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Remarks on solutions to the functional equations of the radical type

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Abstract

This is an expository paper containing remarks on solutions to some functional equations of a form, that could be called of the radical type. Simple natural examples of them are the following two functional equations

$$f\left(\sqrt[n]{x^n+y^n}\right)=f(x)+f(y),$$
$$f\left(\sqrt[n]{x^n+y^n}\right)+f\left(\sqrt[n]{|x^n-y^n|}\right)=2f(x)+2f(y)$$

considered recently in several papers, for real functions and with given positive integer n, in connection with the notion of Ulam (or Hyers-Ulam) stability. We provide a general method allowing to determine solutions to them.

Keywords: functional equation, radical type, Cauchy equation, quadratic equation.
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1. Introduction and preliminaries

During the 16$^{\text{th}}$ International Conference on Functional Equations and Inequalities (Będlewo, Poland, May 17-23, 2015), W. Sintunavarat presented a talk concerning the Ulam type stability (for information and further references concerning this notion see, e.g., [4]) of the so-called radical functional equation

$$f\left(\sqrt{x^2+y^2}\right)=f(x)+f(y), \tag{1.1}$$

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in the class of real functions. A question of J. Schwaiger about the general solution of the equation was answered a bit later by the author of this paper (see [14, p. 196]). Namely, a function $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ stands for the set of reals) satisfies equation (1.1) if and only if it is of the form:

$$f(x) = a(x^2), \qquad x \in \mathbb{R},$$

with a function $a : \mathbb{R} \to \mathbb{R}$ that is additive (i.e., satisfies the condition: $a(x + y) = a(x) + a(y)$ for every $x, y \in \mathbb{R}$). This paper contains some remarks on extensions and generalizations of this result.

Clearly, equation (1.1) is a particular case of the functional equation

$$f\big(\sqrt[n]{x^n + y^n}\,\big) = f(x) + f(y), \tag{1.2}$$

which for $k = 2, 3, 4$ have been considered in [2, 3, 5, 7, 8, 9, 12, 15], and some descriptions of solutions to it have been proposed (not always complete and correct). Moreover, the solutions and the Ulam type stability of the equation

$$f\big(\sqrt{ax^2 + by^2}\,\big) = af(x) + bf(y) \tag{1.3}$$

have been considered in [9, 10], for functions f mapping $\mathbb{R}$ into a real linear space X, with real $a, b > 0$ such that $a + b \neq 1$. The authors have proved that every such solution to (1.3) must be a quadratic function, i.e., a solution to the quadratic functional equation

$$q(x + y) + q(x - y) = 2q(x) + 2q(y). \tag{1.4}$$

A somewhat similar is the Pythagorean mean functional equation

$$f\big(\sqrt{x^2 + y^2}\,\big) = \frac{f(x)f(y)}{f(x) + f(y)}, \tag{1.5}$$

considered in [13] for $f : (0, \infty) \to \mathbb{R}$. It is clear that the cases when

$$f(x) + f(y) = 0$$

must be somehow excluded in (1.5) (which has not been done explicitly in [13]).

Moreover, the equation

$$f\big(\sqrt{x^2 + y^2}\,\big) + f\big(\sqrt{|x^2 - y^2|}\,\big) = 2f(x) + 2f(y) \tag{1.6}$$

and its generalized form

$$f\big(\sqrt{ax^2 + by^2}\,\big) + f\big(\sqrt{|ax^2 - by^2|}\,\big) = 2a^2 f(x) + 2b^2 f(y) \tag{1.7}$$

have been considered in [5, 9, 10, 15] for functions f mapping $\mathbb{R}$ into a real linear space X, with real $a, b > 0$ such that $a + b \neq 1$.

It seems that a useful simple description of solutions to functional equations of similar type is of interest and has not been published so far. Therefore we would like to present some general remarks on the issue of solving such equations and obtain in this way much stronger versions and complements of some of the results presented in [2, 3, 5, 7, 10, 9]. The reasonings that we use are well known and some of them can be even considered to be routine (cf., e.g., [1, 11]).

