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The uniqueness of a positive solution to a higher-order nonlinear fractional differential equation with fractional multi-point boundary conditions

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Abstract

In this paper, we apply the iterative method to establish the existence of the positive solution for a type of nonlinear singular higher-order fractional differential equation with fractional multi-point boundary conditions. Explicit iterative sequences are given to approximate the solutions and the error estimations are also given. The result is illustrated with an example.

Keywords: Positive solution; Uniqueness; Iterative sequence; Green function.
2010 MSC: 34A08;34B10; 34B15.

1. Introduction

In this paper, we are interested in the existence of solutions for the nonlinear fractional differential equation

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

subject to the boundary conditions

$$u^{(i)}(0) = 0,\ 0 \le i \le n-2,\ D_{0+}^{\beta} u(1) = \sum_{j=1}^{p} a_j D_{0+}^{\beta} u(\eta_j), \tag{1.2}$$

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where D_{0+}^{α}, D_{0+}^{β} are the stantard Riemann-Liouville fractional derivative of order $\alpha \in (n-1, n]$, $\beta \in [1, n-2]$ $for\, n \in \mathbb{N}^*$ $and\, n \geq 3$, $f \in C((0,1) \times \mathbb{R}, \mathbb{R})$ is allowed to be singular at $t=0$ and/or $t=1$ and $a_j \in \mathbb{R}^+$, $j = 1, 2, \ldots, p$, $0 < \eta_1 < \eta_2 < \ldots < \eta_p < 1$, $for\, p \in \mathbb{N}^*$.

The first definition of fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus, was mentioned already in 1695 by Leibniz and L'Hospital. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For more details we refer the reader to $[2, 5, 6, 7, 19, 9, 21, 25, 26, 29, 30, 31, 36]$ and the references cited therein.

Boundary value problems for nonlinear differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. A point of central importance in the study of nonlinear boundary value problems is to understand how the properties of nonlinearity in a problem influence the nature of the solutions to the boundary value problems. The multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the sensors located, at intermediate points, see $[8, 10, 18, 32, 35]$ and the references therein.

The techniques of nonlinear analysis, as the main method to deal with the problems of nonlinear fractional differential equations, plays an essential role in the research of this field, such as establishing the existence and the uniqueness or the multiplicity of solutions for nonlinear fractional differential equations, see $[1, 3, 4, 6, 11, 12, 13, 14, 15, 16, 17, 22, 24, 27, 28, 34, 37, 38, 39]$ and the references therein. For instance, Zhang et al. [24] studied the existence of two positive solutions of following singular fractional boundary value problems:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & t \in (0,1), \\ u(0) = 0,\ D_{0+}^{\beta} u(0) = 0,\ D_{0+}^{\beta} u(1) = \sum\limits_{j=1}^{\infty} a_j D_{0+}^{\beta} u(\eta_j), \end{cases}$$

where D_{0+}^{α}, D_{0+}^{β} are the stantard Riemann-Liouville fractional derivative of order $\alpha \in (2,3]$, $\beta \in [1,2]$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ and a_j, $\eta_j \in (0,1)$, $\alpha - \beta \geq 1$ with $\sum\limits_{i=0}^{\infty} a_j \eta_j^{\alpha-\beta-1} < 1$.

In [23], the authors studied the boundary value problems of the fractional order differential equation:

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0,\ D_{0+}^{\beta} u(1) = a D_{0+}^{\beta} u(\eta), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \eta < 1$, $0 < a$, $\beta < 1$, $f \in C\left([0,1] \times \mathbb{R}^2, \mathbb{R}\right)$ and D_{0+}^{α}, D_{0+}^{β} are the stantard Riemann-Liouville fractional derivative of order α. They obtained the multiple positive solutions by the Leray-Schauder nonlinear alternative and the fixed point theorem on cones.

Inspired and motivated by the works mentioned above, we focus on the uniqueness of positive solutions for the nonlocal boundary value problem $(1.1) - (1.2)$ with the iterative method and properties of $f(t, u)$, explicit iterative sequences are given to approximate the solutions and the error estimations are also given. The rest of this paper is organized as follows. After this section, we present some notations and lemmas that will be used to prove our main result in Section 2. We discuss the uniqueness in Section 3. Finally, we give an example to illustrate our result.

