Advances in the Theory of Nonlinear Analysis and its Applications **1** (2017) No. 1, 48–56.
https://doi.org/10.31197/atnaa.379113
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Study the topology of Branciari metric space via the structure proposed by Császár

Dong Zhang[a]

[a] *LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, China.*

Abstract

In this paper, we topologically study the generalized metric space proposed by Branciari [3] via the weak structure proposed by Császár [9, 10], and compare convergent sequences in several different senses. We also introduce the concepts of available points and unavailable points on such structures. Besides, we define the continuous function on structures and investigate further characterizations of continuous functions.

Keywords: Branciari metric space, generalized topology, structure
2010 MSC: 54A05, 54C60.

1. Introduction and Preliminaries

Branciari [3] introduced the concept of a generalized metric space where the triangle inequality is replaced by a rectangular inequality. Many authors studied the fixed point theory on such generalized metric space (cf. [1, 2, 3, 4, 5, 6, 7]). Recall the notion of Branciari metric space.

Definition 1.1. *[3] For a nonempty set X, let $d : X \times X \longrightarrow [0, \infty]$ be a map such that for any $x, y \in X$ and distinct $u, v \in X \setminus \{x, y\}$,*

$$
\begin{array}{ll}
(BMS1) & d(x,y) = 0 \text{ if and only if } x = y, \\
(BMS2) & d(x,y) = d(y,x), \\
(BMS3) & d(x,y) \le d(x,u) + d(u,v) + d(v,y).
\end{array}
$$

Email addresses: `dongzhang@pku.edu.edu` (Dong Zhang), `13699289001@163.com` (Dong Zhang)

Received April 30, 2017, Accepted: June 30, 2017, Online: July 07, 2017.

The map d is called a Branciari metric, and the pair (X, d) is called a Branciari metric space, abbreviated as BMS. The open ball and closed ball are defined respectively by

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\},\ B(x, \varepsilon] = \{y \in X : d(x, y) \leq \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in (X, d) is convergent to x if $d(x_n, x) \to 0$ as $n \to \infty$.

The story in this paper starts from the topology of BMS. In contrast to the metric space, the topology of BMS,

$$\mathcal{T} = \{S \subset X : \forall x \in S, \exists r > 0 \text{ s.t. } B(x, r) \subset S\} \cup \{\varnothing\},$$

is difficult to describe. In the topology space $(X, \mathcal{T})$, an open ball may not be open. Furthermore, a terrible fact is that $x_n \xrightarrow{\mathcal{T}} x$ (i.e., x_n converges to x with respect to the topology $\mathcal{T}$) can not guarantee that $x_n \to x$, i.e., $d(x_n, x) \to 0$ (see Example 2.2 for details).

To remedy this problem, an alternative way is to define a new topology $\widetilde{\mathcal{T}}$ generated by all open balls (as subbase). In this topology, the above problems are solved, that is, every open ball is open, and $x_n \xrightarrow{\widetilde{\mathcal{T}}} x$ implies $x_n \to x$.

However, a new phenomenon arises: $x_n \to x$ can not guarantee $x_n \xrightarrow{\widetilde{\mathcal{T}}} x$ (see Example 2.2).

In some sense, the topology equivalent to convergent sequences with respect to d has no equivalence relation with open balls. How can we directly study the convergent sequence $x_n \to x$ from topological view?

One way to overcome all the difficults is adopting the generalized topology proposed by Császár [9], which removes the intersection property of finite number of open sets. Let $\mathcal{T}' = \{\cup_{B \in \mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B}\}$, where $\mathcal{B} = \{B(x, r) : x \in X, r > 0\}$. Then $\mathcal{T}'$ is a generalized topology on X which contains all the open balls as its generalized topological base. With the aid of the generalized topology, we show an easy way to study the convergent sequence $x_n \to x$ using topological method in Sections 2 and 3.

The generalized topology was extended to weak structure by Császár [10], in which some families of sets (like $\beta(\omega)$, $\rho(\omega)$, $\sigma(\omega)$, $\pi(\omega)$, $\alpha(\omega)$) play very fundamental roles. There have been some further results about these families of sets, such as [11], [12]. In Section 4, we introduce the available points and unavailable points on structures (mentioned by Császár in the introduction of [10]), and define the interior points, accumulation points, isolated points of a set. With the help of these points, we define the interior operator and closure operator, which are equivalent to the corresponding concepts defined by Császár. We also establish the Kuratowski 7-sets theorem and some other results on structures. A main contribution is to characterize the continuity on structures, where Theorems 5.7 and 5.11 are commendatory results in Section 5.

2. Convergent sequences with respect to d, $\mathcal{T}$ and $\widetilde{\mathcal{T}}$

Theorem 2.1. *Let (X, d) be a BMS. Then we have:*

$$x_n \xrightarrow{\widetilde{\mathcal{T}}} x \ \Rightarrow\ x_n \to x \ \Rightarrow\ x_n \xrightarrow{\mathcal{T}} x.$$

The converse is false, i.e., $x_n \xrightarrow{\widetilde{\mathcal{T}}} x \ \not\Leftarrow\ x_n \to x \ \not\Leftarrow\ x_n \xrightarrow{\mathcal{T}} x$.

Proof. Suppose $x_n \xrightarrow{\widetilde{\mathcal{T}}} x$. Then for any $U \in \widetilde{\mathcal{T}}$ with $x \in U$, there exists $N > 0$ such that $x_n \in U$ for any $n > N$. Taking $U = B(x, \epsilon)$, we have $d(x_n, x) < \epsilon$ for $n > N$, which deduces that $x_n \to x$.

