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Fractional Relaxation Equations and a Cauchy Formula for Repeated Integration of the Resolvent

Leigh C. Becker[a], Ioannis K. Purnaras[b]

[a]*Department of Mathematics, Christian Brothers University, 650 E. Parkway South, Memphis, TN 38104, USA*
[b]*Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*

Abstract

Cauchy's formula for repeated integration is shown to be valid for the function

$$R(t) = \lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)$$

where λ and q are given positive constants with $q \in (0, 1)$, Γ is the Gamma function, and $E_{q,q}$ is a Mittag-Leffler function. The function R is important in the study of Volterra integral equations because it is the unique continuous solution of the so-called *resolvent equation*

$$R(t) = \lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1}R(s)\,ds$$

on the interval $(0, \infty)$. This solution, commonly called the *resolvent*, is used to derive a formula for the unique continuous solution of the Riemann-Liouville fractional relaxation equation

$$D^q x(t) = -ax(t) + g(t) \qquad (a > 0)$$

on the interval $[0, \infty)$ when g is a given polynomial. This formula is used to solve a generalization of the equation of motion of a falling body. The last example shows that the solution of a fractional relaxation equation may be quite elementary despite the complexity of the resolvent.

Keywords: Cauchy's formula for repeated integration, fractional differential equations, Mittag-Leffler functions, relaxation equations, resolvents, Riemann-Liouville operators, Volterra integral equations
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Email addresses: `lbecker@cbu.edu` (Leigh C. Becker), `ipurnara@uoi.gr` (Ioannis K. Purnaras)

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1. Introduction

In classical physics, the ordinary differential equation

$$x'(t) = -ax(t) + g(t) \tag{1.1}$$

is sometimes called the *relaxation equation* (cf. [6, p. 138], [11]) when the constant a is positive. A generalization of (1.1) in a fractional calculus setting is the *fractional relaxation equation*

$$D^q x(t) = -ax(t) + g(t), \tag{1.2}$$

where D^q denotes a fractional differential operator of order q with $q \in (0,1)$ (cf. [6, p. 138]; [12, p. 292]; [17, p. 224]).

This paper is a study of (1.2) when $g(t)$ is a polynomial. For given $a > 0$ and g, we will prove that this equation has a unique continuous solution on the half-closed interval $[0,\infty)$ and that necessarily $x(0) = 0$. Furthermore, in Section 7, we will derive a formula that expresses this solution as a sum involving two-parameter Mittag-Leffler functions (cf. (7.13)). Moreover, we will show that each term of this sum can also be expressed as a convolution integral involving the solution of the integral equation

$$R(t) = \lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1} R(s)\, ds, \tag{R_λ}$$

where λ is a positive constant related to the value of the constant a. In fact, it is well-established that (R_λ) has a unique continuous solution on the interval $(0,\infty)$ whenever λ and q are positive constants with $q \in (0,1)$. A proof of this for a more general version of equation (R_λ) can be found in the 1971 monograph by Miller [14, Ch. IV].

There is also the recent paper [3] that investigates (R_λ) directly. Not only is the existence and uniqueness of a continuous solution of (R_λ) on $(0,\infty)$ proven there but also a formula for it is derived, namely

$$R(t) = \lambda\Gamma(q) t^{q-1} E_{q,q}(-\lambda\Gamma(q)t^q), \tag{1.3}$$

where $E_{\alpha,\beta}$ ($\alpha, \beta > 0$) denotes the two-parameter Mittag-Leffler function:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{1.4}$$

We use the following established terminology: *resolvent equation* refers to equation (R_λ) and *resolvent* refers to its solution (1.3).

For given constants $\lambda > 0$ and $q \in (0,1)$, important characteristics of the resolvent (1.3) are:

(i) For all $t > 0$, $0 < R(t) \le \left(\dfrac{q}{q + \lambda t^q}\right) \lambda t^{q-1}$.

(ii) $R(t) \to \infty$ as $t \to 0^+$ and $R(t) \to 0$ as $t \to \infty$.

(iii) The graph of R is decreasing and concave upward on $(0,\infty)$. In fact, R is *completely monotone* on $(0,\infty)$. That is, $R(t)$ is infinitely differentiable on $(0,\infty)$ and $(-1)^k R^{(k)}(t) \ge 0$ for all $t > 0$ and for $k = 0, 1, 2, \ldots$.

(iv) For all $t > 0$,

$$\frac{1}{1 + \frac{q}{\lambda t^q}} \le \int_0^t R(s)\, ds \le 1 - e^{-\frac{\lambda t^q}{q}}.$$

(v) $\int_0^\infty R(s)\, ds = 1$.

(vi) For given $\lambda > 0$, R is the unique continuous solution on $(0, \infty)$ of the initial value problem

$$D^q x(t) = -\lambda\Gamma(q)x(t), \quad \lim_{t\to 0^+} t^{1-q}x(t) = \lambda.$$

Items (i) and (ii) are proved in [3, Cor. 4.6, Thm. 7.3]. In [14, Thm. 7.2], Miller states that the solution of the above-mentioned general version is completely monotone; for a proof of this, he references [7] (cf. [14, p. 243]). A proof for the less general (1.3) that is based on the complete monotonicity of $E_{q,q}(-t)$ for $t \geq 0$ is given in [3, pp. 29–30]. Item (iv) is proved in [3, Thm. 4.5]. Clearly (iv) implies (v). A proof of (vi) is found in [3, Thm. 5.2].

The resolvent (1.3) is also expressed in terms of classical functions of mathematical physics in [3] and [5] for the following values of q. For $q = 1/2$, it is shown in [3, (6.12)] that

$$\begin{aligned} R(t) &= \lambda\Gamma(\tfrac{1}{2})t^{-1/2}E_{\frac{1}{2},\frac{1}{2}}(-\lambda\Gamma(\tfrac{1}{2})t^{1/2}) \\ &= \frac{\lambda}{\sqrt{t}} - \pi\lambda^2 e^{\pi\lambda^2 t}\big(1 - \operatorname{erf}(\lambda\sqrt{\pi t})\big), \end{aligned} \tag{1.5}$$

where $\operatorname{erf}(\cdot)$ is the error function (cf. (4.14)). In [5, (5.7)], after adjusting the notation there to be in accord with this paper, we find for $q = 1/3$ the formula

$$\begin{aligned} R(t) = {} & \frac{\lambda}{(\sqrt[3]{t})^2} - \frac{\sqrt{3}\,\sigma}{2\pi\lambda\sqrt[3]{t}} \\ & + \sigma^3 e^{-\sigma t}\left[1 + \frac{1}{\Gamma(\frac{1}{3})}\gamma(\tfrac{1}{3}, -\sigma t) + \frac{\sqrt{3}}{2\pi}\,\Gamma(\tfrac{1}{3})\gamma(\tfrac{2}{3}, -\sigma t)\right] \end{aligned} \tag{1.6}$$

where $\sigma := \big[\lambda\Gamma(\frac{1}{3})\big]^3$ and $\gamma(\cdot,\cdot)$ denotes the lower incomplete gamma function (cf. (4.20)). In Figure 1 of Section 4 the solid [resp. dashed] concave-upward curve is the graph of (1.5) [resp. (1.6)].

2. Riemann-Liouville Operators

For a function f that is (Riemann) integrable, we employ the integral operator J defined by

$$Jf(t) := \int_0^t f(s)\,ds.$$

Furthermore, for $n \in \mathbb{N}$ (set of natural numbers), let the operator J^n denote the n-fold iterate of J; that is,

$$J^n := JJ^{n-1} \quad \text{for} \quad n \geq 1$$

where $J^0 := I$, the identity operator. For example, taking $n = 2$ and applying J^2 to an integrable function f, we have

$$J^2 f(t) = \int_0^t Jf(s)\,ds = \int_0^t \left(\int_0^s f(u)\,du\right) ds = \int_0^t \left(\int_0^{t_2} f(t_1)\,dt_1\right) dt_2$$

or

$$J^2 f(t) = \int_0^t dt_2 \int_0^{t_2} f(t_1)\,dt_1.$$

In general,

$$J^n f(t) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)\,dt_1. \tag{2.1}$$

This particular iterated integral can be expressed in terms of a single integral with a weighted integrand as in (2.2) below. It is known as *Cauchy's formula for repeated integration* (cf. [16, p. 38]). This formula is found in Abramowitz and Stegun's handbook [1, (25.4.58)]; and in some textbooks, such as [8, p. 487], it appears as an exercise. We omit its proof here because it is basically the mathematical induction argument that is used in the proof of Theorem 3.2 in Section 3.

