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A new Generalization of Wardowski Fixed Point Theorem in Complete Metric Spaces

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Abstract

The aim of this paper is to state and prove Wardowski type fixed point theorem in metric spaces. The paper includes an example which shows that our result is a proper extension of some known results.

Keywords: Wardowski type contraction, fixed point, metric space
2010 MSC: 54A05, 54C60.

1. Introduction and Preliminaries

Starting from one of the fundamental results of fixed point theory known as the Banach contraction principle [5], several authors proved many interesting extensions and generalizations ([1]-[4], [6]-[18]).

In 2012, D. Wardowski [14], using functions $F : \mathbb{R}_+ \to \mathbb{R}$ proved a fixed point theorem concerning a new type of contractions, called $F-$contractions.

Let function $F : \mathbb{R}_+ \to \mathbb{R}$ such that:

$(F1)$ F is strictly increasing, that is, for all $x, y \in \mathbb{R}_+$ if $x < y$ then $F(x) < F(y)$;

$(F2)$ For each sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \left(\alpha^k F(\alpha)\right) = 0$

We denote by $\mathcal{F}$ the family of all that functions.

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Definition 1.1. *[14] Let (X,d) be a metric space. A map $T : X \to X$ is said to be an $F-$contraction on (X,d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)) \tag{1}$$

Theorem 1.2. *[14] Let (X,d) be a complete metric space and $T : X \to X$ be an $F-$contraction. Then T has a unique fixed point x^* and for all $x \in X$ the sequence $\{T^n x\}$ is convergent to x^*.*

Remark 1.3. *From $(F1)$ and (1) it follows that*

$$\begin{aligned} F(d(Tx,Ty)) &\leq F(d(x,y)) - \tau < F(d(x,y)) \Rightarrow \\ &\Rightarrow d(Tx,Ty) < d(x,y) \end{aligned}$$

for all $x, y \in X$ such that $Tx \neq Ty$. Also, T is a continuous operator.

Afterwards, Wardowski and Van Dung [15] have introduced the notion of a $F-$weak contraction, in this way.

Definition 1.4. *[15] Let (X,d) be a metric space. A map $T : X \to X$ is said to be a $F-$weak contraction on (X,d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $d(Tx,Ty) > 0$, the following holds:*

$$\tau + F\left(d(Tx,Ty)\right) \leq F\left(M(x,y)\right) \tag{2}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}.$$

By using this notion, Wardowski and Van Dung [15] have demonstrated a fixed point theorem which generalizes the theorem 1.2 as follows.

Theorem 1.5. *[15] Let (X,d) be a complete metric space and $T : X \to X$ be a $F-$weak contraction. If T or F is continous, then T has a unique fixed point x^* and for all $x \in X$ the sequence $\{T^n x\}$ is convergent to x^*.*

Latter, Piri and Kumam [12] introduced a large class of functions by replacing the condition $(F3)$ in the definition of $F-$contraction with the following

$(F3')$ F is continous on $(0, \infty)$

and they denote the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfies the conditions $(F1)$, $(F2)$, and $(F3')$ by $\mathfrak{F}$.

With this assumptions, Piri and Kumam [12] proved the next fixed point theorem.

Theorem 1.6. *[12].Let (X,d) be a complete metric space and a mapping $T : X \to X$. Suppose there exists $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x, y \in X$*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F\left(d(Tx,Ty)\right) \leq F(d(x,y)).$$

Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to x^*.

In this paper, using the ideea from **[10],** we introduce a new type of $F-$contraction, and prove a fixed point theorem which generalizes some known results.

2. Main results

First, let $\mathcal{F}_E$ denote the familly of all functions $F:\mathbb{R}_+\to\mathbb{R}$ which satisfies the following conditions:
(F_E1) F is strictly increasing, that is, for all $x,y\in\mathbb{R}_+$, if $x<y$ then $F(x)<F(y)$;
(F_E2) There exists $\tau>0$ such that $\tau+\lim_{t\to t_0}\inf F(t)>\lim_{t\to t_0}\sup F(t)$, for every $t_0>0$.

Definition 2.1. *Let (X,d) be a metric space. A map $T:X\to X$ is said to be a F_E-contraction on (X,d) if there exists $F\in\mathcal{F}_E$ and $\tau>0$ such that for all $x,y\in X$*

$$d(Tx,Ty)>0\Rightarrow\tau+F\left(d\left(Tx,Ty\right)\right)\le F\left(E(x,y)\right) \tag{3}$$

where

$$E(x,y)=d(x,y)+\left|d(x,Tx)-d\left(y,Ty\right)\right|. \tag{4}$$

Remark 2.2. *(1) Every F_E- contraction is an $F-$ contraction, but the inverse implication does not hold. (2) Not every $F-$ weak contraction is a F_E contraction .*

The following example shows that the statements from previous remark hold.