Assume $x_n \to x$. For any $V \in \mathcal{T}$ with $x \in V$, there exists $B(x, \epsilon) \subset V$. So, there is $N > 0$ such that $x_n \in B(x, \epsilon) \subset V$ for $n > N$. Accordingly, $x_n \xrightarrow{\mathcal{T}} x$.

See Example 2.2 for the counter-example of the converse. □

Example 2.2. *Let $X = [0,1]$ and let $d : [0,1] \times [0,1] \to [0,+\infty)$ be a symmetric function defined by*

$$d(y,x) = d(x,y) = \begin{cases} |x-y|, & \text{if } x \in [0,1] \cap \mathbb{Q} \text{ and } y \in [0,1] \setminus \mathbb{Q}, \\ 1, & \text{if } x \neq y,\ x,y \in [0,1] \cap \mathbb{Q} \text{ or } x,y \in [0,1] \setminus \mathbb{Q}, \\ 0, & \text{if } x = y. \end{cases}$$

It can be easily verified that (X,d) is a BMS.

We will prove that $\mathcal{T}$ is the standard Euclidean topology on $[0,1]$, and $\widetilde{\mathcal{T}}$ is the discrete topology on $[0,1]$. For $0 < r < 1$, keep

$$B(x,r) = \begin{cases} \{y \in [0,1] \setminus \mathbb{Q} : |y-x| < r\} \cup \{x\}, & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ \{y \in [0,1] \cap \mathbb{Q} : |y-x| < r\} \cup \{x\}, & \text{if } x \in [0,1] \setminus \mathbb{Q}, \end{cases}$$

in mind.

For any $U \in \mathcal{T} \setminus \{\varnothing\}$ and $x \in U$, there exists $r > 0$ such that $B(x,r) \subset U$. Without loss of generality, we may assume $x \in \mathbb{Q} \cap [0,1]$. Then $\{y \in [0,1] \setminus \mathbb{Q} : |y-x| < r\} \subset U$, i.e., $(x-r, x+r) \cap [0,1] \setminus \mathbb{Q} \subset U$. Thus, for any $y \in (x-r,x+r) \cap [0,1] \setminus \mathbb{Q}$, there exists $r_y \leq r - |x-y|$ such that $\{z \in [0,1] \cap \mathbb{Q} : |z-y| < r_y\} \subset U$, i.e., $(y-r_y, y+r_y) \cap [0,1] \cap \mathbb{Q} \subset U$. Therefore,

$$\bigcup_{y \in (x-r,x+r)\cap[0,1]\setminus\mathbb{Q}} (y-r_y, y+r_y) \cap [0,1] \cap \mathbb{Q} \subset U,$$

i.e., $(x-r,x+r) \cap [0,1] \cap \mathbb{Q} \subset U$. Together with $(x-r,x+r) \cap [0,1] \setminus \mathbb{Q} \subset U$, we obtain $(x-r,x+r) \cap [0,1] \subset U$. On the other hand, for any $y \in (x-r,x+r) \cap [0,1]$, let $r' = r - |x-y| > 0$. Then $B(y,r') \subset (x-r,x+r) \cap [0,1]$, which implies that $(x-r,x+r) \cap [0,1] \in \mathcal{T}$. So, $\{(x-r,x+r) \cap [0,1] : x \in [0,1], r > 0\}$ forms a topological base of $\mathcal{T}$. This means that $\mathcal{T}$ is the Euclidean topology on $[0,1]$.

Since $B(x,r) \in \widetilde{\mathcal{T}}$, $\forall x \in [0,1]$ and $r > 0$, we have $B(x,r) \cap B(y,r') \in \widetilde{\mathcal{T}}$, $\forall x,y \in [0,1]$, $\forall r,r' > 0$. For $x \in [0,1] \cap \mathbb{Q}$ and $y \in [0,1] \setminus \mathbb{Q}$,

$$B(x,r) \cap B(y,r') = \begin{cases} \varnothing, & \text{if } r,r' \leq |x-y|, \\ \{x\}, & \text{if } r \leq |x-y| < r', \\ \{y\}, & \text{if } r' \leq |x-y| < r, \\ \{x,y\}, & \text{if } |x-y| < r,r'. \end{cases}$$

Hence, $\{x\}, \{y\} \in \widetilde{\mathcal{T}}$. This deduces that every singleton set is an open set, which means that $\widetilde{\mathcal{T}}$ is a discrete topology.

Since $|\frac{1}{n} - 0| \to 0$ and $d(\frac{1}{n}, 0) = 1 \not\to 0$, we have $\frac{1}{n} \xrightarrow{\mathcal{T}} 0$ and $\frac{1}{n} \not\to 0$.

Since $d(\frac{\sqrt{2}}{n}, 0) = \frac{\sqrt{2}}{n} \to 0$ and $\{0\}$ is an open set in $\widetilde{\mathcal{T}}$, one has $\frac{\sqrt{2}}{n} \to 0$ and $\frac{\sqrt{2}}{n} \overset{\widetilde{\mathcal{T}}}{\not\to} 0$.

3. The second countability and the separability on BMS

We call a Branciari metric space a **strong separable** space if there exists a countable subset A of X such that for any $x \in X$, there is a Cauchy sequence $\{x_n\} \subset A$ with different terms such that $x_n \to x$, $n \to +\infty$ unless x is a isolated point in A. Here A is said to be a **strong dense** set.