Theorem 2.1 (Cauchy's formula for repeated integration)**.** *Let $n \in \mathbb{N}$. If f is integrable on $[0, T]$, then*

$$J^n f(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)\, ds \tag{2.2}$$

for $t \in [0, T]$.

Now let us extend the values of n in (2.2) from $\mathbb{N}$ to $\mathbb{R}^+$ (set of strictly positive real numbers) by replacing $(n-1)!$ with $\Gamma(n)$, where Γ denotes the Gamma function. This leads to the well-known definition of the Riemann-Liouville integral operator of order n.

Definition 2.2. For any $n \in \mathbb{R}^+$, J^n denotes the integral operator

$$J^n f(t) := \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} f(s)\, ds, \tag{2.3}$$

where f denotes a function for which the integral exists. J^n is called the *Riemann-Liouville fractional integral operator of order* n. Furthermore, J and J^0 denote the operators

$$J := J^1 \text{ and } J^0 := I \tag{2.4}$$

where I denotes the identity operator.

Just as the integral operator J^n can be defined for all values of $n \in \mathbb{R}^+$, the same is true of D^n, namely, the classical ordinary differential operator of order $n \in \mathbb{N}$. That is, for an n-times differentiable function f,

$$Df(t) := \frac{d}{dt} f(t),\ D^2 f(t) := \frac{d^2}{dt^2} f(t),\ \ldots\ , D^n f(t) := \frac{d^n}{dt^n} f(t).$$

This can be expressed recursively as follows:

$$D^n := DD^{n-1} \quad \text{for} \quad n \geq 2,$$

where $D^1 := D$ and $D^0 := I$, the identity operator.

In the following extension of the definition of D^n, we employ the *floor function* $\lfloor \cdot \rfloor$, where $\lfloor n \rfloor$ denotes the largest integer less than or equal to n.

Definition 2.3. For a given $n \in \mathbb{R}^+$, D^n denotes the differential operator

$$D^n f := D^m J^{m-n} f \tag{2.5}$$

where $m = \lfloor n \rfloor + 1$ and f denotes a function for which the right-hand side exists. For $n = 0$, $D^n f := f$. D^n is called the *Riemann-Liouville fractional differential operator of order* n (cf. [6, p. 27]).

Remark 2.4. The symbol D^n on the left-hand side of (2.5) denotes the fractional differential operator of order n whereas D^m on the right-hand side denotes the ordinary differential operator d^m/dt^m since $m \in \mathbb{N}$. If $n \in \mathbb{N}$, then $m = \lfloor n \rfloor + 1 = n + 1$; so

$$D^n f = D^{n+1} J^1 f = D^n DJf = D^n If = D^n f.$$

Thus the definition of the operator D^n is well-defined.

Remark 2.5. Combining (2.3) and (2.5), we obtain the form

$$D^n f(t) = \frac{1}{\Gamma(m-n)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-n-1} f(s)\, ds \tag{2.6}$$

that is found in well-known works such as [10, (2.1.10) on p. 70].

Example 2.6. Let $n = q$ where $0 < q < 1$. Then

$$m = \lfloor q \rfloor + 1 = 1 \quad \text{and} \quad m - n = 1 - q.$$

Consequently,

$$D^q f(t) = D^1 J^{1-q} f(t) = DJ^{1-q} f(t)$$

or

$$D^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} f(s)\, ds. \tag{2.7}$$

3. Cauchy's Formula for the Resolvent

Theorem 3.1. *Let $n \in \mathbb{R}^+$. Let R be the resolvent, namely, the unique continuous solution of* (R_λ) *on* $(0, \infty)$. *Then*

$$\int_0^t \int_0^s (s-u)^{n-1} R(u)\, du\, ds = \int_0^t \int_u^t (s-u)^{n-1} R(u)\, ds\, du$$

for all $t > 0$.

Proof. See the proof of Theorem 4.3 in [3], where the Tonelli-Hobson test ([2, p. 415], [15, p. 93]) is employed; and note that the proof is valid not only for $n \in (0, 1)$ but for all $n > 0$. □

We now use this theorem to show that Cauchy's formula for repeated integration can be applied to the resolvent $R(t)$, notwithstanding the singularity at $t = 0$.

Theorem 3.2. *For $n \in \mathbb{N}$,*

$$J^n R(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} R(s)\, ds \tag{3.1}$$

for $t \geq 0$.

Proof. For a given $t \geq 0$, it is well-known (cf. [14, Ch. IV]) that the resolvent integral function

$$JR(t) = \int_0^t R(s)\, ds \tag{3.2}$$

exists. Furthermore, it is proven in [3, Thm. 9.5] that

$$JR(t) = 1 - E_q(-\lambda \Gamma(q) t^q)$$

where E_α, for $\alpha \in \mathbb{R}^+$, denotes the one-parameter Mittag-Leffler function, which is defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{3.3}$$

Since $E_q(z)$ is an entire function of z (cf. [6, Thm. 4.1]) in the complex plane, it follows that

$$J^n R(t) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} R(t_1)\, dt_1$$

exists and is continuous on $[0, \infty)$ for each $n \in \mathbb{N}$.

Note that (3.1) simplifies to (3.2) when $n = 1$. We complete the proof using mathematical induction to establish that (3.1) is true for all $n \in \mathbb{N}$. Suppose that (3.2) is also true when $n = k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} J^{k+1} R(t) = JJ^k R(t) &= \int_0^t J^k R(s)\, ds \\ &= \int_0^t \left[\frac{1}{(k-1)!} \int_0^s (s-u)^{k-1} R(u)\, du \right] ds. \end{aligned}$$

Interchanging the order of integration and appealing to Theorem 3.1, we have

$$\begin{aligned}
J^{k+1}R(t) &= \frac{1}{(k-1)!}\int_0^t \left(\int_u^t (s-u)^{k-1} ds\right) R(u)\,du \\
&= \frac{1}{(k-1)!}\int_0^t \left[\frac{(s-u)^k}{k}\right]_{s=u}^{s=t} R(u)\,du \\
&= \frac{1}{(k-1)!}\int_0^t \left[\frac{(t-u)^k}{k}\right] R(u)\,du = \frac{1}{k!}\int_0^t (t-u)^k R(u)\,du,
\end{aligned}$$

which is precisely (3.1) when $n = k+1$. Thus, as (3.1) is true for $n = 1$, it must be true for all $n \in \mathbb{N}$. □

The following result will be needed in the next section to prove Lemma 4.2. Although the proof is straightforward, it can be found in a number of places (e.g., [6, p. 28]).

Lemma 3.3. *Let $q \in (0,1)$ and $p > -1$. If $p \neq q-1$, then*

$$D^q t^p = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} t^{p-q} \tag{3.4}$$

for $t > 0$. If $p = q-1$, then $D^q t^p = 0$ for $t > 0$.

4. Solution of a fractional relaxation equation

The following proof is an adaptation of a proof in [6, Thm. 2.14].

Theorem 4.1. *Let $n \in \mathbb{R}_0^+$, where $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. If a function f is continuous and absolutely integrable on an interval $(0,T]$, then*

$$D^n J^n f(t) = f(t) \tag{4.1}$$

for all $t \in (0,T]$.