Note that all those equations are simple particular cases of the following general functional equation

$$\begin{aligned} H\Big(f\Big(\sqrt[n]{F_1(x_1^n, \dots, x_m^n)}\,\Big), \dots, f\Big(\sqrt[n]{F_k(x_1^n, \dots, x_m^n)}\,\Big)\Big) \\ = G\big(f(x_1), \dots, f(x_m)\big) \end{aligned} \tag{1.8}$$

for the unknown functions $f : \mathbb{R} \to D$, with given functions $H : D^k \to T$, $G : D^m \to T$, $F_1, \dots, F_k : P^m \to P$, where n, k and m are fixed positive integers with $n > 1$, T and D are nonempty sets, and

$$P := \{x^n : x \in \mathbb{R}\}.$$

2. Main results

In the whole paper n, k, m, P, D, T, H, G, and $F_1, \dots, F_k$ have the same meaning as described at the end of the previous section.

The next theorem is the main result of this paper. Namely, we have the following description of the general solution $f : \mathbb{R} \to D$ to functional equation (1.8).

Theorem 2.1. *Let f be a function mapping $\mathbb{R}$ into D. Assume that one of the following two conditions is valid:*

(i) *n is odd;*

(ii) *there are $e_1, \dots, e_{m-1} \in f(\mathbb{R})$ such that*

$$G(e_1, \dots, e_{m-1}, u) \neq G(e_1, \dots, e_{m-1}, v), \qquad u, v \in D, u \neq v. \tag{2.1}$$

Then f satisfies functional equation (1.8) if and only if there exists a solution $h : P \to D$ of the equation

$$\begin{aligned} H\big(h\big(F_1(x_1, \dots, x_m)\big), \dots, h\big(F_k(x_1, \dots, x_m)\big)\big) \\ = G\big(h(x_1), \dots, h(x_m)\big), \end{aligned} \tag{2.2}$$

such that

$$f(x) = h(x^n), \qquad x \in \mathbb{R}. \tag{2.3}$$

Proof. Assume that f fulfils (1.8). Let

$$h(x) = f\big(\sqrt[n]{x}\,\big), \qquad x \in P. \tag{2.4}$$

We show that (2.2) holds.

To this end take $x_1, \dots, x_m \in P$ and write

$$y_i := \sqrt[n]{x_i}, \qquad i = 1, \dots, m.$$

Then, by (1.8),

$$\begin{aligned} &H\big(h\big(F_1(x_1,\ \dots, x_m)\big), \dots, h\big(F_k(x_1, \dots, x_m)\big)\big) \\ &\quad = H\Big(f\Big(\sqrt[n]{F_1(y_1^n, \dots, y_m^n)}\ \Big), \dots, f\Big(\sqrt[n]{F_k(y_1^n, \dots, y_m^n)}\ \Big)\Big) \\ &\quad = G\big(f(y_1), \dots, f(y_m)\big) = G\big(f\big(\sqrt[n]{x_1}\,\big), \dots, f\big(\sqrt[n]{x_m}\,\big)\big) \\ &\quad = G\big(h(x_1), \dots, h(x_m)\big). \end{aligned} \tag{2.5}$$

Clearly, if n is odd, then $P = \mathbb{R}$ and consequently, by (2.4),

$$f(x) = h(x^n), \qquad x \in \mathbb{R}.$$

So, assume that n is even. Then $P = [0, \infty)$ and, according to (2.4),

$$f(x) = h(x^n), \qquad x \in [0, \infty). \tag{2.6}$$

Next, according to (ii), there exist $e_1, \dots, e_{m-1} \in f(\mathbb{R})$ such that (2.1) is valid. Let $v_1, \dots, v_{m-1} \in f(\mathbb{R})$ be such that

$$e_i = f(v_i), \qquad i = 1, \dots, m.$$

It is easily seen that, for each $x \in \mathbb{R}$,

$$\begin{aligned} G\left(e_1, \ldots, e_{m-1}, f(-x)\right) &= G\big(f(v_1), \ldots, f(v_{m-1}), f(-x)\big) \\ &= H\Big(f\Big(\sqrt[n]{F_1(v_1^n, \ldots, v_{m-1}^n, x^n)}\Big), \ldots, f\Big(\sqrt[n]{F_k(v_1^n, \ldots, v_{m-1}^n, x^n)}\Big)\Big) \\ &= G\big(f(v_1), \ldots, f(v_{m-1}), f(x)\big) = G\big(e_1, \ldots, e_{m-1}, f(x)\big). \end{aligned}$$