To describe the second countability on generalized metric space, we replace 'topology' by 'generalized topology', in which every generalized open set is defined to be the union of a family of balls $B(x,r)$ in X.

We call $\mathcal{U}$ a generalized topological base of X, if any generalized open set V can be written as the union of some generalized open sets from $\mathcal{U}$.

A BMS is said to be a **generalized second countable** BMS if there is a generalized topological base with countable members.

Note that $\rho(x,y) := \inf_{z\in X} d(x,z) + d(z,y) \le d(x,y)$ and $\rho(x,y) \le \rho(x,z) + \rho(z,y)$, for any $x,y,z \in X$ (see [6]). For describing more properties, we introduce the (K) condition as follows:
(K) There is $k \in (0,1)$ such that $\rho(x,y) \ge kd(x,y)$ for any $x,y \in X$.

Theorem 3.1. *Let (X,d) be a Branciari metric space. We have the following results.*

(1) *If X is strong separable, then it must be a generalized second countable space.*
(2) *If X is generalized second countable, then X is separable.*
(3) *If X is generalized second countable with the condition (K), then X is strong separable.*

Proof. (1) Let $A \subset X$ be a countable strong dense subset, and $\mathcal{B} = \{B(a,q) : a \in A,\ q \in \mathbb{Q}^+\}$. We will show that each $B(x,r)$ can be covered by some elements in $\mathcal{B}$, where $r \in \mathbb{R}^+$ and $x \in X$.

If $y \in B(x,r) \cap A$ is an isolated point of A, then there exists $r' \in \mathbb{Q}^+$ such that $B(y,r') \cap A = \{y\}$. If $B(y,r') \setminus A \neq \varnothing$, then for any $z \in B(y,r') \setminus A$, there exists a Cauchy sequence $\{z_n\} \subset A$ satisfying $z_n \to z$ with $z_n \neq z$, $z_n \neq z_m$, $n \neq m$. So $d(y,z_n) \le d(y,z) + d(z,z_m) + d(z_m,z_n) \to d(y,z) < r'$ as $n,m \to +\infty$. That is, $z_n \in B(y,r') \cap A$ for sufficiently large n, which contradicts with $B(y,r') \cap A = \{y\}$. Consequently, $B(y,r') \setminus A = \varnothing$, and thus $B(y,r') = \{y\} \subset B(x,r)$.

Next, we assume $y \in B(x,r)$ is not an isolated point of A, Let δ be a positive rational number with $\delta \le r - d(x,y)$. Since A is strong dense in X, there exists a Cauchy sequence $\{x_n\}$ in A converging to y with $x_n \neq y$ and $x_n \neq x_m$ for any $n \neq m$. Thus, there is some $N \in \mathbb{N}$ such that $d(x_n,y) < \frac{\delta}{2}$ and $d(x_n,x_m) < \frac{\delta}{2}$ for all $n,m > N$. Let m be a natural number with $m > N$. Now for each $z \in B(x_m,\frac{\delta}{2})$, we show that $d(z,x) < r$.

Case I: $z \neq x_m$.

$$\begin{aligned} d(z,x) &\le d(z,x_m) + d(x_m,y) + d(y,x) \\ &< \frac{\delta}{2} + \frac{\delta}{2} + d(y,x) = r. \end{aligned}$$

Case II: $z = x_m$.

$$\begin{aligned} d(x_m,x) &\le d(x_m,x_n) + d(x_n,y) + d(y,x) \\ &< \frac{\delta}{2} + \frac{\delta}{2} + d(y,x) = r. \end{aligned}$$

This proves $y \in B(x_m,\frac{\delta}{2}) \subset B(x,r)$, and hence $\mathcal{B}$ is a countable base for X. Therefore, X is generalized second countable.

(2) Let $\mathcal{U} = \{U_1, U_2, \cdots\}$ be a topological base. Take $x_n \in U_n$, and let $A = \{x_n : n \in \mathbb{N}\}$. Now we show that A is a countable dense set in X. In fact, for any $x \in X$, and $m \in \mathbb{N}^+$, there is a U_{n_m} contained in $B(x,\frac{1}{m})$, so $x_{n_m} \in B(x,\frac{1}{m})$, i.e., $\lim\limits_{m\to+\infty} x_{n_m} = x$.

(3) We only need to show that, under the condition (K), $x_n \to x$ implies that $\{x_n\}$ is Cauchy. Indeed,

$$\begin{aligned} d(x_n,x_m) &\le \frac{1}{k}\rho(x_n,x_m) \le \frac{1}{k}(\rho(x_n,x) + \rho(x_m,x)) \\ &\le \frac{1}{k}(d(x_n,x) + d(x_m,x)) \to 0, n,m \to +\infty. \end{aligned}$$

□

An analogous result of Theorem 3.1 on partial metric space was provided in [8].

4. Structures

Let X be a nonempty set and A be a subset of X. We denote by A^c the complement of A.

