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Existence of periodic solutions for two types of second-order nonlinear neutral integro-differential equations with infinite distributed mixed-delays

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Abstract

We consider two types of second-order neutral functional differential equations with infinite distributed delays and offer existence criteria for periodic solutions. During the process we invert the integro-differential equations into equivalent integral equations and derive suitable fixed point mappings. We show that these mappings fit into the framework of Schauder's fixed point theorem so that periodic solutions are readily obtained.

Keywords: Nonlinear neutral differential equations; periodic solutions; fixed point theorem; distributed delays.

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1. Introduction

All biological systems and processes take time delays to complete. The delays can represent gestation times, incubation periods, or transport delays. In many cases time delays can be substantial such as gestation and maturation or can represent little lags such as acceleration and deceleration in physical processes. Therefore, it become natural to include time delay terms into the differential equations that model population

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dynamics. Such models are referred as delay differential equation models. Thus, it seems clear that ordinary differential models are, at best, approximations of real word problems. In the last fifty years, delay models are becoming more common, appearing in many branches of biological, economical and physical modeling (see [1]-[12], [15]-[18]). This is due to their advantage of combining a simple, intuitive derivation with a wide variety of possible behavior regimes and to the fact that such models operate on an infinite dimensional space consisting of continuous functions that accommodate high dimensional dynamics (see [15]-[16]). More recently investigators have given special attentions to the study of equations in which the delay occurs in the derivative of the state variable as well as in the independent variable, so called neutral differential equations (see, [1]-[10], [12], [13], [17]). As known by Hale [15], Hale and Lunel [16], neutral delay differential equations appear as models of electrical networks which contain lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits.

Existence and periodicity of solutions of functional differential equations are of great interest in mathematics and its applications to the modeling of various practical problems and have been extensively studied in recent times (see [1]-[13], [17]-[18] and references therein).

The study on neutral functional differential equations is more intricate than ordinary delay differential equations. This is why the studies of periodic solutions for neutral differential equations are relatively less than those devoted to ordinary differential equation. Most of the investigations on neutral type equations are confined to first order neutral differential equations. Very recently, Wu and Wang (see [18]) discussed the second order neutral delay differential equation

$$(x(t) - cx(t - \delta))'' + a(t)x(t) = \lambda b(t)f(x(t - \tau(t))),$$

where λ is a positive parameter, δ and c are constants with $|c| \neq 1$, $a(\cdot)$, $b(\cdot)$ are continuous positive functions, $f \in C([0, \infty), [0, \infty))$ and $a(\cdot)$, $b(\cdot)$, $\tau(\cdot)$ are periodic functions.

We consider the following two types of second-order neutral functional integro-differential equations with infinite distributed mixed-delays

$$\begin{aligned} & \frac{d^2}{dt^2} \left(x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ &= a(t)x(t) - b(t) \left(\int_{-\infty}^t p(t, s) h_1(x(s)) ds + \int_t^{\infty} q(t, s) h_2(x(s)) ds \right), \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} & \frac{d^2}{dt^2} \left(x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ &= -a(t)x(t) + b(t) \left(\int_{-\infty}^t p(t, s) h_1(x(s)) ds + \int_t^{\infty} q(t, s) h_2(x(s)) ds \right). \end{aligned} \quad (1.2)$$

Here $a, b \in C(\mathbb{R}, \mathbb{R}^+)$ are T -periodic functions, $p : \Delta^- \rightarrow \mathbb{R}^+$, $q : \Delta^+ \rightarrow \mathbb{R}^+$, $p(t+T, s+T) = p(t, s)$, $q(t+T, s+T) = q(t, s)$, $f(t+T, s+T, x) = f(t, s, x)$ and $g(t+T, s+T, x) = g(t, s, x)$ with

$$\Delta^- := \{(t, s) \mid s \leq t\} \text{ and } \Delta^+ := \{(t, s) \mid t \leq s\}.$$

Also we assume that $f : \Delta^- \times \mathbb{R} \rightarrow \mathbb{R}^+$, $g : \Delta^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$ and $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$ are uniformly continuous functions at x .

