rho_2-1}\frac{P_2(s,t)}{st}dtds\\&\leq\left(\frac{P_1^*(\ln a)^{r_1}(\ln b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}+\frac{P_2^*(\ln a)^{\rho_1}(\ln b)^{\rho_2}}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)}\right)\|v\|_{\mathcal{C}}.\end{aligned}$$

Thus

$$\|v\| \leq L\|v\|_{\mathcal{C}}.$$

From (3.1), we get $\|v\|_{\mathcal{C}} = 0$, that is, $v(x,y) = \beta(V(x,y)) = 0$, for each $(x,y) \in J$ and then by Theorem 2 in [29], V is weakly relatively compact in $\mathcal{C}$. Applying now Theorem 2.11, we conclude that N has a fixed point which is a solution of the coupled system (1.1). □

4. An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots),\ \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

We consider the following coupled system of partial Pettis Hadamard integral equations, for $(x,y) \in [1,e]^2$,

$$\begin{cases} u_n(x,y) = \mu_1(x,y) + \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{f_n(s,t,u(s,t),v(s,t))}{st\Gamma(r_1)\Gamma(r_2)} dtds, \\ \\ v_n(x,y) = \mu_2(x,y) + \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{\rho_1-1} \left(\ln \frac{y}{t}\right)^{\rho_2-1} \frac{g_n(s,t,u(s,t),v(s,t))}{st\Gamma(\rho_1)\Gamma(\rho_2)} dtds, \end{cases} \tag{4.1}$$

where $r_1, r_2, \rho_1, \rho_2 > 0$, $\mu_1(x,y) = x + y^2$, $\mu_2(x,y) = x^2 + y$,

$$f_n(x,y,u(x,y),v(x,y)) = \frac{cxy^2}{1 + \|u(x,y)\|_E + \|v(x,y)\|_E} \left(e^{-7} + \frac{1}{e^{x+y+5}}\right) u_n(x,y)$$

and

$$g_n(x,y,u(x,y),v(x,y)) = \frac{2cx^{2y^{-6}}}{1 + \|u(x,y)\|_E + \|v(x,y)\|_E} v_n(x,y),$$

with

$$u = (u_1, u_2, \ldots, u_n, \ldots),\ v = (v_1, v_2, \ldots, v_n, \ldots),$$

and

$$c := \frac{e^4}{8}\Gamma(1+r_1)\Gamma(1+r_2).$$

Set

$$f = (f_1, f_2, \ldots, f_n, \ldots),\ g = (g_1, g_2, \ldots, g_n, \ldots).$$

Clearly, the functions f and g are continuous.
For each $u, v \in E$ and $(x,y) \in [1,e] \times [1,e]$, we have

$$\|f(x,y,u(x,y),v(x,y))\|_E \leq cxy^2 \left(e^{-7} + \frac{1}{e^{x+y+5}}\right).$$

and

$$\|g(x,y,u(x,y),v(x,y))\|_E \leq cx^2y^{-6}.$$

Hence, the hypothesis (H_3) is satisfied with $P_1^* = P_2^* = 2ce^{-4}$. We shall show that condition (3.1) holds with $a = b = e$. Indeed,

$$\begin{aligned} \frac{P_1^*(\ln a)^{r_1}(\ln b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} + \frac{P_2^*(\ln a)^{\rho_1}(\ln b)^{\rho_2}}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)} &= \frac{2c}{e^4\Gamma(1+r_1)\Gamma(1+r_2)} \\ &+ \frac{2c}{e^4\Gamma(1+\rho_1)\Gamma(1+\rho_2)} \\ &= \frac{1}{2} < 1. \end{aligned}$$

A simple computation shows that all conditions of Theorem 3.2 are satisfied. It follows that the coupled system (4.1) has at least one solution on $[1, e] \times [1, e]$.

5. Declaration of interest

The authors declare that they have no actual nor potential conflicts of interest.

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