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The Norm Of Certain Matrix Operators On The New Block Sequence Space

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Abstract: The purpose of the this study is to introduce the sequence space

$$\ell_p(E, B(r,s)) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} r x_j + \sum_{j \in E_{n+1}} s x_j \right|^p < \infty \right\},$$

where $E = (E_n)$ is a partition of finite subsets of the positive integers, $r, s \in \mathbb{R} \backslash \{0\}$ and $p \geq 1$. The topological and algebraical properties of this space are examined. Furthermore, we establish some inclusion relations. Finally, the problem of finding the norm of certain matrix operators such as Copson and Hilbert from ℓ_p into $\ell_p(E, B(r,s))$ is investigated.

Keywords: Capson operators, Hilbert operators, Matrix domain, Sequence spaces.

1 Introduction

By a sequence space, we understand a linear subspace of ω, the space of all real valued sequences $x = (x_n)$. The domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{ x = (x_k) \in \omega : Ax \in X \}, \tag{1}$$

which is a sequence space. If A is triangle, then one can easily observe that the sequence spaces X_A and X are linearly isomorphic, i.e., $X_A \cong X$. In the past, several authors studied matrix transformations on the sequence spaces that are the matrix domains of triangle matrices in classical spaces ℓ_p, ℓ_∞, c and c_0. For instance, some matrix domains of the difference operator were studied in [1–8]. In these studies, the matrix domains are obtained by triangle matrices, hence these spaces are normed sequence spaces. For more details on the domain of triangle matrices in some spaces, the reader may refer to Chapter 4 of [9]. The matrix domains given in this paper specify by a certain non-triangle matrix, so we should not expect that related spaces are normed sequence spaces.

In this study, we define the sequence space $\ell_p(E, B(r,s))$ and investigate some topological and algebraical properties of this space and derive inclusion relations concerning with its. Moreover, we shall consider the inequality of the form

$$\|Ax\|_{p,E,B(r,s)} \leq U \|x\|_p \ ,$$

for all the sequence $x \in \ell_p$. The costant U not depending on x and we seek the smallest possible value of U. In the study, we examine the problem of finding the upper bound of certain matrix operators from ℓ_p into $\ell_p(E, B(r,s))$ and we consider certain matrix operators such as Copson and Hilbert.

Let $E = (E_n)$ be a partition of finite subsets of the positive integers such that

$$\max E_n < \min E_{n+1}, \tag{2}$$

for $n = 1, 2, ...$ Foroutannia defined the sequence space $\ell_p(E)$ by

$$\ell_p(E) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p < \infty \right\} \quad (1 \leq p < \infty),$$

with the semi-norm $\|.\|_{p,E}$, which is defined in the following way :

$$\|x\|_{p,E} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p \right)^{1/p}.$$

It is significant that in the special case $E_n = \{n\}$ for $n = 1, 2, ...$, we have $\ell_p(E) = \ell_p$ and $\|x\|_{p,E} = \|x\|_p$. For more details on the sequence space $\ell_p(E)$, the reader may refer to [10].

2 The Block Sequence Space $\ell_p(E, B(r,s))$ of Non-Absolute Type

Suppose $E = (E_n)$ is a partition of finite subsets of the positive integers that satisfies the condition (2). We define the sequence space $\ell_p(E, B(r,s))$ by

$$\ell_p(E, B(r,s)) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} r x_j + \sum_{j \in E_{n+1}} s x_j \right|^p < \infty \right\},$$

with the semi-norm $\|.\|_{p,E,B(r,s)}$, which is defined in the following way :

$$\|x\|_{p,E,B(r,s)} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} r x_j + \sum_{j \in E_{n+1}} s x_j \right|^p \right)^{1/p}. \tag{3}$$

It should be noted that the function $\|.\|_{p,E,B(r,s)}$ can not be norm, since $x = (x_j) = \{(-1)^{j+1}\}_{j=1}^{\infty}$ and $E = \{2n-1, 2n\}$ for all n , then $\|x\|_{p,E,B(r,s)} = 0$ while $x \neq 0$.

It is also significant that in the special case $r = 1$ and $s = -1$, we have $\ell_p(E, B(r,s)) = \ell_p(E, \Delta)$ [11].

If the infinite matrix $A = \{a_{nk}\}$ is defined by

$$a_{nk} = \begin{cases} r \ , & \text{if } k \in E_n \\ s \ , & \text{if } k \in E_{n+1} \\ 0 \ , & \text{otherwise} \end{cases}$$

with the notation (1), we can redefine the space $\ell_p(E, B(r,s))$ as follows:

$$\ell_p(E, B(r,s)) = (\ell_p)_A.$$

Throughout this paper, the cardinal number of the set E_k is denoted by $|E_k|$.