Definition 4.1. *Let X be a nonempty set and let $\mathcal{S}$ be a nonempty family of subsets of X, then $\mathcal{S}$ is called a structure on X. The elements of $\mathcal{S}$ are called open sets and the complements of open sets are called closed sets.*

Definition 4.2. *For $x \in X$, $\mathcal{S}_x := \{u \in \mathcal{S} : x \in u\}$ is said to be the open neighbourhood system of x.*

Definition 4.3. *We call points in $X - \bigcup\limits_{u\in\mathcal{S}} u$ the unavailable points and in $\bigcup\limits_{u\in\mathcal{S}} u$ the available points, denoted by $\mathrm{UK}(X)$ and $\mathrm{K}(X)$, respectively.*

Proposition 4.4. *For $x \in X$, $x \in \mathrm{K}(X)$ if and only if $\mathcal{S}_x$ is nonempty.*

Proof. Clearly, by Definitions 4.2 and 4.3, it is easy to see that $\mathcal{S}_x \neq \varnothing$ if and only if $x \in \bigcup\limits_{u\in\mathcal{S}} u = \mathrm{K}(X)$. □

Definition 4.5. *For $x \in A \subset X$, x is called an interior point of A if there exists $u \in \mathcal{S}_x$ such that $u \subset A$. The interior of A is the union of all interior points of A, denoted by $i(A)$. If A has no interior points, we denote $i(A) = \varnothing$.*

Similar to the Lemma 2.2 in [10], we immediately get that $i(A)$ is the union of all open sets contained in A.

Definition 4.6. *For $x \in \mathrm{K}(X)$ and $A \subset X$, we call x an accumulation point of A if $\forall u \in \mathcal{S}_x$, $u \bigcap A - \{x\} \neq \varnothing$. We call x an isolated point of A if $\exists u \in \mathcal{S}_x$, $u \bigcap A = \{x\}$.*

Definition 4.7. *The derived set of A is the union of all accumulation points of A, denoted by $d(A)$. The closure of A is the union of all unavailable points of X, all accumulation points and all isolated points of A, denoted by $c(A)$.*

We simply use iA, dA and cA instead of $i(A)$, $d(A)$ and $c(A)$, respectively.

Remark 4.8. *It follows from Definition 4.7 that $cA = dA \bigcup A \bigcup \mathrm{UK}(X)$.*

Proposition 4.9. *(1) $cA = \{x : \forall u \in \mathcal{S}_x,\ u \bigcap A \neq \varnothing\} \bigcup \mathrm{UK}(X)$.*

(2) If A is a closed set, then $A = cA$. If A is an open set, then $A = iA$.

(3) If the union of any subfamily of $\mathcal{S}$ always belongs to $\mathcal{S}$, then A is closed iff $A = cA$ and A is open iff $A = iA$.

Proof. (1) It follows directly from Remark 4.8.

(2) We only show that A is closed $\Rightarrow A = cA$. Suppose that $cA - A \neq \varnothing$, then we pick $x \in cA - A$. Note that $A^c \in \mathcal{S}_x$, but $A^c \bigcap A = \varnothing$, which is a contradiction, so $cA = A$.

(3) We only need to show $\forall A \subset X,\ iA \in \mathcal{S}$.

For any $x \in iA$, there exists $u \in \mathcal{S}_x$ such that $u \subset A$. If $u - iA \neq \varnothing$, then picking $y \in u - iA$, we have $y \in A \bigcap (iA)^c$. So, $\forall v \in \mathcal{S}_y$, $v \not\subset A$. Note that $u \in \mathcal{S}_y$, so $u \not\subset A$. This is a contradiction.

Hence, we get $u - iA = \varnothing$, which means that $\forall x \in iA$, $\exists u_x \in \mathcal{S}_x$ such that $x \in u_x \subset iA$. Then $iA = \bigcup\limits_{x\in iA} u_x$, where $u_x \in \mathcal{S}$. Thus $iA \in \mathcal{S}$. □

Remark 4.10. *Proposition 4.9 (1) is an equivalent definition of cA.*

If $\mathcal{S}$ is closed under arbitrary union, then iA is the maximal open set contained in A.

Proposition 4.11. *(1) $c\varnothing = \mathrm{UK}(X)$.*

(2) $A \subset cA$.

(3) $cA = ccA$.

(4) $A \subset B \Rightarrow cA \subset cB$. If $\mathcal{S}$ is closed under finite intersection, then $cA \bigcup cB = c(A \bigcup B)$.

Proof. (1) Suppose $c\varnothing \neq \mathrm{UK}(X)$. Then $\forall x \in c\varnothing - \mathrm{UK}(X), \forall u \in \mathcal{S}_x$, $u \bigcap \varnothing \neq \varnothing$, which is a contradiction. Consequently, $c\varnothing = \mathrm{UK}(X)$.

(2) It follows directly from Proposition 4.9 (1).

(3) $\forall x \in ccA$, $\forall u \in \mathcal{S}_x$, $u \bigcap cA \neq \varnothing$. Take $y \in u \bigcap cA$. Then $u \in \mathcal{S}_y$ and thus $u \bigcap A \neq \varnothing$. It follows that $x \in cA$. Accordingly, $ccA \subset cA$ and combining with (2), we get $cA = ccA$.

(4) $\forall x \in cA$, $\forall u \in \mathcal{S}_x$, $u \bigcap A \neq \varnothing$. Thus $u \bigcap B \neq \varnothing$, and then $x \in cB$, i.e., $cA \subset cB$. In consequence, $cA \bigcup cB \subset c(A \bigcup B)$.

Now assume that $\mathcal{S}$ is closed under finite intersection. Suppose $c(A \bigcup B) - cA \bigcup cB \neq \varnothing$. Let $x \in c(A \bigcup B) - cA \bigcup cB$. Then there exist $u_A, u_B \in \mathcal{S}_x$ such that $u_A \bigcap A = \varnothing$, $u_B \bigcap B = \varnothing$. Let $u = u_A \bigcap u_B \in \mathcal{S}_x$. Then $u \bigcap A = \varnothing$, $u \bigcap B = \varnothing$, and thus $u \bigcap (A \bigcup B) = (u \bigcap A) \bigcup (u \bigcap B) = \varnothing$.