Special cases of (1.1) and (1.2) have been considered and investigated by many authors. Particularly, W. Han and J. Ren in [17], Ardjouni et al. in [1], have, by choosing available operators and applying Krasnoselskii's fixed-point theorem, obtained sufficient conditions providing existence of periodic solutions to special cases of equations (1.1) and (1.2).

The main features of this exposition are the following. In first section we introduce some notations and lemmas and state some preliminary results needed in later sections. Then we give the Green’s function of (1.1) and (1.2), which plays an important role in our investigation. Also, we present the inversions of (1.1) and (1.2) and Schauder’s fixed point theorem. For details on Schauder’s theorem we refer the reader to [14]. In the last section, we present our main results on existence of periodic solutions of (1.1) and (1.2).

2. Preliminaries

For $T > 0$, let C_T be the set of all continuous scalar functions x that are periodic in t with period T . Then, endowed with the supremum norm,

$$\|x\| := \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0, T]} |x(t)|,$$

$(C_T, \|\cdot\|)$ is a Banach space. Define

$$C_T^- := \{x \in C_T \mid x < 0\}, \quad C_T^+ = \{x \in C_T \mid x > 0\},$$

and

$$C_T^J := \{\varphi \in C_T \mid \|\varphi\| \leq J\}.$$

Denote

$$M := \sup \{a(t) \mid t \in [0, T]\}, \quad m := \inf \{a(t) \mid t \in [0, T]\}, \quad \beta := \sqrt{M},$$

$$\lambda_1 := \frac{\exp\left(-\frac{\beta T}{2}\right)}{\beta(1 - \exp(-\beta T))}, \quad \mu_1 := \frac{1 + \exp(-\beta T)}{2\beta(1 - \exp(-\beta T))},$$

and

$$\lambda_2 := \frac{\cos\left(\frac{\beta T}{2}\right)}{2\beta \sin\left(\frac{\beta T}{2}\right)}, \quad \mu_2 := \frac{1}{2\beta \sin\left(\frac{\beta T}{2}\right)}.$$

Throughout this section we let

$$H_1(t, s, x) := b(t)p(t, s)h_1(x(s)) - a(t)f(t, s, x(s+t)) \geq 0, \quad \text{on } \Delta^- \times C_T^J,$$

$$H_2(t, s, x) := b(t)q(t, s)h_2(x(s)) - a(t)g(t, s, x(s+t)) \geq 0, \quad \text{on } \Delta^+ \times C_T^J.$$

Also, in order to simplify notation, we define the function H by

$$H(t) := \int_{-\infty}^t H_1(t, s, x(s)) ds + \int_t^{\infty} H_2(t, s, x(s)) ds.$$

Clearly, H is a positive continuous function on \mathbb{R} .

Lemma 2.1 ([13]). *The equation*

$$\frac{d^2}{dt^2}y(t) - My(t) = h(t), \quad h \in C_T^-,$$

has a unique T -periodic solution

$$y(t) = \int_t^{t+T} K_1(t, s)(-h(s)) ds,$$

where

$$K_1(t, s) := \frac{\exp(-\beta(s-t)) + \exp(\beta(s-t-T))}{2\beta(1 - \exp(-\beta T))}, \quad s \in [t, t+T].$$

Lemma 2.2 ([13]). $\lambda_1 \leq K_1(t, s) \leq \mu_1$ and $\int_t^{t+T} K_1(t, s) ds = \frac{1}{M}$ for all $t \in [0, T]$ and $s \in [t, t + T]$.

Define the operator $P_1 : C_T \rightarrow C_T$ by

$$(P_1x)(t) := (I - T_1B_1)^{-1} T_1 \left(\int_{-\infty}^t [-H_1(t, s, x)] ds + \int_t^{\infty} [-H_2(t, s, x)] ds \right), \tag{2.1}$$

where

$$(T_1h)(t) = \int_t^{t+T} K_1(t, s) (-h(s)) ds \text{ and } (B_1y)(t) = [a(t) - M]y(t).$$

Lemma 2.3 ([13]). *The equation*

$$\frac{d^2}{dt^2}y(t) - a(t)y(t) = h(t), h \in C_T^-,$$

has a unique T -periodic solution

$$y(t) = (P_1h)(t) = (I - T_1B_1)^{-1} T_1h(t).$$

Lemma 2.4 ([13]). P_1 satisfies

$$0 < (T_1h)(t) \leq (P_1h)(t) \leq \frac{M}{m} \|T_1h\|, h \in C_T^-.$$

The following lemma is essential for our results on existence of periodic solution of (1.1).