Now we are beginning with the following theorem which is essential in the study.

Theorem 1. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2). The set $\ell_p(E, B(r,s))$ becomes a vector space with coordinatewise addition and scalar multiplication, which is a complete semi-normed space by $\|.\|_{p,E,B(r,s)}$ defined by (3).*

It can easily checked that the absolute property does not hold on the space $\ell_p(E, B(r,s))$, that is $\|x\|_{p,E,B(r,s)} \neq \||x|\|_{p,E,B(r,s)}$ for at least one sequence in the space $\ell_p(E, B(r,s))$, and this says that $\ell_p(E, B(r,s))$ is a sequence space of nonabsolute type, where $|x_k| = (|x_k|)$.

Theorem 2. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2). If*

$$M = \left\{ x = (x_n) : \sum_{j \in E_n} r x_j + \sum_{j \in E_{n+1}} s x_j = 0, \forall n \right\},$$

then we have $\ell_p(E, B(r,s))/M \simeq \ell_p$.

Note that the mapping defined in Theorem 2, T is not injective, while $\|Tx\|_p = \|x\|_{p,E,B(r,s)}$ for all $x \in \ell_p(E, B(r,s))$.

Let us derive some inclusion relations concerning with the space $\ell_p(E, B(r,s))$.

Result 1. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (2). If $\sup_n |E_n| < \infty$, then $\ell_p \subset \ell_p(E)$. Moreover if $|E_n| > 1$ for an infinite number of n, then the inclusion is strict.*

Theorem 3. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (2). Then $\ell_p(E) \subset \ell_p(E, B(r,s))$, furthermore the inclusion is strictly holds.*

Combining Lemma 1 and Theorem 3, we get the following corollary.

Corollary 1. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (2). If $\sup_n |E_n| < \infty$, then $\ell_p \subset \ell_p(E, B(r,s))$. Moreover if $|E_n| > 1$ for an infinite number of n, then the inclusion is strict.*

Theorem 4. *Let $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (2). Except the case $p = 2$, the space $\ell_p(E, B(r,s))$ is not a semi-inner product space.*

Definition 1. *Let X be a semi-normed space with a semi-norm g. A sequence (b_n) of elements of the semi-normed space X is called a Schauder basis (or briefly basis) for X iff, for each $x \in X$ there exists a unique sequence of scalars (α_n) such that*

$$\lim_{n\to\infty} g\left(x - \sum_{k=1}^{n} \alpha_k b_k\right) = 0.$$

The series $\sum_{k=1}^{n} \alpha_k b_k$ which has the sum x, is then called the expansion of x with respect to (b_n) and written as $x = \sum_{k=1}^{n} \alpha_k b_k$. In the following, we give a sequence of points of the space $\ell_p(E, B(r,s))$ which forms a basis for the space $\ell_p E, B(r,s)$.

Theorem 5. *Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2). If the sequence $b^{(k)}(r,s) = \{b_j{}^{(k)}(r,s)\}_{j\in\mathbb{N}}$ is defined such that*

$$\sum_{j\in E_n} b_j{}^{(k)}(r,s) = \begin{cases} 0 & , \ if \ \ n < k \\ \frac{1}{r}(-\frac{s}{r})^n & , \ if \ \ n \geq k \end{cases}$$

and the remaining elements are zero, for $k = 1, 2, \ldots$. Then, the sequence $\left\{b^{(k)}(r,s)\right\}_{k\in\mathbb{N}}$ is a basis for the space $\ell_p(E, B(r,s))$ and any $x \in \ell_p(E, B(r,s))$ has a unique representation of the form

$$x = \sum_k \alpha_k b^{(k)}(r,s) \ ,$$

where $\alpha_k = \sum_{j\in E_k} x_j$ for $k = 1, 2, \ldots$.

3 The Norm of Matrix Operators from ℓ_p into $\ell_p(E, B(r,s))$

In this section, the problem of finding the norm of certain matrix operators such as Copson and Hilbert from ℓ_p into $\ell_p(E, B(r,s))$ is considered, where $p \geq 1$.

Theorem 6. *Let $A = (a_{n,k})$ be a matrix operator and $E = (E_n)$ be a partition that satisfies condition (2). If*

$$M = \sup_k \sum_{n=1}^{\infty} \left| \sum_{i\in E_n} r a_{i,k} + \sum_{i\in E_{n+1}} s a_{i,k} \right| < \infty,$$

then A is a bounded operator from ℓ_1 into $\ell_1(E, B(r,s))$ and $\|A\|_{1,E,B(r,s)} = M$.