Example 2.3. *Let $X=\left[0,\frac{7}{10}\right]\cup\{1\}$ and $d(x,y)=|x-y|$, $x,y\in X$. Then (X,d) is a complete metric space. Define $T:X\to X$ by*

$$Tx=\begin{cases}\frac{x}{2}, & 0\le x\le\frac{7}{10}\\ \frac{1}{4}, & x=1\end{cases}$$

and choosing $F(\alpha)=\ln\alpha$, $\alpha\in(0,\infty)$ and $\tau=\ln 7$.

Since T is not continuous, T is not an $F-$contraction. In addition to that, for $x=\frac{1}{4}$ and $y=1$ we have

$$d\left(T\frac{1}{4},T1\right)=\left|\frac{1}{8}-\frac{1}{4}\right|=\frac{1}{8}>0$$

and

$$\begin{aligned}M\left(\frac{1}{4},1\right)&=\max\left\{d\left(\frac{1}{4},1\right),d\left(\frac{1}{4},T\frac{1}{4}\right),d\left(1,T1\right),\frac{d\left(1,T\frac{1}{4}\right)+d\left(\frac{1}{4},T1\right)}{2}\right\}\\&=\max\left\{\frac{1}{8},\frac{3}{4},\frac{3}{4},\frac{7}{16}\right\}=\frac{3}{4}.\end{aligned}$$

Then,

$$\begin{aligned}\tau+F\left(d\left(T\frac{1}{4},T1\right)\right)&=\ln 7+\ln\left(\frac{1}{8}\right)=\ln\left(\frac{7}{8}\right)\\&\ge\ln\left(\frac{3}{4}\right)=F\left(M\left(\frac{1}{4},1\right)\right)\end{aligned}$$

so T is not a $F-$weak contraction.

For $x\in\left[0,\frac{7}{10}\right]$ and $y=1$, we have

$$d\left(Tx,T1\right)=d\left(\frac{x}{2},\frac{1}{4}\right)=\frac{|2x-1|}{4}$$

and

$$\begin{aligned}E(x,1)&=d\left(x,1\right)+\left|d\left(x,Tx\right)-d\left(1,T1\right)\right|\\&=1-x+\left|\frac{x}{2}-\frac{3}{4}\right|=\frac{7-6x}{4}.\end{aligned}$$

Therefore,

$$\begin{aligned}
\ln 7 + \ln\left(d\left(Tx, T1\right)\right) &\leq \ln\left(E(x,1)\right) \Leftrightarrow \\
\ln 7 + \ln\left(\frac{|2x-1|}{4}\right) &\leq \ln\left(\frac{7-6x}{4}\right) \Leftrightarrow \\
7 \cdot \frac{|2x-1|}{4} &\leq \frac{7-6x}{4}.
\end{aligned}$$

For $x \leq \frac{1}{2}$*,*

$$7 \cdot \frac{1-2x}{4} \leq \frac{7-6x}{4} \Leftrightarrow 7 - 14x \leq 7 - 6x \Leftrightarrow x \geq 0,$$

and for $x > \frac{1}{2}$

$$7 \cdot \frac{2x-1}{4} \leq \frac{7-6x}{4} \Leftrightarrow 14x - 7 \leq 7 - 6x \Leftrightarrow x \leq \frac{7}{10}$$

which prove that T *is a* F_E-*contraction.*

Now we state the main result of the paper.

Theorem 2.4. *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a* F_E- *contraction. Then* T *has a unique fixed point* x^* *and for all* $x_0 \in X$ *the sequence* $\{T^n x_0\}$ *is convergent to* x^**.*

Proof. Let $x_0 \in X$ be arbitrary and fixed and we define $x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, because $x_{n_0+1} = Tx_{n_0}$, we obtain that $Tx_{n_0} = x_{n_0}$, so x_{n_0} is a fixed point of T.

