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Analysis of Numerical Methods in Fractional Order Control Systems with Time Delay and Large Time Coefficient

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ABSTRACT: In recent years, fractional calculus has been used frequently in the field of control engineering. One of the main reasons for this is that it models real world problems more successfully. However, there are some disadvantages. First, it has complex and tedious mathematical calculations. Second, it does not have general analytical solutions. Namely, computing time responses of fractional order systems is still a big problem. Therefore, integer order approximation methods and some numerical methods are used for computation of impulse and step responses. Furthermore, computation accuracy and computation duration of time responses by using Matlab is also important because the computation duration may be too long for some systems such as systems with large time delay and large inertia. In this paper, computation duration and accuracy of time responses is investigated by testing different numerical approximation methods like Grünwald-Letnikov, Fourier Series Method (FSM) and Inverse Fourier Transform Method (IFTM) for fractional order control systems with large time coefficient.
Keywords – *Fractional order system, Grunwald-Letnikov, Fourier series method, inverse Fourier transform method, Time delay, Large time coefficient.*

1. Introduction

The history of fractional calculus is quite old and many studies have been presented in this field since 1695 (Das, 2007; Xue et al., 2009). Basically, a control system with fractional order can be described by a differential equation where the order of the derivative can't be integer. Laplace transform of such a differential equation gives a transfer function. The transfer function is included a fractional order Laplace complex variable such as s^{λ} , $\lambda \in \mathbb{R}$ and referred to as a Fractional Order Transfer Function (FOTF) (Monje et al., 2010). Real world systems are described more adequate by using fractional order differential equations than integer order models (Nonnenmacher and Glöckle, 1991). The time response computation of fractional order systems is one of the major problem since fractional order transfer functions suffer from lacking of analytical inverse Laplace transform. Therefore, one can use approximation methods such as Oustaloup's, Matsuda's, CFE, etc. or numerical methods such as Grünwald-Letnikov (GL), Fourier Series Method (FSM) and Inverse Fourier Transform Method (IFTM) for computation of time responses (Atherton et al., 2014).

Time delay which occurs in many physical systems is also an important problem for control systems. Also, it is well known that the presence of a time delay makes the computation of time responses for control systems more complicated, and approximate results are often used like Pade approximation. Especially, if the control system includes both fractional order and time delay, it is difficult to analyze time responses of systems with large time coefficient. This is mainly due to the inability to perform analytic inverse Laplace transformations of fractional order control systems. Therefore, the time response analysis of fractional order control systems are computed by using either the approximation methods such as Oustaloup's, Matsuda, etc. or the numerical methods such as Grünwald-Letnikov, IFTM and FSM (Atherton et al., 2014). Within these techniques, numerical methods are the least error-prone.

In this study, the computational advantages and disadvantages of Grünwald-Letnikov, IFTM and FSM are presented by analyzing the time response of the fractional order system model with large time coefficient. Some preliminary work on this subject is given in (Yüce and Tan, 2017).

The paper is organized as follows: In Section 2, the fractional order system model forms with large time coefficient are investigated. In Section 3, the time response analysis of fractional order control systems with large time coefficient are examined on numerical example.

2. Fractional Order Control Systems with Large Time Coefficient

A closed loop fractional order control system with time delay is shown in Figure 1. The closed loop control system consists of input $R(s)$, output $Y(s)$, fractional order controller $C_f(s)$ and plant $G(s)$ with time delay θ. The control system is referred to as fractional order since it includes fractional order PI controller $C_f(s)$.