This is a contradiction with $x \in c(A \bigcup B)$. Therefore, $cA \bigcup cB = c(A \bigcup B)$. □

Similarly, we have:

Proposition 4.12. *(1) $i\varnothing = \varnothing$. (2) $iA \subset A$. (3) $iA = iiA$.*
(4) $A \subset B \Rightarrow iA \subset iB$. If $\mathcal{S}$ is closed under finite intersection, then $iA \bigcap iB = i(A \bigcap B)$.

Proposition 4.13. *(1) $d\varnothing = \varnothing$.*
(2) $ddA \subset A \bigcup dA$.
(3) $A \subset B \Rightarrow dA \subset dB$. If $\mathcal{S}$ is closed under finite intersection, then $dA \bigcup dB = d(A \bigcup B)$.

Next we show the relations among these operators, $i(\cdot)$, $c(\cdot)$ and $d(\cdot)$.

Proposition 4.14. *(1) $(cA^c)^c = iA$, $(iA^c)^c = cA$.*
(2) If $x \in dA$, then $c(A - \{x\}) = cA$.
(3) $dA = \{x \in \mathrm{K}(X) : x \in c(A - \{x\})\}$.

Proof. (1) It has been shown in Theorem 2.1 [10].

(2) We only need to show $cA \subset c(A - \{x\})$. Assume there exists $y \in cA - c(A - \{x\}) \subset \mathrm{K}(X)$. Then let $u \in \mathcal{S}_y$ such that $u \bigcap (A - \{x\}) = \varnothing$. It follows that $u \bigcap A \subset \{x\}$, $u \bigcap A \neq \varnothing$, i.e., $u \bigcap A = \{x\}$. It is easy to see that $u \in \mathcal{S}_x$. Then by $x \in d(A)$, we get $u \bigcap A - \{x\} \neq \varnothing$, which is a contradiction.

(3) $\forall x \in d(A)$, $x \in cA = c(A - \{x\})$, so $d(A) \subset \{x \in \mathrm{K}(X) : x \in c(A - \{x\})\}$. On the other hand, if $x \in \mathrm{K}(X) \bigcap c(A - \{x\})$, then $x \in d(A - \{x\}) \bigcup (A - \{x\})$, i.e., $x \in d(A - \{x\}) \subset d(A)$. Hence $\{x \in \mathrm{K}(X) : x \in c(A - \{x\})\} \subset d(A)$. □

Inspired by Proposition 4.14 (3), we can define a dual concept of derived set.

Definition 4.15. *$eA = \{x \in X : x \in i(A \bigcup \{x\})\}$ is called the dual derived set of A.*

From Proposition 4.14 (1), we know that c and i are dual operators. Moreover, the following result concludes that d and e are also dual operators (relative to $\mathrm{K}(X)$).

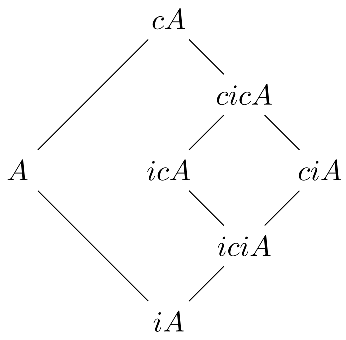
Proposition 4.16. *$eA = d(A^c)^c \cap \mathrm{K}(X)$ and $dA = e(A^c)^c \cap \mathrm{K}(X)$ hold for any subset A of X.*

Proof. We only need to prove $eA = d(A^c)^c \cap \mathrm{K}(X)$. For any $x \in e(A)$, we have $x \in \mathrm{K}(X)$ and $x \in i(A \bigcup \{x\})$. By Proposition 4.14 (1) we obtain $c(A^c - \{x\}) = c((A \bigcup \{x\})^c) = (i(A \bigcup \{x\}))^c$. Hence, $x \notin c(A^c - \{x\})$, that is, $x \notin d(A^c)$. Thus $e(A) \subset d(A^c)^c \cap \mathrm{K}(X)$.

On the other hand, for any $x \in d(A^c)^c \cap \mathrm{K}(X)$, we have $x \in \mathrm{K}(X)$ but $x \notin c(A^c - \{x\}) = (i(A \bigcup \{x\}))^c$, i.e., $x \in i(A \bigcup \{x\})$. So, $x \in eA$ and then $d(A^c)^c \cap \mathrm{K}(X) \subset eA$. □

The following theorem is a counterpart of Kuratowski 7-sets theorem.

Theorem 4.17. *Let $A \subset X$. The number of distinct sets which can be obtained from A by successively taking c and i (in any order) is at most 7. The inclusion relations of the 7 sets are $iA \subset A \subset cA$ and $iA \subset iciA \subset ciA \cap icA \subset ciA \cup icA \subset cicA \subset cA$, which can be written as a Hasse diagram as follows:*



Proof. It is easy to check the result by Theorem 2.1 and Proposition 2.6 in [10]. ☐

As a supplement of Theorem 2.1(c) in [12], we have:

Proposition 4.18. *Let A be a subset of X. Then the following statements are equivalent:*
(1) $cA = cicA$.
(2) For any open set u satisfying $u \bigcap A \neq \varnothing$, we have $u \bigcap icA \neq \varnothing$.