Now, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. So, $d(x_n, x_{n+1}) > 0$, $(\forall)\, n \in \mathbb{N} \cup \{0\}$ and from (3) it follows that, for all $n \in \mathbb{N}$

$$\begin{aligned}
d\left(x_n, x_{n+1}\right) &= d\left(Tx_{n-1}, Tx_n\right) > 0 \Rightarrow \\
&\Rightarrow \tau + F\left(d\left(Tx_{n-1}, Tx_n\right)\right) \leq F\left(E(x_{n-1}, x_n)\right) \\
&\Leftrightarrow \tau + F\left(d\left(x_n, x_{n+1}\right)\right) \leq \\
&\leq F\left(d(x_{n-1}, x_n) + |d(x_{n-1}, Tx_{n-1}) - d(x_n, Tx_n)|\right. \Leftrightarrow \\
&\Leftrightarrow \tau + F\left(d\left(x_n, x_{n+1}\right)\right) \leq \\
&\leq F\left(d(x_{n-1}, x_n) + |d(x_{n-1}, x_n) - d(x_n, x_{n+1})|\right.
\end{aligned}$$

or, if we denote by $d_n = d\left(x_{n-1}, x_n\right)$, we have

$$\tau + F(d_{n+1}) \leq F\left(d_n + |d_n - d_{n+1}|\right). \tag{5}$$

If there exists $n \in \mathbb{N}$ such that $d_{n+1} \geq d_n$, then (5) becomes

$$\tau + F(d_{n+1}) \leq F\left(d_{n+1}\right) \Rightarrow \tau \leq 0.$$

But, this is a contradiction, so, for $d_{n+1} < d_n$ we have

$$\tau + F\left(d_{n+1}\right) \leq F\left(2d_n - d_{n+1}\right) \tag{6}$$

$$\Leftrightarrow F\left(d_{n+1}\right) \leq F\left(2d_n - d_{n+1}\right) - \tau < F\left(2d_n - d_{n+1}\right)$$

and using (F_E1)

$$d_{n+1} < 2d_n - d_{n+1}.$$

Therefore, the sequence $\{d_n\}$ is strictly increasing and bounded.

Now, let $d = \lim_{n\to\infty} d_n$ and we suppose that $d > 0$. Because $d_n \searrow d$ it results that $(2d_n - d_{n+1}) \searrow d$ and taking the limit as $n \to \infty$ in (6), we get

$$\tau + F(d+0) \le F(d+0) \Rightarrow \tau \le 0.$$

It is a contradiction, so

$$d = \lim_{n\to\infty} d_n = \lim_{n\to\infty} d\left(x_{n-1}, x_n\right) = 0. \tag{7}$$

In order to prove that $\{x_n\}$ is a Cauchy sequence in (X, d), we suppose the contrary, that is, there exists $\varepsilon > 0$ and the sequences $\{n(k)\}, \{m(k)\}$ of positive integers, with $n(k) > m(k) > k$ such that

$$d(x_{n(k)}, x_{m(k)}) \ge \varepsilon \text{ and } d\left(x_{n(k)-1}, x_{m(k)}\right) < \varepsilon \tag{8}$$

for any $k \in \mathbb{N}$.

Then, we have

$$\begin{aligned} \varepsilon &\le d(x_{n(k)}, x_{m(k)}) \le d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &< d(x_{n(k)}, x_{n(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ and using (7) it follows

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{9}$$

Furthermore, using the triangle inequality, we obtain that

$$\begin{aligned} 0 &\le \left|d\left(x_{n(k)+1}, x_{m(k)+1}\right) - d\left(x_{n(k)}, x_{m(k)}\right)\right| \\ &= d\left(x_{n(k)+1}, x_{n(k)}\right) + d\left(x_{m(k)}, x_{m(k)+1}\right) \end{aligned}$$

and

$$\begin{aligned} &\lim_{k\to\infty} \left|d\left(x_{n(k)+1}, x_{m(k)+1}\right) - d\left(x_{n(k)}, x_{m(k)}\right)\right| \\ =\ &\lim_{k\to\infty} \left[d\left(x_{n(k)+1}, x_{n(k)}\right) + d\left(x_{m(k)}, x_{m(k)+1}\right)\right] = 0. \end{aligned}$$

So,

$$\lim_{k\to\infty} d\left(x_{n(k)+1}, x_{m(k)+1}\right) = \lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon. \tag{10}$$

On the other hand, because from (7)

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0,$$

there exists $N \in \mathbb{N}$ such that

$$d(x_{n(k)}, Tx_{n(k)}) < \frac{\varepsilon}{4} \text{ and } d(x_{m(k)}, Tx_{m(k)}) < \frac{\varepsilon}{4},\ \ (\forall)\, k \ge N. \tag{11}$$

Assuming by contradiction, that there exists $l \in \mathbb{N}$ such that $d(x_{n(l)+1}, x_{m(l)+1}) = 0$, from (11) and (7) it follows that

$$\begin{aligned} \varepsilon &\le d(x_{n(l)}, x_{m(l)}) \\ &\le d(x_{n(l)}, x_{n(l)+1}) + d(x_{n(l)+1}, x_{m(l)+1}) + d(x_{m(l)+1}, x_{m(l)}) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