Thus, in view of (2.1), we have proved that

$$f(-x) = f(x), \qquad x \in \mathbb{R}, \tag{2.7}$$

whence (on account of (2.6))

$$f(x) = h(x^n), \qquad x \in \mathbb{R}.$$

Now, assume that $f(x) = h(x^n)$ for every $x \in \mathbb{R}$, with some solution $h : P \to D$ of equation (2.2). Then (2.4) holds. We are to show that f is a solution to functional equation (1.8). So, take $y_1, \ldots, y_m \in \mathbb{R}$. Then $y_1^n, \ldots, y_m^n \in P$ and consequently

$$\begin{aligned} H\Big(f\Big(\sqrt[n]{F_1(y_1^n, \ldots, y_m^n)}\Big), \ldots, f\Big(\sqrt[n]{F_k(y_1^n, \ldots, y_m^n)}\Big)\Big) \\ = H\big(h\big(F_1(y_1^n, \ldots, y_m^n)\big), \ldots, h\big(F_k(y_1^n, \ldots, y_m^n)\big)\big) \\ = G\big(h(y_1^n), \ldots, h(y_m^n)\big) = G\big(f(y_1), \ldots, f(y_m)\big). \end{aligned}$$

□

Let $\mathbb{F}$ be a field, $D = T = \mathbb{F}$, $k = 1$, $m = 2$, $F_1(x_1, x_2) \equiv x_1 + x_2$, $G(u, v) \equiv uv$ and $H(u) \equiv u$. Then functional equation (1.8) takes the form

$$f\big(\sqrt[n]{x^n + y^n}\big) = f(x)f(y) \tag{2.8}$$

and Theorem 2.1 implies the following very simple corollary.

Corollary 2.2. *A function $f : \mathbb{R} \to \mathbb{F}$ satisfies functional equation* (2.8) *if and only if there exists a solution $g : P \to \mathbb{F}$ to the equation*

$$g(x + y) = g(x)g(y), \tag{2.9}$$

such that $f(x) = g(x^n)$ for $x \in \mathbb{R}$.

Proof. Let f be a solution to (2.8). If $f(x) \equiv 0$, then it is enough to take $g(x) \equiv 0$. If there is $x \in \mathbb{R}$ with $f(x) \neq 0$, then condition (ii) holds and we can use Theorem 2.1.

The converse also follows from Theorem 2.1. □

Let X denote a linear space over a field $\mathbb{K}$ with $2 \neq 0$, $\alpha, \beta \in \mathbb{K}$, and $a, b \in \mathbb{R}_+$. Let $D = T = X$, $k = 1$, $F_1(x_1, x_2) \equiv ax_1 + bx_2$, $H(u) \equiv u$ and $G(u, v) \equiv \alpha u + \beta v$. Then functional equation (1.8) has the form

$$f\big(\sqrt[n]{ax^n + by^n}\big) = \alpha f(x) + \beta f(y), \tag{2.10}$$

which generalizes simultaneously equations (1.2) and (1.3). Note that Theorem 2.1 implies at once the following:

Corollary 2.3. *Assume that $\alpha \neq 0$ or $\beta \neq 0$. A function $f : \mathbb{R} \to X$ satisfies functional equation* (2.10) *if and only if there exists a solution $g : P \to X$ to the equation*

$$g(ax + by) = \alpha g(x) + \beta g(y) \tag{2.11}$$

such that $f(x) = g(x^n)$ for $x \in \mathbb{R}$.

The next proposition describes solutions $g : P \to X$ to (2.11) ($\mathbb{R}_+$ stands for the set of nonnegative reals).

Proposition 2.4. *Let $P_0 \in \{\mathbb{R}_+, \mathbb{R}\}$. Assume that $\alpha \neq 0$ or $\beta \neq 0$. Then a function $g : P_0 \to X$ satisfies equation* (2.11) *if and only if,*