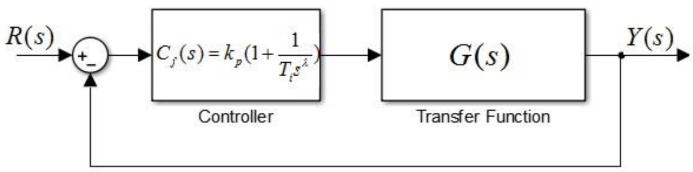

Figure 1. A Control System with Fractional Order PI Controller

The plant form of $G(s)$ is given in (1). The model given in (1) can represent the heater for temperature control systems (Ibrahim, 2002). Temperature system model has large inertia and time delay (Fei and Yongjie, 2014). Therefore, the control systems with such a plant model can be referred to as long settling time systems and its time response computation takes a lot times.

$$G(s) = \frac{Ke^{-s\theta}}{(1+sT)} \tag{1}$$

For example, when T is selected large enough in (1), $G(s)$ can be called as transfer function with large time coefficient or long settling time. Time response analysis of such a control system required an average of 1000 sec. If the controller is integer order, then time response computation is very easy. However, if the controller is fractional, then there is not a general technique for computing the output of a control system with an FOTF or FOTF with time delay for step and impulse response. Therefore, in the next section, we introduced the numerical method such as Grünwald-Letnikov, FSM and IFTM methods for time response computation of fractional order systems with time delay. In this study, the advantages and disadvantages of numerical methods mentioned above are explored on fractional order control system with large time coefficient.

For example, a plant model with large time coefficient and fractional order PI controller are shown in (2) and (3) respectively. The model is a temperature control system model (Ibrahim, 2002).

$$G(s) = \frac{0.844e^{-125s}}{(1+1700s)} \tag{2}$$

$$C_f(s) = 14.502\left[1 + \frac{1}{412.5s^{\lambda}}\right] \tag{3}$$

3. Time Response Analysis of Fractional Order Systems using Numerical Methods

In this section, Grünwald-Letnikov (GL), Inverse Fourier Transform Method (IFTM) and Fourier Series Method (FSM) are briefly introduced for time response computation of the fractional order closed loop control system shown in Figure 1. IFTM and FSM are new techniques in literature and they are numerical methods with minimized error and unaffected by step size (*Δt*).

The open loop transfer function *L(s)* is given in (4) and closed loop transfer function *P(s)* is given in (5) for Figure 1.

$$L(s) = C_f(s)G(s) \tag{4}$$

$$P(s) = \frac{L(s)}{1+L(s)} \tag{5}$$

3.1. Grünwald-Letnikov (GL) Method

Dating back to 1867-1868 the Grünwald-Letnikov (GL) definition can be used in the development of numerical methods (Weilbeer, 2005). The GL numerical approximation method (Chen et al., 2009; Xue et al., 2009) can be used for step response computation of fractional order transfer functions without estimating integer order rational transfer functions. The step and impulse response results obtained from the GL method are generally very accurate as long as the step size, *Δt*, of the simulation time is small enough. The accuracy of the GL method strongly depends on *Δt*.

Podlubny (Podlubny, 1998) and Diethelm (Diethelm, 2010) provide the GL Definition as follows:

Let $y:[a,T]\rightarrow \mathbf{R}$ be $(m+1)$ times continuously differentiable function where m is the greatest integer less than $\alpha>0$. Then, the fractional derivative of $y(t)$ of order α is given by

$$ {}_aD_T^{\alpha} y(t)=\lim_{N\to\infty}\left\{\frac{N}{T-a}\sum_{k=0}^{N}(-1)^k\binom{\alpha}{k}y(t-k\left[\frac{T-a}{N}\right])\right\}. \tag{6}$$

3.2. Inverse Fourier Transform Method (IFTM)

The impulse response of *P(s)* is given by *p(t)=L*$^{-1}$ *(P(s))* where *L*$^{-1}$ denotes the inverse Laplace transform. Assuming the impulse response is that of a stable system so that $\lim_{t\to\infty} p(t)=0$ then the Fourier transform can be mentioned. Thus, unit impulse response for the control system given in Figure 1 can be computed from the closed loop transfer function, *P(s)*, of the system. For this case, (7) and (8) can be used as

$$p(t)=\frac{2}{\pi}\int_0^{\infty}\mathrm{Re}[P(j\omega)]\cos(\omega t)d\omega \tag{7}$$

or

$$p(t)=-\frac{2}{\pi}\int_0^{\infty}\mathrm{Im}[P(j\omega)]\sin(\omega t)d\omega \tag{8}$$

Thus, *p(t)* can be computed by numerical integration using (7) or (8). The numerical integral can be computed easily using Matlab “trapz” command.