Proof. $\Leftarrow$: $\forall x \in cA \bigcap \mathrm{K}(X)$, $\forall u \in \mathcal{S}_x$, $u \bigcap A \neq \varnothing$. So $u \bigcap icA \neq \varnothing$ and hence $x \in cicA$. Therefore, $cA \subset cicA$. Since $cicA \subset cA$ (by Theorem 4.17), we have $cA = cicA$.

$\Rightarrow$: If there exists V such that $V \bigcap A \neq \varnothing$ and $V \bigcap icA = \varnothing$, then $icA \subset V^c$. Note that V^c is closed. So $cicA \subset V^c$, i.e., $cicA \bigcap V = \varnothing$. Since $cA \bigcap V \neq \varnothing$, there exists $x \in cA$ such that $x \notin cicA$. ☐

5. Continuous map, open map and closed map

Definition 5.1. *Let $x \in \mathrm{K}(X)$, $f(x) \in \mathrm{K}(Y)$. $f : X \to Y$ is said to be continuous at x if for all $v \in \mathcal{S}_{f(x)}$, there exists $u \in \mathcal{S}_x$ such that $f(u) \subset v$. We call f a continuous map, if it is continuous at every point in $\mathrm{K}(X) \bigcap f^{-1}(\mathrm{K}(Y))$.*

To get more properties of continuous mapping, we introduce the following concepts.

Definition 5.2. *In $(X, \mathcal{S})$, let $A \subset X$. We call A a generalized closed set if $cA = A$. We call A a generalized open set if $A = iA$.*

Definition 5.3. *Let $\mathcal{S}^\sim = \{iA : A \subset X\}$ and let $i_{\mathcal{S}^\sim} A$ denote the interior of A in $(X, \mathcal{S}^\sim)$. Similarly, the open neighborhood system of x in $(X, \mathcal{S}^\sim)$ is denoted by $\mathcal{S}^\sim_x$.*

Proposition 5.4. *$\mathcal{S}^\sim = \{A : A \subset iA\} = \{A : A = iA\}$ is a set of all generalized open sets in X.*

Proposition 5.5. *$iA = i_{\mathcal{S}^\sim} A$ is open in $(X, \mathcal{S}^\sim)$.*

Proof. We first show that if $\mathcal{S} \subset \mathcal{B}$, then $i_{\mathcal{S}} A \subset i_{\mathcal{B}} A$. Without loss of generality, we assume $iA \neq \varnothing$. Then $\forall x \in iA$, $\exists u \in \mathcal{S}_x \subset i_{\mathcal{B}} A$ and $u \subset A$. So x is an interior point of A in $(X, \mathcal{B})$. Hence $iA \subset i_{\mathcal{B}} A$.

Now we prove that if $\mathcal{B} = \mathcal{S}^\sim$, then $i_{\mathcal{S}^\sim} A \subset iA$. Assume that $i_{\mathcal{S}^\sim} A - iA \neq \varnothing$, then we take $x \in i_{\mathcal{S}^\sim} A - iA$. There exists $u \in \mathcal{S}^\sim$ such that $x \in u \subset A$. Take $v \subset X$ satisfying $u = iv$. By $x \in iv$, there exists $w \in \mathcal{S}$ such that $x \in w \subset iv = u \subset A$. So x is an interior point of A in $(X, \mathcal{S})$, and then $x \in iA$, which is a contradiction. Thus $i_{\mathcal{S}^\sim} A = iA$. $\forall A \subset X$, $i_{\mathcal{S}^\sim} A = iA \in \mathcal{S}^\sim$. So $i_{\mathcal{S}^\sim} A$ is open in $(X, \mathcal{S}^\sim)$. ☐

Proposition 5.6. *$\varnothing \in \mathcal{S}^\sim$ and $\mathcal{S}^\sim$ is closed under arbitrary union.*

Proof. Since $i\varnothing = \varnothing$, we have $\varnothing \in \mathcal{S}^\sim$. For any $\mathcal{B} \subset \mathcal{S}^\sim$ and $\forall A \in \mathcal{B}$,

$$A \subset \bigcup_{B \in \mathcal{B}} B \Rightarrow iA \subset i\left(\bigcup_{B \in \mathcal{B}} B\right) \Rightarrow \bigcup_{A \in \mathcal{B}} iA \subset i\left(\bigcup_{B \in \mathcal{B}} B\right).$$

By $A \in \mathcal{S}^\sim \Leftrightarrow A = iA$, we get $\bigcup_{A \in \mathcal{B}} A \subset i(\bigcup_{A \in \mathcal{B}} A) \subset \bigcup_{A \in \mathcal{B}} A$. So $i(\bigcup_{A \in \mathcal{B}} A) = \bigcup_{A \in \mathcal{B}} A$, and then $\bigcup_{A \in \mathcal{B}} A \in \mathcal{S}^\sim$. ☐

With the aid of operators, i and c, we can study specific structures. For a subset $A \subset X$, let $A \in \alpha(\mathcal{S})$ iff $A \subset iciA$; $A \in \sigma(\mathcal{S})$ iff $A \subset ciA$; $A \in \pi(\mathcal{S})$ iff $A \subset icA$; $A \in \beta(\mathcal{S})$ iff $A \subset cicA$; $A \in \rho(\mathcal{S})$ iff $A \subset ciA \cup icA$.