In particular if

$$\sum_{i\in E_n} r a_{i,k} + \sum_{i\in E_{n+1}} s a_{i,k} \geq 0$$

and $r + s = 0$ for all n, k, then

$$\|A\|_{1,E,B(r,s)} = \sup_k \sum_{i\in E_1} r a_{i,k}$$

The Copson operator C is defined by $y = Cx$, where

$$y_n = \sum_{k=n}^{\infty} \frac{x_k}{k}, (\forall n).$$

It is given by the Copson matrix:

$$c_{n,k} = \begin{cases} \dfrac{1}{k} & , \ if \ \ n \leq k \\ 0 & , \ if \ \ n > k. \end{cases}$$

Corollary 2. *Let C be the Copson operator and $E = (E_n)$ be a partition that satisfies condition (2). If*

$$\sum_{i\in E_n} r c_{i,k} + \sum_{i\in E_{n+1}} s c_{i,k} \geq 0$$

for all n, k and $r + s = 0$, then C is a bounded operator from ℓ_1 into $\ell_1(E, B(r,s))$ and $\|C\|_{1,E,B(r,s)} = r$.

Corollary 3. *Suppose that C is the Copson operator, $rc_{n,k} + sc_{n+1,k} \geq 0$ for all n, k, $r + s = 0$ and $E = \{n\}$ for all n. Then C is a bounded operator from ℓ_1 into $\ell_1(B(r,s))$ and $\|C\|_{1,B(r,s)} = r$.*

Recall that the Hilbert operator H defined by the matrix:

$$h_{n,k}=\frac{1}{n+k}\quad,\qquad (n,k=1,2,...).$$

Corollary 4. *Let H be the Hilbert operator and $E=(E_n)$ be a partition that satisfies the condition (2). If*

$$\sum_{i\in E_n} rh_{i,k}+\sum_{i\in E_{n+1}} sh_{i,k}\geq 0$$

for all n,k and $r+s=0$, then H is a bounded operator from ℓ_1 into $\ell_1(E,B(r,s))$ and

$$\|H\|_{1,E,B(r,s)}=r\left(\frac{1}{2}+...+\frac{1}{\max E_1+1}\right).$$

Corollary 5. *If H is the Hilbert operator, $rh_{n,k}+sh_{n+1,k}\geq 0$ for all n,k and $r+s=0$, then H is a bounded operator from ℓ_1 into $\ell_1(E,B(r,s))$ and $\|H\|_{1,B(r,s)}=\frac{r}{2}$.*

Theorem 7. *([12], Theorem 275) Let $p>1$ and $B=(b_{n,k})$ be a matrix operator with $b_{n,k}\geq 0$ for all n,k. Suppose that K and R are two strictly positive numbers such that*

$$\sum_{n=1}^{\infty} b_{n,k}\leq K,\ \textit{for all}\ \ k,\qquad \sum_{k=1}^{\infty} b_{n,k}\leq R\ \textit{for all}\ \ n,$$

(bounds for column and row sums respectively). Then

$$\|B\|_p\leq R^{(p-1)/p}\ .\ K^{1/p}.$$

Result 2. *If $A=(a_{n,k})$ and $B=(b_{n,k})$ are two matrix operators such that*

$$b_{n,k}=\sum_{i\in E_n} ra_{i,k}+\sum_{i\in E_{n+1}} sa_{i,k},$$

then

$$\|A\|_{p,E,B(r,s)}=\|B\|_p.$$

Hence, if B is a bounded operator on ℓ_p, then A will be a bounded operator from ℓ_p into $\ell_p(E,B(r,s))$.

Theorem 8. *Let C is the Copson matrix operator $p>1$, $r>0$ and $r+s=0$. If N is a positive integer and $E_n=\{nN-N+1,nN-N+2,...,nN\}$ for all n, then C is a bounded operator from ℓ_p into $\ell_p(E,B(r,s))$ and*

$$\|C\|_{p,E,B(r,s)}\leq r\left(N+\frac{N-1}{N+1}+\frac{N-2}{N+2}+...+\frac{1}{2N-1}\right)^{\frac{(p-1)}{p}}.$$

Theorem 9. *Suppose that $p>1$, $r>0$, $r+s=0$, N is a positive integer and $E_n=\{nN-N+1,nN-N+2,...,nN\}$ for all n. If H is the Hilbert matrix operator, then it is a bounded operator from ℓ_p into $\ell_p(E,B(r,s))$ and*

$$\|H\|_{p,E,B(r,s)}\leq r\left(\frac{1}{2}+\frac{2}{3}+...+\frac{N}{N+1}+...+\frac{1}{2N}\right)^{\frac{(p-1)}{p}}\left(\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2N}\right)^{\frac{1}{p}}.$$

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