Proof. This is trivially true for $n = 0$ since by definition $J^0 := I$ and $D^0 := I$. It is also true for $n = 1$ because by the Fundamental Theorem of Calculus

$$D^1 J^1 f(t) = DJf(t) = \frac{d}{dt}\int_0^t f(s)\,ds = f(t)$$

for $0 < t \leq T$. It follows from this and an induction argument that (4.1) is true for all $n \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Now consider (4.1) for a given $n > 0$ when it is not a positive integer. Then, by Definition 2.3,

$$D^n J^n f = D^m J^{m-n} J^n f$$

where $m = \lfloor n \rfloor + 1$. Since f by hypothesis is continuous and absolutely integrable on $(0,T]$ and $m + n \geq 1$, we have

$$J^{m-n} J^n f(t) = J^{(m-n)+n} f(t) = J^m f(t)$$

for $0 \leq t \leq T$ by [4, Lemma 4.8]. As a result, since $m \in \mathbb{N}$,

$$D^n J^n f(t) = D^m J^m f(t) = f(t)$$

for $0 < t \leq T$. □

The following result relates solutions of (1.2) to those of a Volterra integral equation when $g(t) \equiv b$, a constant. It will be extended to all polynomials in Section 7.

Lemma 4.2. *Let a, b, and q be constants with $a > 0$, $b \in \mathbb{R}$, and $q \in (0,1)$. If there is a continuous solution of the fractional relaxation equation*

$$D^q x(t) = -ax(t) + b \tag{4.2}$$

on the interval $[0,\infty)$, then it is also a solution of

$$x(t) = \beta t^q - \lambda \int_0^t (t-s)^{q-1} x(s)\, ds \tag{4.3}$$

on $[0,\infty)$ when β and λ have the values

$$\beta = \frac{b}{\Gamma(q+1)} \quad \text{and} \quad \lambda = \frac{a}{\Gamma(q)}. \tag{4.4}$$

Conversely, let $\beta \in \mathbb{R}$ and $\lambda > 0$ and suppose there is a continuous solution of the integral equation (4.3) on $[0,\infty)$. Then it is also a continuous solution of (4.2) on $[0,\infty)$ when

$$a = \lambda\Gamma(q) \quad \text{and} \quad b = \beta\Gamma(q+1). \tag{4.5}$$

Proof. Let $\beta \in \mathbb{R}$ and $\lambda > 0$ be given constants. Then let a and b be defined by (4.5). Suppose there is a continuous function $x(t)$ that satisfies the integral equation (4.3) on $[0,\infty)$. Expressing this in terms of the Riemann-Liouville integral operator (2.3), we obtain

$$x(t) = \beta t^q - \lambda\Gamma(q) \cdot \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s)\, ds = \beta t^q - aJ^q x(t).$$

Applying the Riemann-Liouville differential operator D^q to this and using Theorem 4.1, we get

$$D^q x(t) = \beta\Gamma(q+1) - ax(t) = b - ax(t)$$

since $D^q t^q = \Gamma(1+q)$ (cf. Lemma 3.3). In other words, the function $x(t)$ must also be a solution of (4.2) on $[0,\infty)$. Note from (4.3) that $x(0) = 0$.

Now let $a > 0$ and $b \in \mathbb{R}$ be given constants. Then define constants $\beta \in \mathbb{R}$ and $\lambda > 0$ by (4.4) and suppose $x(t)$ is a continuous function satisfying (4.2) on $[0,\infty)$. And so

$$DJ^{1-q} x(t) = -ax(t) + b$$

since $D^q = DJ^{1-q}$. For a fixed $t > 0$, let $\eta \in (0,t)$. The integration

$$\int_\eta^t \frac{d}{ds} J^{1-q} x(s)\, ds = \int_\eta^t (-ax(s) + b)\, ds$$

yields

$$J^{1-q} x(t) - J^{1-q} x(\eta) = -a \int_\eta^t x(s)\, ds + b(t-\eta). \tag{4.6}$$

Lemma 3.1 in [3, p. 5] implies that

$$\lim_{\eta \to 0^+} J^{1-q} x(\eta) = \frac{1}{\Gamma(1-q)} \lim_{\eta \to 0^+} \int_0^\eta (\eta - s)^{-q} x(s)\, ds = 0. \tag{4.7}$$

Because of this and the continuity of x on $[0,\infty)$, we obtain

$$J^{1-q} x(t) = -a \int_0^t x(s)\, ds + bt = -aJx(t) + bt$$

upon taking the limit of both sides of (4.6) as $\eta \to 0^+$. Then the application of D^{1-q} yields

$$D^{1-q}J^{1-q}x(t) = -aD^{1-q}Jx(t) + bD^{1-q}t,$$

which because of Theorem 4.1 and (2.5) simplifies to

$$x(t) = -aDJ^qJx(t) + bD^{1-q}t. \tag{4.8}$$

It follows from [4, Lemma 4.8] that

$$DJ^qJx(t) = DJJ^qx(t) = J^qx(t)$$

and from Lemma 3.3 that

$$D^{1-q}t = \frac{1}{\Gamma(q+1)}t^q.$$

Therefore, we conclude from (4.8) that any continuous solution of (4.2) on $[0,\infty)$ must also be a solution of

$$\begin{aligned} x(t) &= -aJ^qx(t) + \frac{b}{\Gamma(q+1)}t^q \\ &= -\frac{a}{\Gamma(q)}\int_0^t (t-s)^{q-1}x(s)\,ds + \beta t^q = \beta t^q - \lambda\int_0^t (t-s)^{q-1}x(s)\,ds. \end{aligned}$$

Moreover, we see from this integral equation that $x(0) = 0$. □

Remark 4.3. Observe in the statement of Lemma 4.2 that no initial condition accompanies the fractional differential equation (4.2). At first this may appear to be an oversight until we realize from the proof that positing the existence of a continuous solution $x(t)$ of (4.2) for $t \geq 0$ implies $x(0) = 0$.

With the next theorem we complete what was initiated with Lemma 4.2 and that is to show that (4.2) and (4.3) do in fact share the same continuous solution on $[0,\infty)$. But first let us dispose of the special case $b = 0$.

Lemma 4.4. *There is one and only one continuous solution of*

$$D^qx(t) = -ax(t) \quad (a > 0) \tag{4.9}$$

on $[0,\infty)$*; it is the trivial solution* $x(t) \equiv 0$.

Proof. It follows from Lemma 4.2 that any continuous solution of (4.9) on $[0,\infty)$ must also be a continuous solution of

$$x(t) = -\lambda\int_0^t (t-s)^{q-1}x(s)\,ds$$

where $\lambda = a/\Gamma(q)$. But the only solution of this integral equation is $x(t) \equiv 0$ (cf. [3, p. 15]). □

Theorem 4.5. *For given constants* $a > 0$, $b \in \mathbb{R}$, *and* $q \in (0,1)$, *the fractional relaxation equation* (4.2) *has one and only one continuous solution on* $[0,\infty)$, *namely*

$$x(t) = \frac{b}{a}\int_0^t R(s)\,ds = \frac{b}{a}\left[1 - E_q(-at^q)\right], \tag{4.10}$$

where R *denotes the resolvent corresponding to* $\lambda = a/\Gamma(q)$. *This is also the unique continuous solution of the integral equation* (4.3) *on* $[0,\infty)$ *when* β *and* λ *have the values given by* (4.4).

Proof. First consider the integral equation (4.3) where $\beta \in \mathbb{R}$ and $\lambda > 0$ are given constants. From [3, Thm. 8.3] we know that if a function f is continuous on the interval $[0, \infty)$, then

$$x(t) = f(t) - \lambda \int_0^t (t-s)^{q-1} x(s)\, ds$$

has a unique continuous solution on $[0, \infty)$. Moreover, this solution is given by the linear variation of parameters formula:

$$x(t) = f(t) - \int_0^t R(t-s) f(s)\, ds.$$

Taking $f(t) = \beta t^q$, this becomes

$$x(t) = \beta t^q - \int_0^t R(t-s) \beta s^q\, ds = \beta t^q - \beta \int_0^t (t-u)^q R(u)\, du. \tag{4.11}$$

In other words, this is the unique continuous solution of (4.3) on $[0, \infty)$.