Lemma 2.5. *If $x \in C_T$, then x is a solution of equation (1.1) if and only if*

$$x(t) = \int_{-\infty}^t f(t, s, x(s+t)) ds + \int_t^{\infty} g(t, s, x(s+t)) ds + (P_1x)(t),$$

where P_1 is the map given by (2.1).

Proof. Let $x \in C_T$ be a solution of (1.1). Equation (1.1) can be rewritten as

$$\begin{aligned} & \frac{d^2}{dt^2} \left(x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & - M \left(x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & = [-M + a(t)] \left(x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & + a(t) \left(\int_{-\infty}^t f(t, s, x(s+t)) ds + \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & - b(t) \left(\int_{-\infty}^t p(t, s) h_1(x(s)) ds + \int_t^{\infty} q(t, s) h_2(x(s)) ds \right). \end{aligned} \tag{2.2}$$

Taking

$$y(t) = x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds.$$

Then, (2.2) is transformed into

$$\begin{aligned} & \frac{d^2}{dt^2}y(t) - My(t) \\ & = (B_1y)(t) + a(t) \left(\int_{-\infty}^t f(t, s, x(s+t)) ds + \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & - b(t) \left(\int_{-\infty}^t p(t, s) h_1(x(s)) ds + \int_t^{\infty} q(t, s) h_2(x(s)) ds \right) \\ & = h(t). \end{aligned} \tag{2.3}$$

From Lemma 2.1, we have

$$y(t) = (T_1 B_1) y(t) + T_1 \left(\int_{-\infty}^t a(t) f(t, s, x(s+t)) ds + \int_t^{\infty} a(t) g(t, s, x(s+t)) ds \right) - T_1 \left(\int_{-\infty}^t b(t) p(t, s) h_1(x(s)) ds + \int_t^{\infty} b(t) q(t, s) h_2(x(s)) ds \right).$$

This yields

$$(I - T_1 B_1) y(t) = T_1 \left(\int_{-\infty}^t [-H_1(t, s, x)] ds + \int_t^{\infty} [-H_2(t, s, x)] ds \right).$$

Therefore, since $\|T_1 B_1\| \leq 1 - \frac{m}{M} < 1$, we have

$$y(t) = (I - T_1 B_1)^{-1} T_1 \left(\int_{-\infty}^t [-H_1(t, s, x)] ds + \int_t^{\infty} [-H_2(t, s, x)] ds \right).$$

Obviously

$$x(t) = \int_{-\infty}^t f(t, s, x(s+t)) ds + \int_t^{\infty} g(t, s, x(s+t)) ds + (I - T_1 B_1)^{-1} T_1 \left(\int_{-\infty}^t [-H_1(t, s, x)] ds + \int_t^{\infty} [-H_2(t, s, x)] ds \right).$$

□

It is clear that $y(t)$ is the unique T -periodic solution of (2.3) for $h \in C_T^-$.

Lemma 2.6 ([13]). *The equation*

$$\frac{d^2}{dt^2} y(t) + M y(t) = h(t), \quad h \in C_T^+,$$

has a unique T -periodic solution

$$y(t) = \int_t^{t+T} K_2(t, s) h(s) ds,$$

where

$$K_2(t, s) = \frac{\cos \beta \left(\frac{T}{2} + t - s \right)}{2\beta \cos \frac{\beta T}{2}}, \quad s \in [t, t + T].$$

Lemma 2.7 ([13]). $\int_t^{t+T} K_2(t, s) ds = \frac{1}{M}$. Furthermore, if $M < \left(\frac{\pi}{T}\right)^2$ then $\lambda_2 \leq K_2(t, s) \leq \mu_2$ for all $t \in [0, T]$ and $s \in [t, t + T]$.