By the counterparts of Theorems 3.1 and 3.2 [10] on structures, and Theorem 4.17 and Proposition 5.6 in the present paper, we immediately get:

Theorem 5.7. *$\mathcal{S}^{\sim}$, $\alpha(\mathcal{S})$, $\sigma(\mathcal{S})$, $\pi(\mathcal{S})$, $\rho(\mathcal{S})$ and $\beta(\mathcal{S})$ are generalized topologies on X and they satisfy:*

$$\mathcal{S} \subset \mathcal{S}^{\sim} \subset \alpha(\mathcal{S}) \subset \sigma(\mathcal{S}) \cap \pi(\mathcal{S}) \subset \sigma(\mathcal{S}) \cup \pi(\mathcal{S}) \subset \rho(\mathcal{S}) \subset \beta(\mathcal{S}).$$

Proposition 5.8. *Assume $x \in \mathrm{K}(X)$ and $f(x) \in \mathrm{K}(Y)$. Then f is continuous at x if and only if for each $v \in \mathcal{S}_{f(x)}$, there exists $iu \in \mathcal{S}_x^{\sim}$ such that $f(iu) \subset v$.*

The following statements are equivalent: (1) f is continuous.

(2) The preimage of every open set is generalized open.

(3) The preimage of every generalized open set is generalized open.

(4) $f^{-1}(iB) \subset if^{-1}(B)$.

Proof. If $f(u) \subset v$, then $f(iu) \subset v$. On the other hand, since $x \in iu$, there exists $w \in \mathcal{S}_x$ such that $w \subset iu$. So $f(w) \subset v$.

(1) $\Rightarrow$ (2): For any open set v, we will prove that $f^{-1}(v)$ is generalized open. For all $x \in f^{-1}(v)$, since f is continous at x, there exists $u \in \mathcal{S}_x$ such that $f(u) \subset v$, which implies $u \subset f^{-1}(v)$. Therefore x is an interior point of $f^{-1}(v)$. This shows that $f^{-1}(v)$ is generalized open.

(2) $\Rightarrow$ (1): Assume that for all $x \in X$, $v \in \mathcal{S}_{f(x)}$. Since $f^{-1}(v)$ is generalized open, there exists $u \in \mathcal{S}_x$ such that $u \subset f^{-1}(v)$, i.e., $f(u) \subset v$. This implies that f is continuous.

(3) $\Rightarrow$ (2): Since open sets are generalized open, it is trivial.

(2) $\Rightarrow$ (3): For any generalized open set $iB \subset Y$, iB can be written as $iB = \bigcup v_i$, where v_i is open. Since $f^{-1}(iB) = \bigcup f^{-1}(v_i)$, and $f^{-1}(v_i)$ is generalized open for any i, we deduce that $f^{-1}(iB)$ is generalized open.

(4) $\Rightarrow$ (3): Let B be a generalized open set. Then $f^{-1}(B) = f^{-1}(iB) \subset if^{-1}(B)$. Hence $f^{-1}(B) = if^{-1}(B)$, and thus $f^{-1}(B)$ is generalized open.

(3) $\Rightarrow$ (4): Note that $f^{-1}(iB)$ is generalized open. So $f^{-1}(iB) = if^{-1}(iB) \subset if^{-1}(B)$. $\square$

Proposition 5.5, 5.6 and 5.8 indicate that we can assume $\mathcal{S}$ is closed under arbitrary union if we only concentrate on continuity and interior. That is, in some sense, the generalized topology is enough.

Definition 5.9. *We say that $f : X \to Y$ is open, if for any open set $u \subset X$, $f(u)$ is generalized open.*

We say that $f : X \to Y$ is closed, if for any closed set $A \subset X$, $f(A)$ is generalized closed.

Theorem 5.10. *Let $f : X \to Y$ be a map. Then we have:*

(1) $\forall A \subset X$, $cf(A) \subset f(cA) \Leftrightarrow f$ is closed.

(2) $\forall A \subset X$, $f(cA) \subset cf(A) \Leftrightarrow f$ is continuous.

(3) $\forall A \subset X$, $f(iA) \subset if(A) \Leftrightarrow f$ is open.

(4) $\forall B \subset Y$, $cf^{-1}(B) \subset f^{-1}(cB) \Leftrightarrow f$ is continuous.

(5) $\forall B \subset Y$, $f^{-1}(cB) \subset cf^{-1}(B) \Leftrightarrow f$ is open.

(6) $\forall B \subset Y$, $if^{-1}(B) \subset f^{-1}(iB) \Leftrightarrow f$ is open.

(7) $\forall B \subset Y$, $f^{-1}(iB) \subset if^{-1}(B) \Leftrightarrow f$ is continuous.

Proof. Since the proofs are standard and similar, we only show (3) and (7).

(3). $\Rightarrow$: For any open set $V \subset X$, $f(V) = f(iV) \subset if(V) \subset f(V)$. So $f(V) = if(V)$, i.e., $f(V)$ is open. Thus, f is open.

$\Leftarrow$: If f is open, then $\forall A \subset X$, $f(iA) = if(iA) \subset if(A)$.

(7). $\Leftarrow$: If f is continuous, then $\forall B \subset Y$, $f^{-1}(iB)$ is generalized open. Note that $f^{-1}(iB) \subset f^{-1}(B)$. Thus $f^{-1}(iB) \subset if^{-1}(B)$.