But we can simplify (4.11) as follows: integrating the resolvent equation (R_λ) and interchanging the order of integration (cf. Thm. 3.1), we obtain

$$\int_0^t R(s)\, ds = \frac{\lambda}{q} t^q - \frac{\lambda}{q} \int_0^t (t-u)^q R(u)\, du.$$

Thus,

$$\int_0^t (t-u)^q R(u)\, du = t^q - \frac{q}{\lambda} \int_0^t R(s)\, ds.$$

Substituting this into (4.11) and defining a and b by (4.5), we get

$$\begin{aligned} x(t) &= \beta t^q - \beta \left[t^q - \frac{q}{\lambda} \int_0^t R(s)\, ds \right] = \beta q \cdot \frac{1}{\lambda} \int_0^t R(s)\, ds \\ &= \frac{bq}{\Gamma(q+1)} \cdot \frac{\Gamma(q)}{a} \int_0^t R(s)\, ds = \frac{b}{a} \int_0^t R(s)\, ds. \end{aligned}$$

In [3, Thm. 9.5], we find the formula

$$\int_0^t R(s)\, ds = 1 - E_q(-at^q).$$

Therefore (4.10) is the unique continuous solution of (4.3). Moreover, Lemma 4.2 implies that it is also the unique continuous solution of (4.2) on $[0, \infty)$. □

Remark 4.6. We have shown that there is one and only one continuous solution $x(t)$ of the fractional relaxation equation (4.2) on the half-closed interval $[0, \infty)$. Moreover, from (4.10) we see that $x(0) = 0$. Thus, the initial value problem

$$D^q x(t) = -ax(t) + b, \quad x(0) = x_0$$

has no continuous solution on $[0, \infty)$ unless $x_0 = 0$.

Also, observe that if we formally let $q = 1$ in (4.10), then it simplifies to

$$x(t) = \frac{b}{a} [1 - E_1(-at)] = \frac{b}{a} \left(1 - e^{-at}\right) \tag{4.12}$$

since $E_1(z) = e^z$ (cf. (3.3)). Note that this is the unique continuous solution of the classical initial value problem

$$x'(t) = -ax(t) + b, \quad x(0) = 0.$$

Corollary 4.7. *If $b \neq 0$, then solution (4.10) has the following properties:*

(i) $x(0) = 0$.

(ii) $\lim_{t\to\infty} x(t) = b/a$.

(iii) *If $b > 0$ ($b < 0$), then $x(t)$ is strictly increasing (decreasing) on $[0, \infty)$ and $x(t) > 0$ ($x(t) < 0$) for all $t > 0$.*

(iv) *If $b > 0$ ($b < 0$), then $x(t)$ is concave downward (upward) on $(0, \infty)$.*

(v) *If $b > 0$, then the derivative $x'(t)$ is completely monotone on $(0, \infty)$, whereas $-x'(t)$ is completely monotone on $(0, \infty)$ if $b < 0$.*

Proof. Properties (i) and (ii) follow from (4.10) from which we see that $x(0) = 0$ and

$$\lim_{t\to\infty} x(t) = \frac{b}{a} \lim_{t\to\infty} \int_0^t R(s)\, ds = \frac{b}{a} \int_0^\infty R(s)\, ds = \frac{b}{a}.$$

Since the derivative of (4.10) is

$$x'(t) = \frac{b}{a} \frac{d}{dt} \int_0^t R(s)\, ds = \frac{b}{a} R(t), \tag{4.13}$$

it follows that $x'(t) > 0$ if $b > 0$. And so $x(t)$ is strictly increasing on $[0, \infty)$. This together with $x(0) = 0$ implies that $x(t) > 0$ for $t > 0$. Likewise, if $b < 0$, then $x(t)$ is strictly decreasing on $[0, \infty)$ and $x(t) < 0$ for $t > 0$. This concludes the proof of (iii).

To prove (iv), we use the result stated in Section 1 that the resolvent R is a completely monotone function on $(0, \infty)$. Thus,

$$x''(t) = \frac{b}{a} R'(t).$$

And so $x''(t) \leq 0$ if $b > 0$ and $x''(t) \geq 0$ if $b < 0$.

Finally, (v) follows from (4.13) and the complete monotonicity of R. □

Example 4.8. We illustrate some of the properties of solutions of (4.2) that are enumerated in Corollary 4.7 by choosing two different values of q and graphing the corresponding solutions (4.10). For both values, let $a = b = 1$.

First let $q = 1/2$. Then (4.10) is

$$x(t) = 1 - E_{1/2}(-\sqrt{t}).$$

According to [10, (1.8.6)],

$$E_{1/2}(z) = e^{z^2} [1 + \operatorname{erf}(z)]$$

where $\operatorname{erf}(z)$ is the *error function*:

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2}\, du. \tag{4.14}$$

Thus,

$$x(t) = 1 - e^t \left[1 + \operatorname{erf}(-\sqrt{t})\right] = 1 - e^t + e^t \operatorname{erf}(\sqrt{t}) \tag{4.15}$$

is the unique continuous solution of

$$D^{1/2} x(t) = -x(t) + 1 \tag{4.16}$$

on $[0, \infty)$. The graph of (4.15) is the solid concave-downward curve in Figure 1. (All the graphs in this paper were created with Maple$^{\text{TM}}$ 17.)

Now let $q = 1/3$. By (4.10) the unique continuous solution of

$$D^{1/3} x(t) = -x(t) + 1 \tag{4.17}$$

on $[0, \infty)$ is

$$x(t) = 1 - E_{1/3}(-\sqrt[3]{t}). \tag{4.18}$$

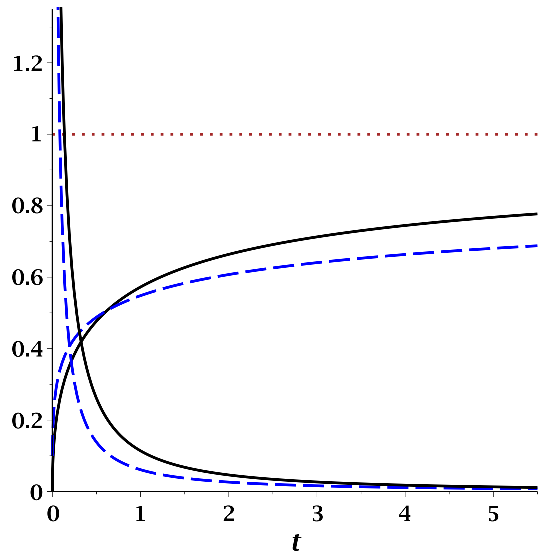

Figure 1: Solutions (1.5), (1.6), (4.15) and (4.21).

The second term can be calculated with the help of formula (1.8.5) in [10]: for $m = 2, 3, 4, \ldots,$

$$E_{1/m}(z) = e^{z^m}\left[1 + m\int_0^z e^{-u^m}\left(\sum_{k=1}^{m-1}\frac{u^{k-1}}{\Gamma(k/m)}\right)du\right].$$

Consequently,

$$\begin{aligned} E_{1/3}(t) &= e^{t^3}\left[1 + 3\int_0^t e^{-u^3}\left(\frac{1}{\Gamma(1/3)} + \frac{u}{\Gamma(2/3)}\right)du\right] \\ &= e^{t^3}\left[1 + \frac{3}{\Gamma(1/3)}\int_0^t e^{-u^3}\,du + \frac{3}{\Gamma(2/3)}\int_0^t u\,e^{-u^3}\,du\right]. \end{aligned} \tag{4.19}$$

We can also express the solution (4.18) in terms of the *lower incomplete gamma function* $\gamma(a, z)$, namely

$$\gamma(a, z) := \int_0^z u^{a-1}e^{-u}\,du. \tag{4.20}$$

Changing the variable of integration to $z = u^3$, we obtain

$$\int_0^t e^{-u^3}\,du = \frac{1}{3}\int_0^{t^3} z^{-2/3}e^{-z}\,dz = \frac{1}{3}\gamma(1/3, t^3).$$

Likewise, the same change of variable yields

$$\int_0^t u\,e^{-u^3}\,du = \frac{1}{3}\int_0^{t^3} z^{-1/3}e^{-z}\,dz = \frac{1}{3}\gamma(2/3, t^3).$$

Thus,

$$E_{1/3}(t) = e^{t^3}\left[1 + \frac{\gamma(1/3, t^3)}{\Gamma(1/3)} + \frac{\gamma(2/3, t^3)}{\Gamma(2/3)}\right].$$

Therefore, an alternative form of (4.18) is

$$x(t) = 1 - e^{-t}\left[1 + \frac{\gamma(1/3, -t)}{\Gamma(1/3)} + \frac{\gamma(2/3, -t)}{\Gamma(2/3)}\right]. \tag{4.21}$$

Its graph is the concave-downward dashed curve in Figure 1.