(a) *in the case $\alpha + \beta \neq 1$, there exists a solution $h : \mathbb{R} \to X$ to the additive Cauchy equation*

$$h(x + y) = h(x) + h(y), \tag{2.12}$$

such that

$$h(ax) = \alpha h(x), \qquad h(by) = \beta h(y), \qquad x \in \mathbb{R}, \tag{2.13}$$

and

$$g(x) = h(x), \qquad x \in P_0. \tag{2.14}$$

(b) *in the case $\alpha + \beta = 1$, there are $w \in X$ and a solution $h : \mathbb{R} \to X$ to equation* (2.12) *such that* (2.13) *holds and*

$$g(x) = h(x) + w, \qquad x \in P_0. \tag{2.15}$$

Proof. Let $g_0 : P_0 \to X$ satisfy functional equation (2.11) and $g_0(0) = 0$. Taking $x = 0$ and next $y = 0$ in (2.11), we get

$$g_0(ax) = \alpha g_0(x), \qquad g_0(by) = \beta g_0(y), \qquad x \in P_0, \tag{2.16}$$

whence

$$g_0(ax + by) = \alpha g_0(x) + \beta g_0(y) = g_0(ax) + g_0(by), \qquad x \in P_0. \tag{2.17}$$

Clearly, (2.17) means that

$$g_0(x + y) = g_0(x) + g_0(y), \qquad x \in P_0. \tag{2.18}$$

Take $x, y, z, w \in P_0$ with $x - w = z - y$. Then $x + y = z + w$ and, by (2.18),

$$g_0(x) + g_0(y) = g_0(x + y) = g_0(z + w) = g_0(z) + g_0(w),$$

which implies that

$$g_0(x) - g_0(w) = g_0(z) - g_0(y).$$

Consequently, we can define $h : \mathbb{R} \to X$ by

$$h(x - y) = g_0(x) - g_0(y), \qquad x, y \in P_0.$$

Note that

$$h(x) = h(x - 0) = g_0(x) - g_0(0) = g_0(x), \qquad x \in P_0, \tag{2.19}$$

and

$$h(-x) = h(0 - x) = g_0(0) - g_0(x) = -g_0(x), \qquad x \in P_0. \tag{2.20}$$

Hence, by (2.16), we get

$$h(ax) = \alpha h(x), \qquad h(by) = \beta h(y), \qquad x \in \mathbb{R}. \tag{2.21}$$

We are yet to show that h fulfils the additive Cauchy equation (2.12). To this end take $u, v \in \mathbb{R}$. There exist $x_1, x_2, y_1, y_2 \in P_0$ with $u = x_1 - x_2$ and $v = y_1 - y_2$. Note that $x_1 + y_1, x_2 + y_2 \in P_0$ and, by (2.18),

$$\begin{aligned} h(u+v) &= h(x_1 - x_2 + y_1 - y_2) \\ &= h(x_1 + y_1 - (x_2 + y_2)) \\ &= g_0(x_1 + y_1) - g_0(x_2 + y_2) \\ &= g_0(x_1) + g_0(y_1) - (g_0(x_2) + g_0(y_2)) \\ &= g_0(x_1) - g_0(x_2) + g_0(y_1) - g_0(y_2) \\ &= h(x_1 - x_2) + h(y_1 - y_2) = h(u) + h(v). \end{aligned}$$

Now, assume that $g : P_0 \to X$ satisfies equation (2.11). First consider the case $\alpha + \beta \neq 1$. Then, with $x = y = 0$ in (3.2), we deduce that $g(0) = 0$. Consequently the reasoning presented above, with $g_0 = g$, ends the proof of the necessary condition.

If $\alpha + \beta = 1$, we write

$$g_0(x) := h(x) - h(0), \qquad x \in \mathbb{R}_+.$$

Then $g_0(0) = 0$ and, as we have shown above, there is a solution $h : \mathbb{R} \to X$ to equation (2.12) such that (2.13) holds. Hence statement (b) is true with $w := g_0(0)$.

The converse is easy to check. □

Remark 2.5. If $\mathbb{K} = \mathbb{R}$ and a function $h : \mathbb{R} \to X$ satisfies equation (2.12) and conditions (2.13), then it is easily seen that

$$h(a^n x) = \alpha^n h(x), \qquad h(b^n y) = \beta^n h(y), \qquad x \in \mathbb{R}, n \in \mathbb{N}.$$

Consequently, if $r := a^{n_0} \in \mathbb{Q}$ (rationals) for some $n_0 \in \mathbb{N}$, then

$$rh(x) = h(rx) = h(a^{n_0} x) = \alpha^{n_0} h(x), \qquad x \in \mathbb{R}.$$

Hence $a^{n_0} = r = \alpha^{n_0}$ or $h(x) \equiv 0$. The same is true for b and β.