$\Rightarrow$: For any open set $V \subset Y$, $f^{-1}(V) = f^{-1}(iV) \subset if^{-1}(V) \subset f^{-1}(V)$. So $f^{-1}(V) = if^{-1}(V)$, i.e., $f^{-1}(V)$ is generalized open. In consequence, f is continuous. $\square$

Theorem 5.11. *Let $f : X \to Y$ be a surjection. Assume $\forall A \subset X$, $if(A) \subset f(iA)$. Then f is continuous.*

Proof. Consider the set

$$H = \{h : f(h(y)) = y, \forall y \in Y, \text{ where } h : Y \to X\}.$$

Clearly, H is nonempty since $f : X \to Y$ is a surjection. For any open set $B \subset Y$, $f^{-1}(B) = \bigcup_{h \in H} h(B)$, we only need to prove that $h(B)$ is generalized open. Accroding to $if(h(B)) \subset f(ih(B)) \subset f(h(B))$ and $f(h(B)) = B = iB$, we get $f(ih(B)) = f(h(B))$. Since $f|_{h(B)}$ is an injection, we have $h(B) = ih(B)$ and thus $h(B)$ is generalized open. Therefore, $f^{-1}(B) = \bigcup_{h \in H} h(B)$ is generalized open. It follows from Proposition 5.8 (2) that f is continuous. □

Remark 5.12. *(1) The conditions of Theorem 5.11 are all necessary. In fact, if we remove the condition that f is a surjection, then Theorem 5.11 is false. Two examples are shown in Examples 5.13 and 5.14.*

(2) The converse of Theorem 5.11 is not true, that is, $if(A) \subset f(iA)$ is not always true when f is a continuous surjection (see Example 5.15).

Example 5.13. *Let $X = \{1, 2\}$, $T_X = \{\varnothing, \{1\}, X\}$, $Y = \{1, 2, 3\}$ and $T_Y = \{\varnothing, \{1\}, \{2, 3\}, Y\}$. Suppose $f : X \to Y$ satisfying $f(1) = 1$ and $f(2) = 2$. Then f is an injection.*

Note that $f^{-1}(\{2, 3\}) = \{2\}$ is not open, which means that f is not continuous.

Since $if(1) = \{1\} = f(i\{1\})$, $if(2) = \{2\}^o = \varnothing \subset f(i\{2\})$ and $if(\{1, 2\}) = i\{1, 2\} = \{1\} \subset \{1, 2\} = f(i\{1, 2\})$, we get that f satisfies $if(A) \subset f(iA), \forall A \subset X$.

Example 5.14. *Let $f(x) = \begin{cases} 0, & -1 \le x < 0, \\ 1, & 0 \le x \le 1. \end{cases}$ Then $f : [-1, 1] \to \mathbb{R}$ is not continuous. Note that $\forall A \subset [-1, 1]$, $f(A) \subset \{0, 1\}$. So $if(A) \subset i\{0, 1\} = \varnothing \subset f(iA)$.*

Example 5.15. *Let $X = \{1, 2\}$, $T_X = \{\varnothing, \{1\}, X\}$, $Y = \{1\}$ and $T_Y = \{\varnothing, Y\}$. Set $f : X \to Y$ with $f(1) = 1$ and $f(2) = 1$.*

Note that $f^{-1}(1) = \{1, 2\} = X$ is open, which deduces that f is continuous. Since $if(2) = i\{1\} = \{1\} \not\subset \varnothing = f(\varnothing) = f(i\{2\})$, $if(A) \subset f(iA)$ fails to hold.

References

[1] H. Aydi, E. Karapınar, B. Samet, *Fixed points for generalized (α, ψ)-contractions on generalized metric spaces*, J. Ineq. Appl., **2014**(2014), Article ID 229.

[2] H. Aydi, E. Karapınar, and D. Zhang, On common fixed points in the context of Brianciari metric spaces, Results in Mathematics, **71** (2017), 73–92.

[3] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen., **57** (2000), 31-37.

[4] N. Bilgili, E. Karapınar, *A note on "common fixed points for (ψ, α, β)-weakly contractive mappings in generalized metric spaces"*, Fixed Point Theory Appl., **2013** (2013), Article ID 287.

[5] W.A. Kirk, N. Shahzad, *Generalized metrics and Caristi's theorem*, Fixed Point Theory Appl., **2013** (2013), Article ID 129.

[6] C. Li and D.Zhang, On generalized metric spaces and generalized convex contractions, Fixed point theory (accepted)

[7] Selma Gulyaz Ozyurt, On some α-admissible contraction mappings on Branciari b-metric spaces, Adv. Theory Nonlinear Anal. Appl. (2017) 1–13.

[8] Suzhen Han, Jianfeng Wu, Dong Zhang, Properties and principles on partial metric spaces, Topology Appl., 230 (2017), 77–98.

[9] A. Csaszar, Generalized topology, generized continuity, Acta Math. Hungar. (2002), 351–357.

[10] A. Császár, Weak structures, Acta Math. Hungar., 131 (2011), 193–195, doi: 10.1007/s10474-010-0020-z.

[11] E. Ekici, On weak structures due to Császár, Acta Math. Hungar., 134 (2012), 565–570, doi:10.1007/s10474-011-0145-8.

[12] M. Navaneethakrishnan and S. Thamaraiselvi, On weak structures of Császár, Acta Math. Hungar., 137 (3) (2012), 224–229, DOI: 10.1007/s10474-012-0218-3

Dong Zhang
LMAM and School of Mathematical Sciences, Peking University,
Beijing 100871, China.
e-mail: dongzhang@pku.edu.edu & 13699289001@163.com