Next, define the operator $P_2 : C_T \rightarrow C_T$ by

$$(P_2 \varphi)(t) := (I - T_2 B_2)^{-1} T_2 \left(\int_{-\infty}^t H_1(t, s, \varphi(s)) ds + \int_t^{\infty} H_2(t, s, \varphi(s)) ds \right), \tag{2.4}$$

where

$$(T_2 h)(t) = \int_t^{t+T} K_2(t, s) h(s) ds \text{ and } (B_2 y)(t) = [M - a(t)] y(t).$$

Lemma 2.8 ([13]). *The equation*

$$\frac{d^2}{dt^2}y(t) + a(t)y(t) = h(t), h \in C_T^+,$$

has a unique T-periodic solution

$$y(t) = (P_2h)(t) = (I - T_2B_2)^{-1}T_2h(t).$$

Lemma 2.9 ([13]). *If $M < (\frac{\pi}{T})^2$ then*

$$0 < (T_2h)(t) \leq (P_2h)(t) \leq \frac{M}{m} \|T_2h\|, h \in C_T^+.$$

The following lemma is essential for our results on existence of periodic solution of (1.2).

Lemma 2.10. *If $x \in C_T$ then x is a solution of equation (1.2) if and only if*

$$x(t) = \int_{-\infty}^t f(t, s, x(s+t)) ds + \int_t^{\infty} g(t, s, x(s+t)) ds + (P_2x)(t),$$

where P_2 is the map given by (2.4).

Proof. Let $x \in C_T$ be a solution of (1.2). Equation (1.2) can be rewritten as

$$\begin{aligned} & \frac{d^2}{dt^2} \left(x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & + M \left(x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & = (M - a(t)) \left(x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & - a(t) \left(\int_{-\infty}^t f(t, s, x(s+t)) ds + \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & + b(t) \left(\int_{-\infty}^t p(t, s) h_1(x(s)) ds + \int_t^{\infty} q(t, s) h_2(x(s)) ds \right). \end{aligned} \tag{2.5}$$

Taking

$$y(t) = x(t) - \int_{-\infty}^t f(t, s, x(s)) ds - \int_t^{\infty} g(t, s, x(s)) ds,$$

then (2.5) is transformed into

$$\begin{aligned} & \frac{d^2}{dt^2}y(t) + My(t) \\ & = (B_2y)(t) - a(t) \left(\int_{-\infty}^t f(t, s, x(s+t)) ds + \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & + b(t) \left(\int_{-\infty}^t p(t, s) h_1(x(s)) ds + \int_t^{\infty} q(t, s) h_2(x(s)) ds \right) \\ & = h(t). \end{aligned} \tag{2.6}$$

From Lemma 2.6, we have

$$\begin{aligned} y(t) &= (T_2B_2)y(t) \\ & - T_2 \left(\int_{-\infty}^t a(t) f(t, s, x(s+t)) ds + \int_t^{\infty} a(t) g(t, s, x(s+t)) ds \right) \\ & + T_2 \left(\int_{-\infty}^t b(t) p(t, s) h_1(x(s)) ds + \int_t^{\infty} b(t) q(t, s) h_2(x(s)) ds \right). \end{aligned}$$

This yields

$$(I - T_2 B_2) y(t) = T_2 \left(\int_{-\infty}^t H_1(t, s, x(s)) ds + \int_t^\infty H_2(t, s, x(s)) ds \right).$$

Therefore, since $\|T_2 B_2\| \leq 1 - \frac{m}{M} < 1$, we obtain

$$y(t) = (I - T_2 B_2)^{-1} T_2 \left(\int_{-\infty}^t H_1(t, s, x(s)) ds + \int_t^\infty H_2(t, s, x(s)) ds \right).$$

Obviously

$$\begin{aligned} x(t) &= \int_{-\infty}^t f(t, s, x(s+t)) x(s) ds + \int_t^\infty g(t, s, x(s+t)) ds \\ &\quad + (I - T_2 B_2)^{-1} T_2 \left(\int_{-\infty}^t H_1(t, s, x(s)) ds + \int_t^\infty H_2(t, s, x(s)) ds \right). \end{aligned}$$

□

It is obvious that $y(t)$ is the unique T -periodic solution of (2.6) for $h \in C_T^+$.