2. Preliminaries

In this section, we recall some definitions and facts which will be used in the later analysis.

Definition 2.1. [31]. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. [21]. The Riemann-Liouville fractional derivative order $\alpha > 0$, $n-1 < \alpha < n$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is defined as

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds, \quad t > 0,$$

provided that the right integral converges.

Lemma 2.3. ([21])*(i) If $u \in L^p(0,1)$, $1 \le p \le +\infty$, $\beta > \alpha > 0$, then* $D_{0^+}^{\alpha} I_{0^+}^{\beta} u(t) = I_{0^+}^{\beta-\alpha} u(t)$, $D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$, $I_{0^+}^{\alpha} I_{0^+}^{\beta} u(t) = I_{0^+}^{\alpha+\beta} u(t)$.

(ii) If $\beta > \alpha > 0$, then $D_{0^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta) t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$.

Lemma 2.4. ([21]). *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then the differential equation*

$$D_{0^+}^{\alpha} u(t) = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, where $n-1 < \alpha < n$.

Lemma 2.5. ([21]). *Let $\alpha > 0$. Then the following equality holds for any $u \in L^1(0,1)$, $D_{0^+}^{\alpha} u \in L^1(0,1)$;*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, where $n-1 < \alpha \le n$.

In the following Lemma, we present the Green function of fractional differential equation boundary value problem

Lemma 2.6. *Let $\sum\limits_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1} \in [0,1)$, $\alpha \in (n-1, n]$, $\beta \in [1, n-2]$, $n \ge 3$ and $y \in C[0,1]$. Then the solution of the fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + y(t) = 0, \\ u^{(i)}(0) = 0, \ 0 \le i \le n-2, \\ D_{0^+}^{\beta} u(1) = \sum\limits_{j=1}^{p} a_j D_{0^+}^{\beta} u(\eta_j), \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds, \tag{2.2}$$

where

$$G(t,s) = g(t,s) + \frac{t^{\alpha-1}}{d}\sum_{j=1}^{p} a_j h(\eta_j, s), \tag{2.3}$$

$$g(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

$$h(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.5}$$

where $d = 1 - \sum\limits_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1}$.

Proof. By using Lemma 2.4, the solution of the equation $D_{0+}^{\alpha} u(t) + y(t) = 0$ is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_1, c_2..., c_n$ are arbitrary real constants.
By the boundary condition (2.1), one can $c_2 = c_3... = c_{n-2} = c_{n-1} = c_n = 0$ and

$$c_1 = \frac{1}{d\Gamma(\alpha)} \left[\int_0^1 (1-s)^{\alpha-\beta-1} y(s)\,ds - \sum_{j=1}^{p} a_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-\beta-1} y(s)\,ds \right].$$

Then, the unique solution of the problem (2.1) is given by

$$u(t) = \frac{t^{\alpha-1}}{d\Gamma(\alpha)} \left[\int_0^1 (1-s)^{\alpha-\beta-1} y(s)\,ds - \sum_{j=1}^{p} a_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-\beta-1} y(s)\,ds \right] - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds,$$

$$= \frac{1}{\Gamma(\alpha)} \left[\int_0^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1} y(s)\,ds - \int_0^t (t-s)^{\alpha-1} y(s)\,ds \right.$$

$$\left. + \frac{\sum\limits_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1}}{1 - \sum\limits_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1}} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1} y(s)\,ds + \sum_{j=1}^{p} a_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-\beta-1} y(s)\,ds \right],$$

$$= \frac{1}{\Gamma(\alpha)} \left[\int_0^t \left[t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \right] y(s)\,ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1} y(s)\,ds \right.$$

$$\left. + \frac{\sum\limits_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1}}{1 - \sum\limits_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1}} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1} y(s)\,ds + \sum_{j=1}^{p} a_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-\beta-1} y(s)\,ds \right],$$

$$= \int_0^1 g(t,s) y(s)\,ds + \frac{t^{\alpha-1}}{d} \sum_{j=1}^{p} a_j \left[\int_{\eta_j}^1 \eta_j^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} y(s)\,ds \right.$$