5. Generalization of the equation of motion of a falling body

Consider the vertical downward motion of a body of mass m when it is released from rest above the ground. Besides the force due to gravity and the resisting force (*drag force*) exerted on the body as it moves through the air, assume that all other forces acting on the body are negligible. Also, let us suppose that the drag force is proportional to the velocity v of the body. Since the direction of the drag force is opposite that of the velocity of the body (downward, which we take as negative), the drag force F_d points upward. Thus $F_d = -kmv$, where the proportionality constant $k > 0$. Since the gravitational force acting on the body is $F_g = -mg$, Newton's second law of motion yields

$$m\frac{dv}{dt} = F_g + F_d = -mg - kmv.$$

Hence, we obtain the familiar classical equation of motion

$$\frac{dv}{dt} = -kv - g, \quad v(0) = 0 \tag{5.1}$$

that is found in most undergraduate physics textbooks, such as [13, p. 68]. Solving (5.1) by either separating variables or using the integrating factor e^{kt}, we obtain the solution

$$v(t) = -\frac{g}{k} + \frac{g}{k}e^{-kt}. \tag{5.2}$$

Now suppose we generalize the equation of motion (5.1) by replacing the classical differential operator d/dt with the Riemann-Liouville operator D^q. But this does not make complete sense due to the dimensional inconsistency of the units, where we see from (5.1) that k has the dimension of inverse time. However, we can rectify this with the replacement

$$\frac{d}{dt} \to k^{1-q}D^q$$

suggested by Rosales et al. in [18, p. 519]. (Actually they use the Caputo fractional derivative; however, since $q \in (0, 1)$ and the initial condition is $v(0) = 0$, the Caputo and Riemann-Liouville derivatives are equivalent.) Consequently, the fractional generalization of (5.1) is

$$k^{1-q}D^q v = -kv - g, \quad v(0) = 0$$

or

$$D^q v = -k^q v - k^{q-1}g, \quad v(0) = 0. \tag{5.3}$$

From Theorem 4.5 we see that the unique continuous solution of (5.3) is

$$v(t) = -\frac{g}{k} + \frac{g}{k}E_q(-(kt)^q). \tag{5.4}$$

This agrees with the velocity formula in [18]. Note that by formally letting $q = 1$, (5.4) simplifies to (5.2) because of (3.3).

6. Repeated integration of the resolvent

Lemma 6.1. *Let $R(t)$ be the resolvent, namely, the unique continuous solution of* (R_λ) *on* $(0, \infty)$. *Then*

$$\int_0^t R(s)\, ds = \frac{\lambda}{q}\left[t^q - \int_0^t (t-u)^q R(u)\, du\right] \tag{6.1}$$

for $t \geq 0$.

Proof. Integrating (R_λ) and interchanging the order of integration (cf. Thm. 3.1), we get

$$\begin{aligned}\int_0^t R(s)\,ds &= \lambda\int_0^t s^{q-1}\,ds - \lambda\int_0^t\int_0^s (s-u)^{q-1}R(u)\,du\,ds\\ &= \frac{\lambda}{q}t^q - \lambda\int_0^t\left(\int_u^t (s-u)^{q-1}\,ds\right)R(u)\,du\\ &= \frac{\lambda}{q}t^q - \frac{\lambda}{q}\int_0^t (t-u)^q R(u)\,du.\end{aligned}$$

□

Theorem 6.2. *Let $n\in\mathbb{N}$. The nth repeated integral of the resolvent $R(t)$, namely*

$$J^nR(t) = \int_0^t dt_n\int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} R(t_1)\,dt_1,$$

is given by the formula

$$J^nR(t) = \frac{\lambda\Gamma(q)}{\Gamma(q+n)}\left[t^{q+n-1} - \int_0^t (t-u)^{q+n-1}R(u)\,du\right] \tag{6.2}$$

for $t\geq 0$.

Proof. It follows from Lemma 6.1 that formula (6.2) holds for $n=1$. Let us show via a proof by induction that it holds for all $n\in\mathbb{N}$.

Suppose for some $k\in\mathbb{N}$ that (6.2) holds for $n=k$. Then

$$\begin{aligned}J^{k+1}R(t) &= \int_0^t J^kR(s)\,ds\\ &= \frac{\lambda\Gamma(q)}{\Gamma(q+k)}\int_0^t\left[s^{q+k-1} - \int_0^s (s-u)^{q+k-1}R(u)\,du\right]ds\\ &= \frac{\lambda\Gamma(q)}{\Gamma(q+k)}\left[\frac{t^{q+k}}{q+k} - \int_0^t\int_0^s (s-u)^{q+k-1}R(u)\,du\,ds\right].\end{aligned}$$

Interchanging the order of integration as in Theorem 3.1, we obtain

$$\begin{aligned}J^{k+1}R(t) &= \frac{\lambda\Gamma(q)}{\Gamma(q+k)}\left[\frac{t^{q+k}}{q+k} - \int_0^t\left(\int_u^t (s-u)^{q+k-1}\,ds\right)R(u)\,du\right]\\ &= \frac{\lambda\Gamma(q)}{\Gamma(q+k)}\left[\frac{1}{q+k}t^{q+k} - \frac{1}{q+k}\int_0^t (t-u)^{q+k}R(u)\,du\right]\\ &= \frac{\lambda\Gamma(q)}{\Gamma(q+k+1)}\left[t^{q+k} - \int_0^t (t-u)^{q+k}R(u)\,du\right].\end{aligned}$$

This shows that (6.2) holds for $n=k+1$ if it holds for $n=k$. Therefore, by induction, (6.2) holds for all $n\in\mathbb{N}$. □

Corollary 6.3. *Let $m\in\mathbb{N}_0$, $\lambda>0$, and $q\in(0,1)$. Let $R(t)$ be the resolvent corresponding to the parameter λ, i.e., the unique continuous solution of* (R_λ). *Then*

$$\int_0^t (t-s)^{q+m}R(s)\,ds = t^{q+m} - \frac{1}{\lambda}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)\Gamma(m+1)}\int_0^t (t-s)^m R(s)\,ds. \tag{6.3}$$

Proof. From Theorems 3.2 and 6.2, we have two different formulas for $J^nR(t)$. As a result, setting $n=m+1$ in (3.1) and (6.2), we get

$$\frac{\lambda\Gamma(q)}{\Gamma(q+m+1)}\left[t^{q+m} - \int_0^t (t-s)^{q+m}R(s)\,ds\right] = \frac{1}{m!}\int_0^t (t-s)^m R(s)\,ds.$$

Now solve this for the integral on the left-hand side. □

7. Solution of (1.2) for a given polynomial g

The first result of this section generalizes Lemma 4.2.

Lemma 7.1. *Let $a > 0$, $b \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $q \in (0,1)$. If there is a continuous solution of*

$$D^q x(t) = -ax(t) + bt^m \tag{7.1}$$

on $[0,\infty)$, then it is also a solution of

$$x(t) = \beta t^{q+m} - \lambda \int_0^t (t-s)^{q-1} x(s)\, ds \tag{7.2}$$

on $[0,\infty)$ when

$$\beta = \frac{bm!}{\Gamma(q+m+1)} \quad \text{and} \quad \lambda = \frac{a}{\Gamma(q)}. \tag{7.3}$$

Conversely, let $\beta \in \mathbb{R}$ and $\lambda > 0$ and suppose there is a continuous solution of (7.2) on $[0,\infty)$. Then it is also a continuous solution of (7.1) on $[0,\infty)$ when

$$a = \lambda\Gamma(q) \quad \text{and} \quad b = \frac{\beta}{m!}\Gamma(q+m+1). \tag{7.4}$$

Proof. Equation (7.2), written in terms of the Riemann-Liouville integral operator, is

$$x(t) = \beta t^{q+m} - \lambda\Gamma(q) J^q x(t).$$

Applying the differential operator D^q, we get

$$\begin{aligned} D^q x(t) &= \beta D^q t^{q+m} - \lambda\Gamma(q) D^q J^q x(t) = \beta\, \frac{\Gamma(q+m+1)}{\Gamma(m+1)} t^m - ax(t) \\ &= \frac{\beta}{m!}\Gamma(q+m+1)t^m - ax(t) = -ax(t) + bt^m, \end{aligned}$$

where we have used Theorem 4.1, Lemma 3.3, and (7.4). Thus, if a continuous solution of (7.2) exists for $t \geq 0$, it must also be a solution of (7.1) when a and b have the values given by (7.4).