This is a contradiction. So we proved that the inequality occurs

$$d(Tx_{n(k)}, Tx_{m(k)}) = d(x_{n(k)+1}, x_{m(k)+1}) > 0 \tag{12}$$

for all $k \geq N$, and using (3), there exists $\tau > 0$ such that

$$\tau + F\left(d(Tx_{n(k)}, Tx_{m(k)})\right) \leq F\left(E(x_{n(k)}, x_{m(k)})\right)$$

for any k, where

$$\begin{aligned} E(x_{n(k)}, x_{m(k)}) &= d(x_{n(k)}, x_{m(k)}) + \left|d\left(x_{n(k)}, Tx_{n(k)}\right) - d\left(x_{m(k)}, Tx_{m(k)}\right)\right| \\ &= d(x_{n(k)}, x_{m(k)}) + \left|d\left(x_{n(k)}, x_{n(k)+1}\right) - d\left(x_{m(k)}, x_{m(k)+1}\right)\right|. \end{aligned}$$

Hence $\lim\limits_{k\to\infty} E(x_{n(k)}, x_{m(k)}) = \varepsilon$ and by (10) we have

$$\begin{aligned} \tau + \lim_{k\to\infty} \inf F\left(d(Tx_{n(k)}, Tx_{m(k)})\right) &\leq \liminf_{k\to\infty} F\left(E(x_{n(k)}, x_{m(k)})\right) \\ &\leq \limsup_{k\to\infty} F\left(E(x_{n(k)}, x_{m(k)})\right) \Leftrightarrow \\ &\Leftrightarrow \tau + F(\varepsilon+) \leq F(\varepsilon+) \end{aligned}$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and by completeness of X there converges to some point $x^* \in X$.

Next, we show that x^* is a fixed point of T. We consider two cases:

(1) For any $n \in \mathbb{N}$ there exists $k_n > k_{n-1}$, $k_0 = 1$ and $x_{k_n+1} = Tx^*$. Then, $x^* = \lim\limits_{n\to\infty} x_{k_n+1} = Tx^*$, so x^* is fixed point of T.

(2) There exists $m \in \mathbb{N}$ such that for all $n \geq m$, $d(Tx_n, Tx^*) > 0$. Substituting $x = x_n$ and $y = x^*$ in (3), there exists $\tau > 0$ such that

$$\begin{aligned} \tau + F(d(Tx_n, Tx^*) &\leq F(E(x_n, x^*)) \Leftrightarrow \\ \tau + F(d(x_{n+1}, Tx^*)) &\leq F(d(x_n, x^*) + |d(x_n, Tx_n) - d(x^*, Tx^*)|) \Leftrightarrow \\ \tau + F(d(x_{n+1}, Tx^*)) &\leq F(d(x_n, x^*) + |d(x_n, x_{n+1}) - d(x^*, Tx^*)|). \end{aligned}$$

We suppose that $x^* \neq Tx^*$. letting $n \to \infty$, from(7) we obtain

$$\tau + \liminf_{t\to d(x^*,Tx^*)} F(t) < \liminf_{t\to d(x^*,Tx^*)} F(t) < \limsup_{t\to d(x^*,Tx^*)} F(t)$$

which contradicts $(F_E 2)$ of the hypothesis. Hence $Tx^* = x^*$.

Now, let us show that T must have only one fixed point. If there exists another point $y^* \in X$, $x^* = y^*$ such that $Ty^* = y^*$, then $d(x^*, y^*) = d(Tx^*, Ty^*) > 0$ and we get

$$\begin{aligned} \tau + F(d(Tx^*, Ty^*) &\leq F(E(x^*, y^*)) \Leftrightarrow \\ \tau + F(d(x^*, y^*)) &\leq F(d(x^*, y^*) + |d(x^*, Tx^*) - d(y^*, Ty^*)|) \Leftrightarrow \\ \tau + F(d(x^*, y^*)) &\leq F(d(x^*, y^*) + |d(x^*, x^*) - d(y^*, y^*)|). \Leftrightarrow \\ \tau + F(d(x^*, y^*)) &\leq F(d(x^*, y^*)) \end{aligned}$$

which is a contradiction. □

Example 2.5. *Let T be given as in Example 2.3. Since T is not a contraction, Theorem 1.2 is not applicable to T and because T is not a F-weak contraction, Theorem 1.6 can not be applied. On the other hand let F and τ be given as in Example 2.3. Then T is an F_E contraction, and Theorem 2.4 can be applicable to T and the unique fixed point of T is 0.*

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