$$+\int_0^{\eta_j}\left[\eta_j^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}-(\eta_j-s)^{\alpha-\beta-1}\right]y(s)\,ds\Bigg],$$

$$=\int_0^1 g(t,s)\,y(s)\,ds+\frac{t^{\alpha-1}}{d}\sum_{j=1}^{p}a_j\int_0^1 h(\eta_j,s)\,y(s)\,ds,$$

$$=\int_0^1 G(t,s)\,y(s)\,ds.$$

The proof is complete. $\square$

Lemma 2.7. *Let* $\sum_{j=1}^{p}a_j\eta_j^{\alpha-\beta-1}\in[0,1)$, $\alpha\in(n-1,n]$, $\beta\in[1,n-2]$, $n\geq 3$. *Then, the functions* $g(t,s)$ *and* $h(t,s)$ *defined by* (2.4) *and* (2.5) *have the following properties:*
(i) *The functions* $g(t,s)$ *and* $h(t,s)$ *are continuous on* $[0,1]\times[0,1]$ *and for all* $t,\ s\in(0,1)$

$$g(t,s)>0,\ h(t,s)>0.$$

(ii) $g(t,s)\leq\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ *for all* $t,\ s\in[0,1]$.
(iii) $g(t,s)\geq t^{\alpha-1}g(1,s)$ *for all* $t,\ s\in[0,1]$, *where*

$$g(1,s)=\frac{1}{\Gamma(\alpha)}\left[(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}\right].$$

Proof. It is easy to chek that (i) holds. So we prove that (ii) is true. Note that (2.4) and $0\leq(1-s)^{\alpha-\beta-1}\leq 1$. It follows that $g(t,s)\leq\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for all $t,\ s\in[0,1]$. It remains to prove (iii). We divide the proof into two cases and by (2.4), we have
Case1. When $0\leq s\leq t\leq 1$, we have

$$g(t,s)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}\left[(1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-1}\right]\geq t^{\alpha-1}g(1,s).$$

Case2. When $0\leq t\leq s\leq 1$, we have

$$g(t,s)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1}\geq t^{\alpha-1}g(1,s).$$

Hence $g(t,s)\geq t^{\alpha-1}g(1,s)$ for all $t,\ s\in[0,1]$. $\square$

3. Existence results

First, for the uniqueness results of problem $(1.1)-(1.2)$, we need the following assumptions.
(A_1) $f(t,u_1)\leq f(t,u_2)$ for any $0<t<1,\ 0\leq u_1\leq u_2$.
(A_2) For any $r\in(0,1)$, there exists a constant $q\in(0,1)$ such that

$$f(t,ru)\geq r^q f(t,u),\quad (t,u)\in(0,1)\times[0,\infty).\tag{3.1}$$

(A_3) $0<\int_0^1 f\left(s,s^{\alpha-1}\right)ds<\infty$.
We shall consider the Banach space $E=C[0,1]$ equipped with the norm $\|u\|=\underset{0\leq t\leq 1}{max}|u(t)|$ and let

$$D=\left\{u\in C^+[0,1]:\exists M_u\geq m_u\geq 0,\ such\,that\,m_u t^{\alpha-1}\leq u(t)\leq M_u t^{\alpha-1},\ for\,t\in[0,1]\right\},\tag{3.2}$$

where

$$C^{+}[0,1]=\{u\in E\,:\,u(t)\geq 0,\ t\in[0,1]\}.$$

In view of Lemma 2.5, we define an operator T as

$$(Tu)(t)=\int_0^1 G(t,s)\,y(s)\,ds, \tag{3.3}$$

where $G(t,s)$ is given by (2.3).
By (A_1) it is easy to see that the operator $T\,:\,D\to C^{+}[0,1]$ is increasing.
Observe that the BVP (1.1) – (1.2) has a solution if and only if the operator T has a fixed point.
Obviously, from (A_1) we obtain

$$f(t,ru)\leq r^{q}f(t,u),\quad \forall r>1,\ q\in(0,1),\quad (t,u)\in(0,1)\times[0,\infty).$$