3. Periodic solutions

In this section we offer existence criteria for the periodic solutions of the second-order nonlinear neutral integro-differential equations with infinite distributed delay. Lastly in this section we state the Schauder fixed point theorem which enables us to prove the existence of periodic solutions to (1.1) and (1.2). For its proof we refer the reader to [14]. This section is devoted to results concerning the condition (3.2) below. We already know, from Lemma (2.5), that the existence of periodic solutions for (1.1) is equivalent to the existence of solutions for the operator equation $x = D_1 x$ (3.1), that is, the fixed points in C_T^J of D_1 . So we assume, for any $J > 0$, that there are continuous functions $F_J : \Delta^- \rightarrow \mathbb{R}^+$ and $G_J : \Delta^+ \rightarrow \mathbb{R}^+$ such that

$$\begin{aligned} |f(t, s, \varphi)| &\leq F_J(t, s), \quad (t, s, \varphi) \in \Delta^- \times C_T^J, \\ F_J(t + T, s + T) &= F_J(t, s), \quad (t, s) \in \Delta^-, \end{aligned}$$

and

$$\begin{aligned} |g(t, s, \varphi)| &\leq G_J(t, s), \quad (t, s, \varphi) \in \Delta^+ \times C_T^J, \\ G_J(t + T, s + T) &= G_J(t, s), \quad (t, s) \in \Delta^+. \end{aligned}$$

Whenever necessary, we shall consider

$$\int_{-\infty}^t H_1(t, s, \varphi(s)) ds + \int_t^\infty H_2(t, s, \varphi(s)) ds \leq \eta.$$

We define an operator D_1 on C_T as follows, $\varphi \in C_T$ implies that

$$(D_1 \varphi)(t) = \int_{-\infty}^t f(t, s, \varphi(s+t)) ds + \int_t^\infty g(t, s, \varphi(s+t)) ds + (P_1 \varphi)(t). \tag{3.1}$$

It is easy to check that $(D_1 \varphi)(t + T) = (D_1 \varphi)(t)$.

We are now ready to state existence T -periodic solution criteria for (1.1)–(1.2).

Proposition 3.1. *Assume that*

$$\int_{-\infty}^{t-\tau} F_J(t, s) ds + \int_{t+\tau}^\infty G_J(t, s) ds \text{ converges to } 0, \tag{3.2}$$

as $\tau \rightarrow \infty$, uniformly for $t \in \mathbb{R}$. Then, the set $\{D_1 \varphi \mid \varphi \in C_T^J\}$ is equicontinuous.

Proof. For this purpose, we show that for each $\varepsilon > 0$ there is a continuous increasing positive function $\delta = \delta_J(\varepsilon) : (0, \infty) \rightarrow (0, \infty)$ with

$$|(D_1\varphi)(t_1) - (D_1\varphi)(t_2)| \leq \varepsilon, \text{ if } \varphi \in C_T^J \text{ and } t_1 < t_2 < t_1 + \delta. \tag{3.3}$$

First we prove that there is a continuous increasing positive function $\delta^- = \delta_J^-(\varepsilon) : (0, \infty) \rightarrow (0, \infty)$ with

$$|(A^-\varphi)(t_1) - (A^-\varphi)(t_2)| \leq \frac{\varepsilon}{3} \text{ for } \varphi \in C_T^J \text{ and } t_1 < t_2 < t_1 + \delta^-, \tag{3.4}$$

where A^- is defined by

$$(A^-\varphi)(t) := \int_{-\infty}^t f(t, s, \varphi(s+t)) ds, \quad t \in \mathbb{R}.$$

From (3.2), for any $\varepsilon > 0$, there is a $\tau > 0$ such that

$$(A^-\varphi)(t) \leq \int_{-\infty}^{t-\tau} F_J(t, s) ds \leq \frac{\varepsilon}{12}, \quad t \in \mathbb{R}.$$