For some further comments and references concerning similar issues we refer to [11, ch. XIII, §10].

Remark 2.6. As far as we know, the only published description of solutions $h : \mathbb{R} \to X$ (with $\mathbb{K} = \mathbb{R}$) to functional equation (1.3) (i.e., to (2.10) with $n = 2$, $\alpha = a$ and $\beta = b$) states that if $a + b \neq 1$, then f must be a quadratic function (see [9, Theorem 2.3]). Clearly, this description follows at once from Proposition 2.4 (a). Certainly, Proposition 2.4 provides much more general and precise information.

3. Further applications

In this section, as before, $\mathbb{R}_+$ stands for the set of nonnegative reals, $P := \{x^n : x \in \mathbb{R}\}$, X denotes a linear space over a field $\mathbb{K}$ with $2 \neq 0$, $\alpha, \beta \in \mathbb{K}$, $a, b \in (0, \infty)$, and $n \in \mathbb{N}$. We always assume that $\alpha \neq 0$ or $\beta \neq 0$.

Clearly, if $D = T = X$, $k = 2$, $F_1(x_1, x_2) \equiv ax_1 + bx_2$, $F_2(x_1, x_2) \equiv |ax_1 - bx_2|$, $H(u, v) \equiv u + v$ and $G(u, v) \equiv \alpha u + \beta v$, then functional equation (1.8) takes the form

$$f\left(\sqrt[n]{ax^n + by^n}\,\right) + f\left(\sqrt[n]{|ax^n - by^n|}\,\right) = \alpha f(x) + \beta f(y), \tag{3.1}$$

which is a generalization of equations (1.6) and (1.7). Consequently, Theorem 2.1 implies the following:

Corollary 3.1. *A function $f : \mathbb{R} \to X$ satisfies functional equation* (3.1) *if and only if there exists a solution $h : P \to X$ to the equation*

$$h(ax + by) + h(|ax - by|) = \alpha h(x) + \beta h(y) \tag{3.2}$$

such that $f(x) = h(x^n)$ for $x \in \mathbb{R}$.

We provide descriptions of solutions to (3.2) in the next proposition and corollary. To this end, let us recall that $q : \mathbb{R} \to X$ is quadratic if it satisfies functional equation (1.4) (see also Remark 3.6).

Theorem 3.2. *Let $P_0 \in \{\mathbb{R}_+, \mathbb{R}\}$ and $h : P_0 \to X$ be such that $h(0) = 0$. Then h satisfies functional equation (3.2) if and only if there is a quadratic function $q : \mathbb{R} \to X$ such that*

$$q(ax) = \frac{\alpha}{2} q(x), \qquad q(bx) = \frac{\beta}{2} q(x), \qquad x \in \mathbb{R}, \tag{3.3}$$

and

$$h(x) = q(x), \qquad x \in P_0. \tag{3.4}$$

Proof. Taking $x = 0$ and next $y = 0$ in (3.2), we get

$$h(ax) = \frac{\alpha}{2} h(x), \qquad h(bx) = \frac{\beta}{2} h(x), \qquad x \in \mathbb{R}_+. \tag{3.5}$$

Consequently

$$\begin{aligned} h(ax + by) + h(|ax - by|) &= 2\,\frac{\alpha}{2} h(x) + 2\,\frac{\beta}{2} h(y) \\ &= 2h(ax) + 2h(by), \qquad x, y \in \mathbb{R}_+, \end{aligned}$$

whence

$$h(u + v) + h(|u - v|) = 2h(u) + 2h(v), \qquad u, v \in \mathbb{R}_+. \tag{3.6}$$

Define $q : \mathbb{R} \to X$ by

$$q(x) = h(|x|), \qquad x \in \mathbb{R}. \tag{3.7}$$

Clearly,

$$h(x) = q(x), \qquad x \in \mathbb{R}_+, \tag{3.8}$$

and (3.5) implies

$$q(ax) = \frac{\alpha}{2} q(x), \qquad q(bx) = \frac{\beta}{2} q(x), \qquad x \in \mathbb{R}_+. \tag{3.9}$$

Next, q is even (in view of (3.7)), so (3.9) implies (3.3) also when $P_0 = \mathbb{R}$. We show that q is quadratic.