Conversely, suppose there exists a continuous solution $x(t)$ of (7.1) on $[0,\infty)$; hence

$$DJ^{1-q}x(t) = -ax(t) + bt^m.$$

Integrating, as in the proof of Lemma 4.2, we have

$$\int_\eta^t \frac{d}{ds} J^{1-q} x(s)\, ds = \int_\eta^t (-ax(s) + bs^m)\, ds$$

or

$$J^{1-q}x(t) - J^{1-q}x(\eta) = \int_\eta^t (-ax(s) + bs^m)\, ds. \tag{7.5}$$

Taking the limit of both sides as $\eta \to 0^+$, we obtain

$$J^{1-q}x(t) = \int_0^t (-ax(s) + bs^m)\, ds = -aJx(t) + \frac{b}{m+1} t^{m+1}$$

since $J^{1-q}x(\eta) \to 0$ (cf. (4.7)). Applying D^{1-q}, we get

$$D^{1-q}J^{1-q}x(t) = -aDJ^q Jx(t) + \frac{b}{m+1} DJ^q t^{m+1}$$

or

$$x(t) = -aJ^q x(t) + \frac{b}{m+1} \cdot \frac{\Gamma(m+2)}{\Gamma(m+q+1)} t^{m+q}$$

because of Lemma 3.3 and Theorem 4.1. Since $m \in \mathbb{N}_0$, this simplifies to

$$x(t) = -aJ^q x(t) + \frac{bm!}{\Gamma(m+q+1)} t^{m+q}.$$

We conclude that if a continuous solution $x(t)$ of (7.1) exists, then it must also be a solution of

$$x(t) = \frac{bm!}{\Gamma(m+q+1)} t^{m+q} - \frac{a}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s)\, ds. \qquad \square$$

In the next theorem we prove that (7.1) does have a unique continuous solution on $[0, \infty)$. Moreover, with the following formula, which is found in [17, p. 25], we show how to express it in terms of a Mittag-Leffler function.

Lemma 7.2. *Let $\gamma \in \mathbb{R}$ and $\alpha, \beta, p \in \mathbb{R}^+$. Then*

$$\int_0^t (t-s)^{p-1} E_{\alpha,\beta}(\gamma s^\alpha) s^{\beta-1}\, ds = \Gamma(p) t^{p+\beta-1} E_{\alpha,\beta+p}(\gamma t^\alpha) \tag{7.6}$$

for $t > 0$.

Proof. Let us use (1.4) to write the integrand as the sum

$$(t-s)^{p-1} E_{\alpha,\beta}(\gamma s^\alpha) s^{\beta-1} = (t-s)^{p-1} \left(\sum_{k=0}^{\infty} \frac{(\gamma s^\alpha)^k}{\Gamma(k\alpha+\beta)} \right) s^{\beta-1} = \sum_{k=0}^{\infty} g_k(s),$$

where

$$g_k(s) := (t-s)^{p-1} \frac{\gamma^k}{\Gamma(k\alpha+\beta)}\, s^{k\alpha+\beta-1}.$$

Using an integration formula in [4, (4.4)] (or [6, p. 229]), we find for $t > 0$ that

$$\begin{aligned} \int_0^t |g_k(s)|\, ds &= \frac{|\gamma|^k}{\Gamma(k\alpha+\beta)} \int_0^t (t-s)^{p-1} s^{k\alpha+\beta-1}\, ds \\ &= \frac{|\gamma|^k}{\Gamma(k\alpha+\beta)}\, t^{p+k\alpha+\beta-1} \frac{\Gamma(p)\Gamma(k\alpha+\beta)}{\Gamma(p+k\alpha+\beta)} = \Gamma(p) t^{p+\beta-1} \frac{(|\gamma| t^\alpha)^k}{\Gamma(k\alpha+\beta+p)} < \infty. \end{aligned}$$

It then follows from a generalization of Levi's theorem for series ([2, p. 269]) that

$$\begin{aligned} \int_0^t (t-s)^{p-1} E_{\alpha,\beta}(\gamma s^\alpha) s^{\beta-1}\, ds &= \int_0^t \sum_{k=0}^{\infty} g_k(s)\, ds = \sum_{k=0}^{\infty} \int_0^t g_k(s)\, ds \\ &= \Gamma(p) t^{p+\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma t^\alpha)^k}{\Gamma(k\alpha+\beta+p)} = \Gamma(p) t^{p+\beta-1} E_{\alpha,\beta+p}(\gamma t^\alpha). \qquad \square \end{aligned}$$

With the integration formulas involving the resolvent that we found in Section 6.1 and the variation of parameters formula that was used earlier in the proof of Theorem 4.5, we can now establish the existence of continuous solutions of equations (7.1) and (7.2) on $[0, \infty)$ and their uniqueness.

Theorem 7.3. *Let $a > 0$, $b \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $q \in (0, 1)$. The fractional relaxation equation* (7.1) *has the unique continuous solution*

$$x(t) = \frac{b}{a}\int_0^t (t-s)^m R(s)\,ds = bm!\,t^{q+m}E_{q,q+m+1}(-at^q) \tag{7.7}$$

on $[0, \infty)$, where R denotes the resolvent corresponding to $\lambda = a/\Gamma(q)$. It is also the unique continuous solution of the integral equation (7.2) *on $[0, \infty)$ when β and λ have the values given by* (7.3).

Proof. First consider the integral equation (7.2) for given values of $\beta \in \mathbb{R}$ and $\lambda > 0$. By the variation of parameters formula, the function

$$x(t) = \beta t^{q+m} - \int_0^t R(t-s)\beta s^{q+m}\,ds = \beta t^{q+m} - \beta\int_0^t (t-s)^{q+m}R(s)\,ds$$

is the unique continuous solution of (7.2) on $[0, \infty)$. Because of Corollary 6.3, this solution can be simplified as follows: first define a and b by (7.4). Then

$$\begin{aligned} x(t) &= \beta t^{q+m} - \beta\left[t^{q+m} - \frac{1}{\lambda}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)\Gamma(m+1)}\int_0^t (t-s)^m R(s)\,ds\right] \\ &= \frac{\beta}{\lambda}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)\Gamma(m+1)}\int_0^t (t-s)^m R(s)\,ds \\ &= \frac{bm!}{\Gamma(q+m+1)}\cdot\frac{\Gamma(q)}{a}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)\Gamma(m+1)}\int_0^t (t-s)^m R(s)\,ds \\ &= \frac{b}{a}\int_0^t (t-s)^m R(s)\,ds. \end{aligned} \tag{7.8}$$

In short, we have shown that (7.8) is the unique continuous solution of (7.2) on $[0, \infty)$ if a and b have the values given by (7.4). Furthermore, we can see from Lemma 7.1 that it is also the unique continuous solution of the fractional differential equation (7.1) on $[0, \infty)$.

Finally, let us show how to express (7.8) in terms of a Mittag-Leffler function. From (1.3) we find that the resolvent of (R_λ) corresponding to $\lambda = a/\Gamma(q)$ is

$$R(t) = \lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q) = at^{q-1}E_{q,q}(-at^q).$$

Hence, from (7.8) we have

$$x(t) = \frac{b}{a}\int_0^t (t-s)^m R(s)\,ds = b\int_0^t (t-s)^m s^{q-1}E_{q,q}(-as^q)\,ds.$$

Then, by setting $p = m+1$, $\alpha = \beta = q$, and $\gamma = -a$ in Lemma 7.2, we obtain

$$x(t) = b\Gamma(m+1)t^{(m+1)+q-1}E_{q,q+m+1}(-at^q) = bm!t^{m+q}E_{q,q+m+1}(-at^q)$$

for $t > 0$. Note this formula is also valid for $t = 0$ since from (7.8) we see that $x(0) = 0$. □

Remark 7.4. According to (7.7), the solution of (7.1) when $m = 0$ is

$$x(t) = \frac{b}{a}\int_0^t R(s)\,ds = b\,t^q E_{q,q+1}(-at^q).$$

From equations (9.7) and (9.8) in [3], we see that

$$t^q E_{q,q+1}(-at^q) = \frac{1}{a}\left[1 - E_q(-at^q)\right].$$

Thus,

$$x(t) = \frac{b}{a}\left[1 - E_q(-at^q)\right],$$

which is precisely what was stated in Theorem 4.5.