In order to prove the accuracy of IFTM program, the step responses are plotted using both Matlab and IFTM for closed loop control system given in Figure 1 with the plant given in (2) and integer order PI controller for λ=1 given in (3). The step responses and error plot are given in Figure 2. As a result of the comparison, it is shown in Figure 2 that the error value between Matlab and IFTM is in 10^{-4} and one can say IFTM is accurate technique for fractional order transfer functions.

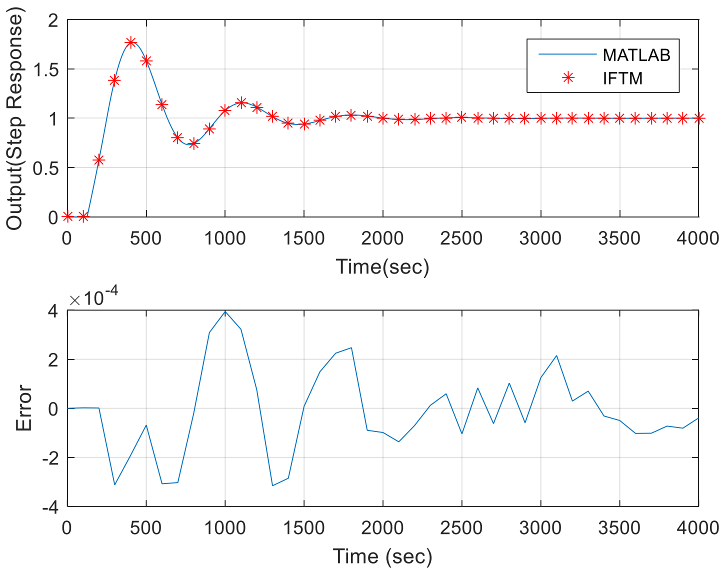

Figure 2. IFTM and Matlab Comparison

3.3. Fourier Series Method (FSM)

The Fourier series for the square wave of -1 to 1 with frequency $\omega_s = 2\pi / T$ can be written as

$$r(t) = \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \frac{1}{k} \sin(k\omega_s t) \tag{9}$$

where *T* indicates the period of the square wave. If *r(t)* is the input of the control system given in Figure 1 means it passes through the transfer function *P(s)* then the output, which is the unit step response if *T* is selected sufficiently large, can be written as

$$y_s(t) = \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \left(\frac{1}{k} \mathrm{Re}\left[P(jk\omega_s)\right] \sin(k\omega_s t) + \frac{1}{k} \mathrm{Im}[P(jk\omega_s)] \cos(k\omega_s t) \right) \tag{10}$$

As $T \to \infty$ and $\omega_s \to 0$ the numerator of the imaginary part of $P(jk\omega_s)$ is multiplied by ω_s so that $\lim_{\omega_s \to 0} \mathrm{Im}\, P(jk\omega_s) = 0$ and (10) become,

$$y_s(t) \cong \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \frac{1}{k} \mathrm{Re}\left[P(jk\omega_s)\right] \sin(k\omega_s t) \tag{11}$$

which is the unit step response of *P(s)*. Similarly, the impulse response, which is the derivative of the step response, is given by

$$y_i(t) = \frac{dy_s(t)}{dt} \cong \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \omega_s \,\mathrm{Re}\left[P(jk\omega_s)\right] \cos(k\omega_s t) \tag{12}$$

Thus, (11) and (12) are obtained to compute the step and impulse responses of *P(s)* respectively.