For any $\varphi \in C_T^J$ and t_1 and t_2 with $t_1 < t_2$, we have

$$\begin{aligned} & |(A^-\varphi)(t_1) - (A^-\varphi)(t_2)| \\ &= \left| \int_{-\infty}^{t_1} f(t_1, s, \varphi(s+t_1)) ds - \int_{-\infty}^{t_2} f(t_2, s, \varphi(s+t_2)) ds \right| \\ &\leq \int_{t_1-\tau}^{t_1} |f(t_1, s, \varphi(s+t_1)) - f(t_2, s, \varphi(s+t_2))| ds + \int_{-\infty}^{t_1-\tau} F_J(t_1, s) ds \\ &\quad + \int_{-\infty}^{t_1-\tau} F_J(t_2, s) ds + \int_{t_1}^{t_2} F_J(t_2, s) ds \\ &\leq \int_{t_1-\tau}^{t_1} |f(t_1, s, \varphi(s+t_1)) - f(t_2, s, \varphi(s+t_2))| ds + \frac{\varepsilon}{6} + \int_{t_1}^{t_2} F_J(t_2, s) ds. \end{aligned}$$

Since $f(t, s, \varphi)$ is uniformly continuous on $U = \{(t, s, \varphi) \mid t - \tau \leq s \leq t, \text{ and } |\varphi| \leq J\}$, one can, for the $\varepsilon > 0$ previously fixed, find a δ_1 such that $\delta_1 \in (0, 1)$ and

$$|f(t_1, s, \varphi) - f(t_2, s, \varphi)| \leq \frac{\varepsilon}{12\tau} \text{ if } (t_1, s, \varphi), (t_2, s, \varphi) \in U \text{ with } |t_1 - t_2| \leq \delta_1.$$

Hence, if $t_1 < t_2 < t_1 + \delta_1$, then we obtain

$$\int_{t_1-\tau}^{t_1} |f(t_1, s, \varphi(s+t_1)) - f(t_2, s, \varphi(s+t_2))| ds \leq \frac{\varepsilon}{12}.$$

Now let $l := \sup \{F_J(t_2, s) \mid t_1 \leq s \leq t_2\}$. Then, for the ε , there is a δ_2 such that $0 < \delta_2 < \min(\frac{\varepsilon}{l}, 1)$ and

$$\int_{t_1}^{t_2} F_J(t_2, s) ds \leq \frac{\varepsilon}{12} \text{ if } t_1 < t_2 < t_1 + \delta_2.$$

Thus, letting $\delta_3 = \min(\delta_1, \delta_2)$, we see that (3.4) is valid with $\delta^- = \delta_3$. Since we may assume that $\delta_3(\varepsilon)$ is nondecreasing, we can easily conclude that there is a continuous increasing function $\delta^- = \delta_J^- : (0, \infty) \rightarrow (0, \infty)$ which satisfies (3.4).

Similarly, one can prove that there is a continuous increasing function $\delta^+ = \delta_J^+ : (0, \infty) \rightarrow (0, \infty)$ with

$$|(A^+\varphi)(t_1) - (A^+\varphi)(t_2)| \leq \frac{\varepsilon}{3} \text{ if } t_1 < t_2 < t_1 + \delta^+, \tag{3.5}$$

where A^+ is defined by

$$(A^+\varphi)(t) := \int_t^\infty g(t, s, \varphi(s+t)) ds, \quad t \in \mathbb{R}.$$

Finally, it is clear that there is a continuous increasing function $\delta_4 = \delta_4(\varepsilon) : (0, \infty) \rightarrow (0, \infty)$ with

$$|(P_1\varphi)(t_1) - (P_1\varphi)(t_2)| \leq \frac{\varepsilon}{3} \text{ if } t_1 < t_2 < t_1 + \delta_4 \text{ and } \delta_4 < T.$$