In what follows, we first prove $T\,:\,D\to D$. In fact, for any $u\in D$, there exist a positive constants $0<m_u<1<M_u$ such that

$$m_u s^{\alpha-1}\leq u(s)\leq M_u s^{\alpha-1},\quad s\in[0,1].$$

Then, from (A_1), $f(t,u)$ non-decreasing respect to u and (A_2), we can imply that for $s\in(0,1)$, $q\in(0,1)$

$$(m_u)^{q}\,f\left(s,s^{\alpha-1}\right)\leq f(s,u(s))\leq (M_u)^{q}\,f\left(s,s^{\alpha-1}\right),\quad s\in(0,1). \tag{3.4}$$

From (3.3) and Lemma 2.1, we obtain

$$\begin{aligned}
Tu(t)&=\int_0^1 g(t,s)\,f(s,u(s))\,ds+\frac{t^{\alpha-1}}{d}\sum_{j=1}^{p}a_j\int_0^1 h(\eta_j,s)\,f(s,u(s))\,ds,\\
&\leq t^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 f(s,u(s))\,ds+\frac{1}{d}\sum_{j=1}^{p}a_j\int_0^1 h(\eta_j,s)\,f(s,u(s))\,ds\right],\\
&\leq t^{\alpha-1}\left[\frac{(M_u)^{q}}{\Gamma(\alpha)}\int_0^1 f\left(s,s^{\alpha-1}\right)ds+\frac{(M_u)^{q}}{d}\sum_{j=1}^{p}a_j\int_0^1 h(\eta_j,s)\,f\left(s,s^{\alpha-1}\right)ds\right],\ t\in[0,1],
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
Tu(t)&=\int_0^1 g(t,s)\,f(s,u(s))\,ds+\frac{t^{\alpha-1}}{d}\sum_{j=1}^{p}a_j\int_0^1 h(\eta_j,s)\,f(s,u(s))\,ds,\\
&\geq t^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 g(1,s)\,f(s,u(s))\,ds+\frac{1}{d}\sum_{j=1}^{p}a_j\int_0^1 h(\eta_j,s)\,f(s,u(s))\,ds\right],\\
&\geq t^{\alpha-1}\left[\frac{(m_u)^{q}}{\Gamma(\alpha)}\int_0^1 g(1,s)\,f\left(s,s^{\alpha-1}\right)ds+\frac{(m_u)^{q}}{d}\sum_{j=1}^{p}a_j\int_0^1 h(\eta_j,s)\,f\left(s,s^{\alpha-1}\right)ds\right],\ t\in[0,1].
\end{aligned} \tag{3.6}$$

Equations (3.5), (3.6) and assumption (A_3) imply that $T\,:\,D\to D$.
Now, we are in the position to give the main result of this work.

Theorem 3.1. *Suppose $(A_1)-(A_3)$ hold. Then problem $(1.1)-(1.2)$ has a unique, nondecreasing solution $u^* \in D$, moreover, constructing successively the sequence of functions*

$$h_n(t)=\int_0^1 G(t,s)\, f(s,h_{n-1}(s))\, ds, \quad t\in[0,1], \quad n=1,2,\ldots, \tag{3.7}$$

for any initial function $h_0(t)\in D$, then $\{h_n(t)\}$ must converge to $u^*(t)$ uniformly on $[0,1]$ and the rate of convergence is

$$\max_{t\in[0,1]} |h_n(t)-u^*(t)| = O\left(1-\theta^{q^n}\right), \tag{3.8}$$

where $0<\theta<1$, which depends on the initial function $h_0(t)$.