Remark 7.5. If we disregard the hypothesis that $q \in (0,1)$ and set $q = 1$, then (7.7) becomes

$$x(t) = bm!\, t^{1+m} E_{1,2+m}(-at).$$

Can this formal substitution be justified? In [6, p. 69] and [17, p. 18], we find the formulas:

$$E_{1,1}(x) = e^x \text{ and } E_{1,n}(x) = \frac{1}{x^{n-1}} \left(e^x - \sum_{k=0}^{n-2} \frac{x^k}{k!} \right) \text{ for } n = 2, 3, \ldots.$$

Setting $n = 2 + m$ and $x = -at$, we obtain

$$\begin{aligned} x(t) &= bm!\, t^{1+m} \frac{1}{(-at)^{m+1}} \left(e^{-at} - \sum_{k=0}^{m} \frac{(-at)^k}{k!} \right) \\ &= (-1)^{m+1} \frac{bm!}{a^{m+1}}\, e^{-at} - (-1)^{m+1} \frac{bm!}{a^{m+1}} \sum_{k=0}^{m} \frac{(-at)^k}{k!} \\ &= (-1)^{m+1} \frac{bm!}{a^{m+1}}\, e^{-at} + \frac{b}{a} \sum_{k=0}^{m} (-1)^{m+2+k} \frac{m!}{a^{m-k} k!} t^k. \end{aligned}$$

Thus,

$$x(t) = \frac{b}{a} \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!\, a^{m-k}} t^k + \frac{b}{a} (-1)^{m+1} \frac{m!}{a^m} e^{-at}. \tag{7.9}$$

Writing out some of the terms of (7.9), we have

$$x(t) = \frac{b}{a} \left[(-1)^m \frac{m!}{a^m} + (-1)^{m-1} \frac{m!}{a^{m-1}} t + \cdots + t^m \right] + (-1)^{m+1} \frac{bm!}{a^{m+1}}\, e^{-at}.$$

Now note that when we evaluate this at $t = 0$, we get

$$x(0) = (-1)^m \frac{bm!}{a^{m+1}} + (-1)^{m+1} \frac{bm!}{a^{m+1}} = 0,$$

which agrees with the value of (7.7) at $t = 0$. Also note that the fractional differential operator D^q is defined to be the ordinary first-order differential operator D when $q = 1$ (cf. Def. 2.3). So the formal substitution $q = 1$ in (7.7) suggests that (7.9) is the solution of the classical initial value problem

$$x'(t) = -ax(t) + bt^m, \quad x(0) = 0. \tag{7.10}$$

Let us see if this is truly the case.

By the classical variation of parameters formula, the solution of (7.10) is

$$x(t) = e^{-at} x(0) + \int_0^t e^{-a(t-s)} b s^m \, ds, \tag{7.11}$$

which simplifies to

$$x(t) = be^{-at} \int_0^t s^m e^{as} \, ds$$

since $x(0) = 0$. Integrating by parts or consulting a table of integrals, such as [9, 2.321], we find that

$$\int s^m e^{as} \, ds = e^{as} \sum_{k=0}^{m} (-1)^k \frac{k!}{a^{k+1}} \binom{m}{k} s^{m-k}.$$

Hence,

$$x(t) = be^{-at}\int_0^t s^m e^{as}\,ds = b\sum_{k=0}^{m}(-1)^k\frac{k!}{a^{k+1}}\binom{m}{k}t^{m-k} - b(-1)^m\frac{m!}{a^{m+1}}e^{-at}$$
$$= \frac{b}{a}\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!a^k}t^{m-k} + \frac{b}{a}(-1)^{m+1}\frac{m!}{a^m}e^{-at}.$$

With an appropriate change in the index of summation, we see that this is equivalent to (7.9). In sum, we have shown that (7.9) does in fact solve the classical initial value problem (7.10).

Because of Corollary 6.3 and Theorem 7.3, we can express the convolution of t^p for $p > -1$ and the resolvent R in terms of two-parameter Mittag-Leffler functions. This is the content of the next result.

Corollary 7.6. *Let $m \in \mathbb{N}_0$. Let R be the resolvent of* (R_λ)*; that is,*

$$R(t) = \lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)$$

where $\lambda > 0$ and $0 < q < 1$. Then for $p > -1$,

$$\int_0^t (t-s)^p R(s)\,ds$$
$$= \begin{cases} t^{q-1} - \Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q) & \text{if} \quad p = q-1 \\ \lambda\Gamma(q)m!\,t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q) & \text{if} \quad p = m \\ t^{q+m} - \Gamma(q+m+1)t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q) & \text{if} \quad p = m+q. \end{cases}$$

Proof. Suppose $p = q-1$. Then it follows from (R_λ) and (1.3) that

$$\int_0^t (t-s)^{q-1}R(s)\,ds = t^{q-1} - \frac{1}{\lambda}R(t) = t^{q-1} - \frac{1}{\lambda}\left[\lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)\right]$$
$$= t^{q-1} - \Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q).$$

Now suppose $p = m$ where $m \in \mathbb{N}_0$. Then from Theorem 7.3 we have

$$\int_0^t (t-s)^m R(s)\,ds = am!\,t^{q+m}E_{q,q+m+1}(-at^q)$$
$$= \lambda\Gamma(q)m!\,t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q)$$

since $a = \lambda\Gamma(q)$.

Finally consider the case when $p = m+q$. Then it follows from the previous case and Corollary 6.3 that

$$\int_0^t (t-s)^{m+q}R(s)\,ds$$
$$= t^{q+m} - \frac{1}{\lambda}\cdot\frac{\Gamma(q+m+1)}{\Gamma(q)m!}\cdot\lambda\Gamma(q)m!\,t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q)$$
$$= t^{q+m} - \Gamma(q+m+1)t^{q+m}E_{q,q+m+1}(-\lambda\Gamma(q)t^q). \qquad \square$$

Our final result employs Theorem 7.3 to obtain the unique continuous solution of

$$D^q x(t) = -ax(t) + g(t) \tag{1.2}$$

on the interval $[0,\infty)$ when $g(t)$ is a given polynomial.

Theorem 7.7. *Let $q \in (0,1)$ and $a > 0$. Let $n \in \mathbb{N}_0$ and $b_m \in \mathbb{R}$ for $m = 0,1,2,\ldots,n$. The fractional relaxation equation*

$$D^q x(t) = -ax(t) + \sum_{m=0}^{n} b_m t^m \tag{7.12}$$

has one and only one continuous solution on $[0,\infty)$, namely,

$$x(t) = \sum_{m=0}^{n} b_m m! t^{q+m} E_{q,q+m+1}(-at^q). \tag{7.13}$$

Proof. For $m = 0,1,\ldots,n$, let x_m denote the continuous solution of

$$D^q x(t) = -ax(t) + b_m t^m$$

on $[0,\infty)$, whose existence and uniqueness was established with Theorem 7.3. It is clear from (2.7) that D^q is a linear operator. Consequently,

$$\begin{aligned} D^q\left(\sum_{m=0}^{n} x_m(t)\right) &= \sum_{m=0}^{n} D^q x_m(t) = \sum_{m=0}^{n}(-ax_m(t) + b_m t^m) \\ &= -a\sum_{m=0}^{n} x_m(t) + \sum_{m=0}^{n} b_m t^m = -a\sum_{m=0}^{n} x_m(t) + g(t) \end{aligned}$$

where

$$g(t) := \sum_{m=0}^{n} b_m t^m. \tag{7.14}$$

Thus $x(t) := \sum_{m=0}^{n} x_m(t)$ is a continuous solution of (7.12) on $[0,\infty)$.