So, fix $u, v \in \mathbb{R}$. If $u, v \in \mathbb{R}_+$, then

$$\begin{aligned} q(u + v) + q(u - v) &= h(u + v) + h(|u - v|) \\ &= 2h(u) + 2h(v) = 2q(u) + 2q(v). \end{aligned}$$

If $u, v \in (-\infty, 0)$, then

$$\begin{aligned} q(u + v) + q(u - v) &= h(-u - v) + h(| - u - (-v)|) \\ &= 2h(-u) + 2h(-v) = 2q(u) + 2q(v). \end{aligned}$$

Further, if $u \geq 0$ and $v < 0$, then

$$\begin{aligned} q(u + v) + q(u - v) &= q(u - (-v)) + q(u + (-v)) \\ &= h(|u - (-v)|) + h(u + (-v)) \\ &= 2h(u) + 2h(-v) = 2q(u) + 2q(v). \end{aligned}$$

Finally, if $u < 0$ and $v \geq 0$, then $q(u-v) = h(|u-v|) = q(v-u)$, whence

$$\begin{aligned} q(u+v) + q(u-v) &= q(v+u) + q(v-u) \\ &= q(v-(-u)) + q(v+(-u)) \\ &= h(|v-(-u)|) + h(v+(-u)) \\ &= 2h(v) + 2h(-u) = 2q(u) + 2q(v). \end{aligned}$$

If $P_0 = \mathbb{R}_+$, then (3.8) is just equality (3.4), whence this finishes the proof of the necessary condition.

So, it remains to prove that (3.4) holds also in the case $P_0 = \mathbb{R}$. To this end fix $y \in (-\infty, 0)$. There is $x \in \mathbb{R}_+$ with $ax + by > 0$ and $ay + bx > 0$ and consequently, by (3.2), (3.3) and (3.8),

$$\begin{aligned} \beta h(y) &= h(ax+by) + h(|ax-by|) - \alpha h(x) \\ &= q(ax+by) + q(ax-by) - \alpha q(x) \\ &= q(ax+by) + q(ax-by) - 2q(ax) \\ &= 2q(by) = \beta q(y), \end{aligned}$$

$$\begin{aligned} \alpha h(y) &= h(ay+bx) + h(|ay-bx|) - \beta h(x) \\ &= q(ay+bx) + q(ay-bx) - \beta q(x) \\ &= q(ay+bx) + q(ay-bx) - 2q(bx) \\ &= 2q(ay) = \alpha q(y), \end{aligned}$$

which means that $h(y) = q(y)$ (whether $\alpha = 0$ or $\beta = 0$). This completes the proof of the necessary condition also for $P_0 = \mathbb{R}$.

Now, we prove the sufficient condition. So, assume that there is a quadratic function $q : \mathbb{R} \to X$ such that (3.3) and (3.4) are valid. Then, with $x = 0$ in (1.4), we get

$$q(y) = q(-y), \qquad y \in \mathbb{R}, \tag{3.10}$$

and consequently

$$\begin{aligned} h(ax+by) + h(|ax-by|) &= q(ax+by) + q(|ax-by|) \\ &= q(ax+by) + q(ax-by) \\ &= 2q(ax) + 2q(by) = \alpha q(x) + \beta q(y) \\ &= \alpha h(x) + \beta h(y), \qquad x, y \in P_0. \end{aligned}$$

□

Corollary 3.3. *Let $P_0 \in \{\mathbb{R}_+, \mathbb{R}\}$. Then $h : P_0 \to X$ satisfies functional equation* (3.2) *if and only if,*

(a) *in the case $\alpha + \beta \neq 2$, there is a quadratic function $q : \mathbb{R} \to X$ such that* (3.3) *and* (3.4) *are valid;*

(b) *in the case $\alpha + \beta = 2$, there are $w \in X$ and a quadratic function $q : \mathbb{R} \to X$ such that* (3.3) *holds and*

$$h(x) = q(x) + w, \qquad x \in P_0. \tag{3.11}$$

Proof. Let $h : P_0 \to X$ be a solution to (3.2).

First consider the case $\alpha + \beta \neq 2$. Then $x = y = 0$ in (3.2) yield $f(0) = 0$. Hence we can simply apply Theorem 3.2.