Proof. For any $h_0\in D$, we let

$$l_{h_0}=\sup\{l>0 : lh_0(t)\le (Th_0)(t),\ t\in[0,1]\}, \tag{3.9}$$

$$L_{h_0}=\inf\{L>0 : Lh_0(t)\ge (Th_0)(t),\ t\in[0,1]\}, \tag{3.10}$$

$$m=\min\left\{1,\ (l_{h_0})^{\frac{1}{1-q}}\right\}, \quad M=\max\left\{1,\ (L_{h_0})^{\frac{1}{1-q}}\right\}, \tag{3.11}$$

and

$$u_0(t)=mh_0(t), \quad v_0(t)=Mh_0(t), \tag{3.12}$$

$$u_n(t)=Tu_{n-1}(t), \quad v_n(t)=Tv_{n-1}(t), \quad n=0,1,\ldots,. \tag{3.13}$$

Since the operator T is increasing, $(A_1),(A_2)$ and $(3.9)-(3.13)$ imply that there exist iterative sequences $\{u_n\},\ \{v_n\}$ satisfying

$$u_0(t)\le u_1(t)\le\ldots\le u_n(t)\le\ldots\le v_n(t)\le\ldots\le v_1(t)\le v_0(t),\ t\in[0,1]. \tag{3.14}$$

In fact, from (3.12) and (3.13), we have

$$u_0(t)\le v_0(t), \tag{3.15}$$

$$\begin{aligned} u_1(t)=Tu_0(t) &= \int_0^1 G_1(t,s)\, f(s,mh_0(s))\, ds+\frac{t^{\alpha-1}}{d}\sum_{i=1}^n a_j\int_0^1 G_2(\eta_j,s)\, f(s,mh_0(s))\, ds, \\ &\ge m^q\left[\int_0^1 G_1(t,s)\, f(s,h_0(s))\, ds+\frac{t^{\alpha-1}}{d}\sum_{i=1}^n a_j\int_0^1 G_2(\eta_j,s)\, f(s,h_0(s))\, ds\right], \\ &\ge m^q Th_0(t)\ge mh_0(t)=u_0(t), \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} v_1(t)=Tv_0(t) &= \int_0^1 G_1(t,s)\, f(s,Mh_0(s))\, ds+\frac{t^{\alpha-1}}{d}\sum_{i=1}^n a_j\int_0^1 G_2(\eta_j,s)\, f(s,Mh_0(s))\, ds, \\ &\le M^q\left[\int_0^1 G_1(t,s)\, f(s,h_0(s))\, ds+\frac{t^{\alpha-1}}{d}\sum_{i=1}^n a_j\int_0^1 G_2(\eta_j,s)\, f(s,h_0(s))\, ds\right] \\ &\le M^q Th_0(t)\le Mh_0(t)=v_0(t). \end{aligned} \tag{3.17}$$

Then, by $(3.15)-(3.17)$ and induction, the iterative sequences $\{u_n\},\ \{v_n\}$ satisfy

$$u_0(t)\le u_1(t)\le\ldots\le u_n(t)\le\ldots\le v_n(t)\le\ldots\le v_1(t)\le v_0(t),\ \forall t\in[0,1].$$

Note that $u_0(t) = \frac{m}{M} v_0(t)$, from $(A_1), (3.3), (3.12)$ and (3.13), it can obtained by induction that

$$u_n(t) \geq \theta^{q^n} v_n(t), \; t \in [0,1], \; n = 0, 1, 2, \ldots, \tag{3.18}$$

where $\theta = \frac{m}{M}$.
From (3.14) and (3.18) we know that

$$0 \leq u_{n+p}(t) - u_n(t) \leq v_n(t) - u_n(t) \leq \left(1 - \theta^{q^n}\right) M h_0(t), \; \forall n, p \in \mathbb{N}, \tag{3.19}$$

and since $\left(1 - \theta^{q^n}\right) M h_0(t) \to 0, \; as \; n \to \infty,$
this yields that there exists $u^* \in D$ such that

$$u_n(t) \to u^*(t), \quad (uniformly \, on \; [0,1]).$$

Moreover, from (3.19) and

$$\begin{aligned} 0 \leq v_n(t) - u^*(t) &= v_n(t) - u_n(t) + u_n(t) - u^*(t), \\ &\leq \left(1 - \theta^{q^n}\right) M h_0(t) \to 0, \quad as \; n \to \infty, \end{aligned}$$

we have

$$v_n(t) \to u^*(t), \quad (uniformly \, on \; [0,1]),$$

so,

$$u_n(t) \to u^*(t), \; v_n(t) \to u^*(t), \quad (uniformly \, on \; [0,1]). \tag{3.20}$$