As for uniqueness, suppose that $y(t)$ is also a continuous solution. Applying the operator D^q to

$$z(t) := x(t) - y(t),$$

we get

$$D^q z(t) = -ax(t) + g(t) - [-ay(t) + g(t)] = -a[x(t) - y(t)] = -az(t)$$

for $t \geq 0$. It follows from Lemma 4.4 that $z(t) \equiv 0$. In other words, $y(t) \equiv x(t)$ on $[0,\infty)$.

Finally, we obtain (7.13) from (7.7). □

Example 7.8. The equation

$$D^{1/2} x(t) = -x(t) + 1 - 3t - 2t^2 + t^3 \tag{7.15}$$

has the unique continuous solution

$$x(t) = t^3 - \frac{16}{5\sqrt{\pi}} t^{5/2} + t^2 - \frac{8}{3\sqrt{\pi}} t^{3/2} - t + \frac{2}{\sqrt{\pi}} t^{1/2} \tag{7.16}$$

on the interval $[0,\infty)$.

Proof. Referring to (7.12) and (7.14), we have $q = 1/2$, $a = 1$, and

$$g(t) = \sum_{m=0}^{3} b_m t^m = 1 - 3t - 2t^2 + t^3 \tag{7.17}$$

where

$$b_0 = 1,\ b_1 = -3,\ b_2 = -2,\ b_3 = 1.$$

Accordingly, we see from (7.13) that

$$\begin{aligned} x(t) &= \sum_{m=0}^{3} b_m m! t^{\frac{1}{2}+m} E_{\frac{1}{2},\frac{1}{2}+m+1}(-\sqrt{t}) \\ &= t^{\frac{1}{2}} E_{\frac{1}{2},\frac{3}{2}}(-\sqrt{t}) - 3t^{\frac{3}{2}} E_{\frac{1}{2},\frac{5}{2}}(-\sqrt{t}) - 4t^{\frac{5}{2}} E_{\frac{1}{2},\frac{7}{2}}(-\sqrt{t}) + 6t^{\frac{7}{2}} E_{\frac{1}{2},\frac{9}{2}}(-\sqrt{t}) \end{aligned}$$

is the unique continuous solution of (7.15) on $[0, \infty)$.

Each of these four terms can be expressed as a finite sum of powers of t and a constant multiple of $e^t \operatorname{erf}(\sqrt{t})$. For instance, consider the second term. From (1.4) we have

$$E_{\frac{1}{2},\frac{5}{2}}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\frac{1}{2}k + \frac{5}{2})}. \tag{7.18}$$

By the Cauchy-Hadamard formula for convergence and Stirling's formula for the Gamma function, the power series (1.4) defining $E_{\alpha,\beta}(z)$ converges absolutely for all z in the complex plane (cf. [6, p. 68]), and a fortiori for all real values of z. Consequently, we can rearrange the terms of (7.18) as follows:

$$E_{\frac{1}{2},\frac{5}{2}}(t) = \sum_{k=2}^{\infty} \frac{t^{2k-3}}{\Gamma(k+1)} + \sum_{k=2}^{\infty} \frac{t^{2k-4}}{\Gamma(k+\frac{1}{2})}.$$

In [1, (6.1.12)] we find the formula

$$\Gamma\left(k + \tfrac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)}{2^k} \Gamma\left(\tfrac{1}{2}\right).$$

Thus,

$$E_{\frac{1}{2},\frac{5}{2}}(t) = \sum_{k=2}^{\infty} \frac{t^{2k-3}}{k!} + \frac{1}{\sqrt{\pi}} \sum_{k=2}^{\infty} \frac{2^k t^{2k-4}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)}.$$

It then follows that

$$\begin{aligned} t^{\frac{3}{2}} E_{\frac{1}{2},\frac{5}{2}}(-\sqrt{t}) &= -\sum_{k=2}^{\infty} \frac{t^k}{k!} + \frac{1}{\sqrt{\pi}} \sum_{k=2}^{\infty} \frac{2^k t^{k-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)} \\ &= 1 + t - \sum_{k=0}^{\infty} \frac{t^k}{k!} - \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{2^k t^{k-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)} \\ &= 1 + t - e^t - \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} + \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{2^{k-1} t^{k-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2k-1)}. \end{aligned}$$

Changing the index of summation, we have

$$t^{\frac{3}{2}} E_{\frac{1}{2},\frac{5}{2}}(-\sqrt{t}) = 1 + t - e^t - \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{2^n t^{n+\frac{1}{2}}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n+1)}.$$

Employing the series expansion

$$e^{t^2} \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{2^n t^{2n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n+1)}$$

found in [1, (7.1.6)], we see that

$$-3t^{\frac{3}{2}} E_{\frac{1}{2},\frac{5}{2}}(-\sqrt{t}) = -3 - 3t + 3e^t + \frac{6}{\sqrt{\pi}} t^{\frac{1}{2}} - 3e^t \operatorname{erf}(\sqrt{t}). \tag{7.19}$$

Similar calculations yield the following:

$$t^{\frac{1}{2}}E_{\frac{1}{2},\frac{3}{2}}(-\sqrt{t}) = 1 - e^t + e^t\,\mathrm{erf}(\sqrt{t}), \tag{7.20}$$

$$-4t^{\frac{5}{2}}E_{\frac{1}{2},\frac{7}{2}}(-\sqrt{t}) = -4 - 4t - 2t^2 + 4e^t + \frac{16}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{8}{\sqrt{\pi}}\,t^{\frac{1}{2}} - 4e^t\,\mathrm{erf}(\sqrt{t}), \tag{7.21}$$

and

$$\begin{aligned} 6t^{\frac{7}{2}}E_{\frac{1}{2},\frac{9}{2}}(-\sqrt{t}) &= 6 + 6t + 3t^2 + t^3 - 6e^t - \frac{16}{5\sqrt{\pi}}t^{\frac{5}{2}} - \frac{8}{\sqrt{\pi}}\,t^{\frac{3}{2}} \\ &\quad - \frac{12}{\sqrt{\pi}}\,t^{\frac{1}{2}} + 6e^t\,\mathrm{erf}(\sqrt{t}). \end{aligned} \tag{7.22}$$

Adding together the terms (7.19)–(7.22), we obtain (7.16). □

As in Remark 7.5, let us compare the solution (7.16) of the fractional relaxation equation (7.15) with the solution of the initial value problem

$$y'(t) = -y(t) + 1 - 3t - 2t^2 + t^3, \quad y(0) = 0. \tag{7.23}$$

Applying the variation of parameters formula or simply multiplying the differential equation by the integrating factor e^t and then integrating by parts and using the initial condition $y(0) = 0$, we obtain the solution

$$y(t) = -6 + 7t - 5t^2 + t^3 + 6e^{-t}. \tag{7.24}$$

The graph of the solution $y(t)$ (dashed curve) is shown in Figure 2. The solid curve is the graph of the solution (7.16) of the fractional relaxation equation (7.15). The curve that begins at $(0, 1)$ (dotted curve) is the graph of the polynomial (7.17).

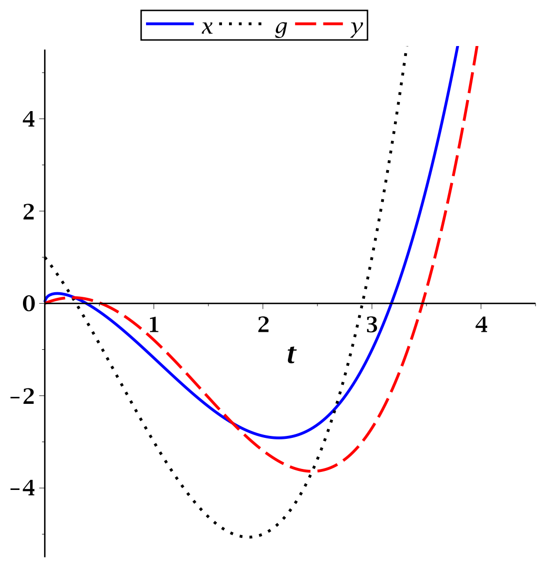

Figure 2: Graphs of (7.16), (7.17), and (7.24).

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