So, assume that $\alpha + \beta = 2$ and write

$$h_0(x) := h(x) - h(0), \qquad x \in P_0.$$

Then $h_0(0) = 0$ and

$$\begin{aligned} h_0(ax+by) + h_0(|ax-by|) &= h(ax+by) - h(0) + h(|ax-by|) - h(0) \\ &= \alpha h(x) + \beta h(y) - 2h(0) \\ &= \alpha(h(x) - h(0)) + \beta(h(y) - h(0)) \\ &= \alpha h_0(x) + \beta h_0(y), \qquad x, y \in P_0. \end{aligned}$$

Hence using again Theorem 3.2, but with h replaced by h_0, we obtain (3.11) with $w = h(0)$.

The converse is easy to check in view of (3.10). □

Corollary 3.4. *A function $h : \mathbb{R} \to \mathbb{R}$ satisfies functional equation* (3.2) *if and only if it is a solution to the equation*

$$h(ax+by) + h(ax-by) = \alpha h(x) + \beta h(y). \tag{3.12}$$

Proof. Let $h_0 : \mathbb{R} \to \mathbb{R}$ be a solution to equation (3.12) with $h_0(0) = 0$. Taking first $x = 0$ and next $y = 0$ in (3.12) gives

$$h_0(ax) = \frac{\alpha}{2} h_0(x), \qquad h_0(ax) = \frac{\beta}{2} h_0(x), \qquad x \in \mathbb{R}. \tag{3.13}$$

Hence

$$\begin{aligned} h_0(ax+by) + h_0(ax-by) &= \alpha h_0(x) + \beta h_0(y) \\ &= 2h_0(ax) + 2h_0(ay), \qquad x, y \in \mathbb{R}, \end{aligned}$$

which means that

$$h_0(x+y) + h_0(x-y) = 2h_0(x) + 2h_0(y), \qquad x, y \in \mathbb{R}. \tag{3.14}$$

Suppose that $h : \mathbb{R} \to \mathbb{R}$ is a solution to equation (3.12). If $\alpha + \beta \neq 2$, then (3.12) with $x = y = 0$ gives $h(0) = 0$, whence (3.14) holds for $h_0 = h$. Consequently, by Corollary 3.3, h is a solution to (3.2).

If $\alpha + \beta = 2$, then the function $h_0 : \mathbb{R} \to \mathbb{R}$, given by $h_0(x) := h(x) - h(0)$ for $x \in \mathbb{R}$, also is a solution to (3.12) and $h_0(0) = 0$. Hence (3.14) is valid, which means that h_0 is quadratic. Since $h(x) \equiv h_0(x) + h(0)$, h is a solution to (3.2) (again by Corollary 3.3).

If $h : \mathbb{R} \to \mathbb{R}$ is a solution to equation (3.2), then it has the form described in Corollary 3.3 and it is easy to check that h fulfils also (3.12). □

Corollary 3.5. *Let n be odd. A function $f : \mathbb{R} \to X$ fulfils the equation*

$$f\big(\sqrt[n]{ax^n + by^n}\,\big) + f\big(\sqrt[n]{|ax^n - by^n|}\,\big) = \alpha f(x) + \beta f(y) \tag{3.15}$$

if and only if it is a solution to the functional equation

$$f\big(\sqrt[n]{ax^n + by^n}\,\big) + f\big(\sqrt[n]{ax^n - by^n}\,\big) = \alpha f(x) + \beta f(y). \tag{3.16}$$

Proof. According to Corollary 3.1, a function $f : \mathbb{R} \to X$ satisfies functional equation (3.15) if and only if there exists a solution $g : \mathbb{R} \to X$ to the equation (3.2) such that $f(x) \equiv g(x^n)$.

Analogously, by Theorem 2.1, a function $f : \mathbb{R} \to X$ satisfies functional equation (3.16) if and only if there exists a solution $g : \mathbb{R} \to X$ to the equation (3.12) such that $f(x) \equiv g(x^n)$.