Indeed, note that

$$H(s) = \int_{-\infty}^t H_1(t, s, x(s)) ds + \int_t^\infty H_2(t, s, x(s)) ds,$$

and let $\varepsilon > 0$. Choosing $0 < \delta_4 < 1$ such that for $t_1 < s < t_2 < t_1 + \delta_4$,

$$|K_1(t_1, s) - K_1(t_2, s)| \leq \frac{m\varepsilon}{6\eta MT}, \quad t_1 < s < t_2 < t_1 + \delta_4,$$

and

$$|K_1(t_1, s) - K_1(t_2, s+T)| \leq \frac{m\varepsilon}{6\eta M}, \quad t_1 < s < t_2 < t_1 + \delta_4.$$

Therefore

$$\begin{aligned} & |(P_1\varphi)(t_1) - (P_1\varphi)(t_2)| \\ & \leq \frac{M}{m} \left| \int_{t_1}^{t_2} K_1(t_1, s) |H(s)| ds - \int_{t_1+T}^{t_2+T} K_1(t_2, s) |H(s)| ds \right. \\ & \quad \left. + \int_{t_2}^{t_1+T} |K_1(t_1, s) - K_1(t_2, s)| |H(s)| ds \right|. \end{aligned}$$

Since $H(s+T) = H(s)$, then

$$\int_{t_1+T}^{t_2+T} K_1(t_2, s) H(s) ds = \int_{t_1}^{t_2} K_1(t_2, s+T) H(s) ds.$$

Consequently,

$$\begin{aligned} & |(P_1\varphi)(t_1) - (P_1\varphi)(t_2)| \\ & \leq \eta \frac{M}{m} \int_{t_1}^{t_2} |K_1(t_1, s) - K_1(t_2, s+T)| ds \\ & \quad + \eta \frac{M}{m} \int_{t_2}^{t_1+T} |K_1(t_1, s) - K_1(t_2, s)| ds \\ & \leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}. \end{aligned} \tag{3.6}$$

It now follows, from (3.4) (3.5) and (3.6), that (3.3) holds for $\delta = \min(\delta_4, \delta^-, \delta^+)$. □

Theorem 3.2 (Schauder). [14, p. 31] *Let S be a closed convex bounded subset of a Banach space X . Assume that $A : S \rightarrow S$ is a compact operator. Then, A has at least one fixed point in S .*

Theorem 3.3. *Let the conditions of Proposition 3.1 holds. Suppose that there is an $\alpha > 0$ such that for $\varphi \in C_T^J$*

$$\int_{-\infty}^t H_1(t, s, \varphi(s)) ds + \int_t^\infty H_2(t, s, \varphi(s)) ds \leq m\alpha, \tag{3.7}$$

and

$$\int_{-\infty}^t F_J(t, s) ds + \int_t^\infty G_J(t, s) ds + \alpha \leq J. \tag{3.8}$$

Then, (1.1) has a T -periodic solution.

Proof. First, we see that C_T^J is a closed convex bounded subset of the Banach space C_T . Let $\varphi \in C_T^J$ and consider the map D_1 on C_T^J . We show that $D_1 : C_T^J \rightarrow C_T^J$. First the equation $(D_1\varphi)(t+T) = (D_1\varphi)(t)$ is not difficult to check. Let $\varphi \in C_T^J$

$$\begin{aligned} & |(D_1\varphi)(t)| \\ & \leq \int_{-\infty}^t F_J(t,s) ds + \int_t^\infty G_J(t,s) ds \\ & + \left\| (I - T_1B_1)^{-1} \right\| \max \int_t^{t+T} K_1(t,s) \left[\int_{-\infty}^s |-H_1(s,u,\varphi(u))| du \right. \\ & \left. + \int_s^\infty |-H_2(s,u,\varphi(u))| du \right] ds. \end{aligned}$$

Therefore, since $\|T_1B_1\| \leq 1 - \frac{m}{M} < 1$ we have $\left\| (I - T_1B_1)^{-1} \right\| \leq \frac{1}{1 - \|T_1B_1\|}$ but $1 - \|T_1B_1\| \geq 1 - 1 + \frac{m}{M}$ thus $\frac{1}{1 - \|T_1B_1\|} \leq \frac{M}{m}$. So, from the Lemma 2.4, (3.7) and from what we deduced above, it follows that

$$|(D_1\varphi)(t)| \leq \int_{-\infty}^t F_J(t,s) ds + \int_t^\infty G_J(t,s) ds + \frac{1}{m}m\alpha \leq J.$$