Therefore

$$u_n(t) \leq u^*(t) \leq v_n(t), \quad t \in [0,1], \; n = 0, 1, 2, \ldots,, \tag{3.21}$$

From $(A_1), (3.12)$ and (3.13), we have

$$u_{n+1}(t) = T u_n(t) \leq T u^*(t) \leq T v_n(t) = v_{n+1}(t), \; n = 0, 1, 2, \ldots,.$$

This together with (3.20) and uniqueness of limit imply that u^* satisfy $u^* = T u^*$, that is $u^* \in D$ is a solution of BVP $(1.1) - (1.2)$.

From $(3.12) - (3.14)$ and (A_1), we obtain

$$u_n(t) \leq h_n(t) \leq v_n(t), \; n = 0, 1, 2, \ldots,. \tag{3.22}$$

It follows from $(3.19), (3.20), (3.21)$ and (3.22) that

$$\begin{aligned} |h_n(t) - u^*(t)| &\leq |h_n(t) - u_n(t)| + |u_n(t) - u^*(t)|, \\ &\leq |h_n(t) - u_n(t)| + |u^*(t) - u_n(t)|, \\ &\leq 2 |v_n(t) - u_n(t)|, \\ &\leq 2M \left(1 - \theta^{q^n}\right) |h_0(t)|. \end{aligned}$$

Therefore

$$\max_{t \in [0,1]} |h_n(t) - u^*(t)| \leq 2M \left(1 - \theta^{q^n}\right) \max_{t \in [0,1]} |h_0(t)|.$$

Hence, (3.8) holds. Since $h_0(t)$ is arbitrary in D we know that $u^*(t)$ is the unique solution of the boundary value problem $(1.1) - (1.2)$ in D.

□

We construct an example to illustrate the applicability of the result presented.

Example 3.2. Consider the following boundary value problem

$$D_{0+}^{\frac{5}{2}}u(t)+\frac{(u)^{\frac{2}{3}-\frac{1}{6}cos(t)}}{\sqrt{t}}=0,\ t\in(0,1),$$

$$u(0)=u'(0)=0,\ u'(1)=\frac{\sqrt{2}}{2}u'\left(\frac{1}{2}\right),$$

where $\alpha=\frac{5}{2}$, $\beta=1$, $a_1=\frac{\sqrt{2}}{2}$, $\eta_1=\frac{1}{2}$ and $f(t,u)=\frac{(u)^{\frac{2}{3}-\frac{1}{6}cos(t)}}{\sqrt{t}}$ is increasing function with respect to u for all $t\in(0,1)$, so, assumption (A_1) satisfied.
By simple calculation we have $d=1-\frac{\sqrt{2}}{2}\left(\sqrt{\frac{1}{2}}\right)=\frac{1}{2}$.
For any $r\in(0,1)$, there exists $q=\frac{1}{2}\in(0,1)$ such that

$$f(t,ru)=\frac{(ru)^{\frac{2}{3}-\frac{1}{6}cos(t)}}{\sqrt{t}}\geq r^{\frac{1}{2}}\frac{(u)^{\frac{2}{3}-\frac{1}{6}cos(t)}}{\sqrt{t}}=r^{\frac{1}{2}}f(t,u),$$

thus, $f(t,u)$ satisfies (A_2) and is singular at $t=0$.
On the other hand,

$$\int_0^1 f\left(t,t^{2,5-1}\right)dt\leq\int_0^1 t^{\frac{1}{4}}dt=\frac{4}{5}<\infty,$$

so, assumption (A_3) is satisfied.
Hence, all the assumptions of Theorem 3.1 are satisfied. Which implies that the boundary value $(1.1)-(1.2)$ has an unique, nondecreasing solution $u^*\in D$.

4. Conclusion

This paper is concerned with the boundary value problem of a class of fractional differential equations involving the Riemann-Liouville fractional derivative with nonlocal boundary conditions. The existence of a positive solution was shown by using the properties of the Green's function and the monotone iteration technique and two successively iterative sequences to approximate the solutions were constructed.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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