Since, in view of Corollary 3.4, equations (3.2) and (3.12) have the same solutions $g : \mathbb{R} \to X$, this completes the proof. □

Remark 3.6. It is well known (see, e.g., [1]) that a function $q : \mathbb{R} \to X$ is quadratic if and only if there exists $L : \mathbb{R}^2 \to X$ that is symmetric (i.e., $L(x,y) = L(y,x)$ for all $x, y \in \mathbb{R}$) and biadditive (i.e., $L(x, y+z) = L(x,y) + L(x,z)$ for all $x, y, z \in \mathbb{R}$) such that

$$q(x) = L(x,x), \qquad x \in \mathbb{R}.$$

Clearly, conditions (3.3) are equivalent to

$$L(ax,ax) = \frac{\alpha}{2} L(x,x), \qquad L(bx,bx) = \frac{\beta}{2} L(x,x), \qquad x \in \mathbb{R}. \tag{3.17}$$

Next, note that

$$4L(x,y) = L(x+y, x+y) - L(x-y, x-y), \qquad x, y \in \mathbb{R}.$$

Hence (3.3) (or (3.17)) implies the following two conditions

$$\begin{aligned} 4L(ax,ay) &= q(a(x+y)) - q(a(x-y)) \\ &= \frac{\alpha}{2}(q(x+y) - q(x-y)) \\ &= 2\alpha L(x,y), \qquad x, y \in \mathbb{R}, \end{aligned} \tag{3.18}$$

$$4L(bx,by) = 2\beta L(x,y), \qquad x, y \in \mathbb{R}. \tag{3.19}$$

So, conditions (3.18) and (3.19) are equivalent to (3.17) and, in view of Corollary 3.3, we can state the following:

Corollary 3.7. *Let $P_0 \in \{\mathbb{R}_+, \mathbb{R}\}$. Then $h : P_0 \to X$ satisfies functional equation* (3.2) *if and only if,*

(a) *in the case $\alpha + \beta \neq 2$, there is a symmetric and biadditive function $L : \mathbb{R}^2 \to X$ such that*

$$L(ax,ay) = \frac{\alpha}{2} L(x,y), \qquad L(bx,by) = \frac{\beta}{2} L(x,y), \qquad x, y \in \mathbb{R}, \tag{3.20}$$

and

$$h(x) = L(x,x), \qquad x \in P_0; \tag{3.21}$$

(b) *in the case $\alpha + \beta = 2$, there are $w \in X$ and a symmetric and biadditive function $L : \mathbb{R}^2 \to X$ such that* (3.20) *is valid and*

$$h(x) = L(x,x) + w, \qquad x \in P_0. \tag{3.22}$$

Finally, let us observe that from Corollaries 3.1 and 3.7 we can easily deduce the following:

Corollary 3.8. *A function $f : \mathbb{R} \to X$ satisfies functional equation* (3.1) *if and only if,*

(a) *in the case $\alpha + \beta \neq 2$, there is a symmetric and biadditive function $L : \mathbb{R}^2 \to X$ such that* (3.20) *is valid and*

$$f(x) = L(x^n, x^n), \qquad x \in \mathbb{R}; \tag{3.23}$$

(b) *in the case $\alpha + \beta = 2$, there are $w \in X$ and a symmetric and biadditive function $L : \mathbb{R}^2 \to X$ such that* (3.20) *is valid and*

$$f(x) = L(x^n, x^n) + w, \qquad x \in \mathbb{R}. \tag{3.24}$$

Note that from Corollary 3.8 it results that the description of solutions for the equation

$$f\left(\sqrt[n]{x^n+y^n}\right)+f\left(\sqrt[n]{|x^n-y^n|}\right)=2f(x)+2f(y), \tag{3.25}$$

which is a generalization of (1.6), is quite simple. Namely, we have the following:

Corollary 3.9. *A function $f:\mathbb{R}\to X$ satisfies functional equation* (3.25) *if and only if there is a symmetric and biadditive function $L:\mathbb{R}^2\to X$ such that $f(x)=L(x^n,x^n)$ for $x\in\mathbb{R}$.*

Remark 3.10. According to our best knowledge, the only published so far description (see [9, Theorem 2.3]) of solutions $f:\mathbb{R}\to X$ (with $\mathbb{K}=\mathbb{R}$) of functional equation (1.7) (i.e., of (3.1) with $n=2$, $\alpha=2a^2$ and $\beta=2b^2$) states that if $a^2+b^2\neq 1$, then f must be a solution to the functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{3.26}$$

It is easy to check that this description (with $n=2$) follows from Corollary 3.8 (a), which provides much more general and precise information.

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