Remains to show D_1 is continuous on C_T^J . For that, let $\{\varphi_n\}$ be a sequence of elements in C_T^J such that $\varphi_n \rightarrow \varphi \in C_T^J$ as $n \rightarrow \infty$. We prove that the sequence $(D_1\varphi_n)$ converges to $(D_1\varphi) \in C_T^J$ as $n \rightarrow \infty$. To see this, let $\tau > 0$ be given. Since

$$\begin{aligned} & \int_{-\infty}^t |f(t,s,\varphi_n)| ds + \int_t^\infty |g(t,s,\varphi_n)| ds \\ & \leq \int_{-\infty}^t F_J(t,s) ds + \int_t^\infty G_J(t,s) ds \leq J - \alpha, \end{aligned}$$

and

$$\begin{aligned} & \int_{-\infty}^t |f(t,s,\varphi_n)| ds + \int_t^\infty |g(t,s,\varphi_n)| ds \\ & = \int_{-\infty}^{t-\tau} |f(t,s,\varphi_n)| ds + \int_{t-\tau}^t |f(t,s,\varphi_n)| ds \\ & + \int_t^{t+\tau} |g(t,s,\varphi_n)| ds + \int_{t+\tau}^\infty |g(t,s,\varphi_n)| ds, \end{aligned}$$

then, for all $n \geq 0$,

$$\begin{aligned} & \int_{t-\tau}^t |f(t,s,\varphi_n)| ds + \int_t^{t+\tau} |g(t,s,\varphi_n)| ds \\ & \leq \int_{-\infty}^t F_J(t,s) ds + \int_t^\infty G_J(t,s) ds \leq J - \alpha. \end{aligned}$$

Thus, by the dominated convergence theorem we deduce that

$$\int_{t-\tau}^t |f(t,s,\varphi_n)| ds + \int_t^{t+\tau} |g(t,s,\varphi_n)| ds \rightarrow \int_{t-\tau}^t |f(t,s,\varphi)| ds + \int_t^{t+\tau} |g(t,s,\varphi)| ds,$$

because f and g are continuous. Proposition 3.1 shows that the set $\{D_1\varphi \mid \varphi \in C_T^J\}$ is equicontinuous and the function D_1 is continuous. Thus D_1 is compact operator. The conditions of Schauder’s theorem are satisfied and there exists $x \in C_T^J$ with $D_1x = x$. Thus, x is a T -periodic solution of (1.1). \square

Example 3.4. Consider the following second-order neutral functional integro-differential equations with infinite distributed mixed-delays

$$\begin{aligned} & \frac{d^2}{dt^2} \left(x(t) - \int_{-\infty}^t f(t, s, x(s+t)) ds - \int_t^{\infty} g(t, s, x(s+t)) ds \right) \\ & = a(t)x(t) - b(t) \left(\int_{-\infty}^t p(t, s) h_1(x(s)) ds + \int_t^{\infty} q(t, s) h_2(x(s)) ds \right), \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} T &= \pi, \quad a(t) = \frac{1}{4} \sin^2(t) + 1, \quad b(t) = 0.4 \times \left(\frac{1}{4} \sin^2(t) + 1 \right), \\ p(t, s) &= e^{s-t}, \quad q(t, s) = e^{t-s}, \quad h_1(x) = \sin(x), \quad h_2(x) = \cos(x), \\ f(t, s, x) &= 0.5 \sin(x), \quad g(t, s, x) = 0.2 \cos(x). \end{aligned}$$

Clearly, all the hypotheses of Theorem 3.3 are satisfied. Then (3.9) has a π -periodic solution.

Using Lemmas 2.9 and 2.10 and reasoning as in last section, it is possible to establish the following theorem for the existence of T -periodic solution to (1.2).

Theorem 3.5. *Assume that the hypotheses of Theorem 3.3 hold. Then, (1.2) has a T -periodic solution.*

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