To show the accuracy of FSM, the step responses are plotted using both Matlab and FSM for closed loop control system given in Figure 1 with the plant given in (2) and integer order PI controller for λ=1 given in (3). The results are given in Figure 3. It is shown that FSM and IFTM have approximately same error. For this reason, it can be seen that FSM gives also accurate results like IFTM.

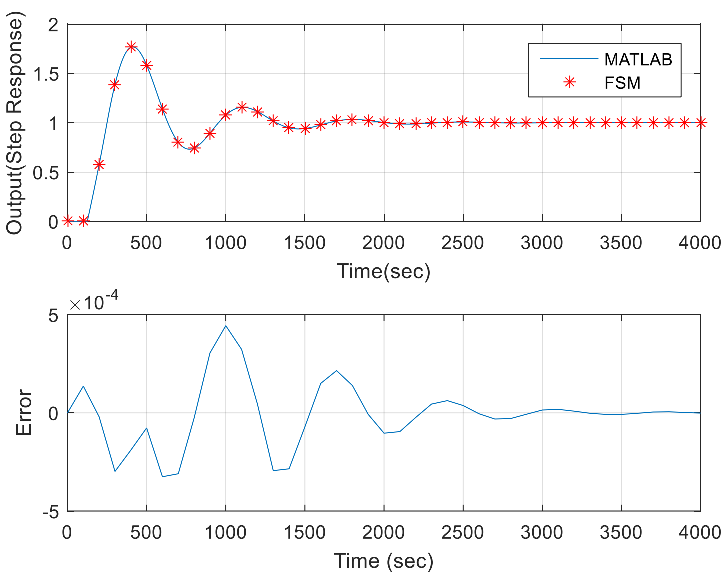

Figure 3. FSM and Matlab Comparison

4. Numerical Example

Consider the system model with large time delay given in (13) and fractional PI controller given in (14).

$$G(s) = \frac{1}{(480s+1)} e^{-200s} \tag{13}$$

$$C_f(s) = 2.021\left[1 + \frac{1}{392.7s^{0.9}}\right] \tag{14}$$

$$L(s) = C_f(s)G(s) \tag{15}$$

The open loop transfer function is obtained in (15) and given in (16).

$$L(s) = \frac{793.4s^{0.9} + 2.021}{1.885.10^5 s^{1.9} + 392.7s^{0.9}} e^{-200s} \tag{16}$$

In this paper, the Matlab toolbox for fractional order system identification and control (FOMCON) is used for step response computation with GL method. The closed loop step response of open loop transfer function with time delay cannot be drawn directly using FOMCON tool. Therefore, 6/6 Pade approximation method is used for converting time delay to transfer function and the transfer function is given in (17). The Pade approximation method is used only in GL computation.

$$e^{-200s} = \frac{s^6 - 0.21s^5 + 0.021s^4 - 0.00126s^3 + 4.725e-05s^2 - 1.0395e-06s + 1.0395e-08}{s^6 + 0.21s^5 + 0.021s^4 + 0.00126s^3 + 4.725e-05s^2 + 1.0395e-06s + 1.0395e-08} \tag{17}$$

$$P(s) = \frac{L(s)}{1 + L(s)} \tag{18}$$

The closed loop transfer function is computed using (18) and the transfer function obtained by Pade approximation is given in (19).

$$P(s) = \frac{\begin{array}{l} 793.4s^{6.9} - 2.021s^6 + 166.61s^{5.9} - 0.42441s^5 + 16.661s^{4.9} \\ +0.042441s^4 - 0.99968s^{3.9} - 0.0025465s^3 + 0.037488s^{2.9} + 9.5492e-05s^2 \\ -0.00082474s^{1.9} - 2.1008e-06s + 8.2474e-06s^{0.9} + 2.1008e-08 \end{array}}{\begin{array}{l} 1.885e+05s^{7.9} + 40771s^{6.9} + 2.021s^6 + 3874.4s^{5.9} - 0.42441s^5 + 262.42s^{4.9} \\ +0.042441s^4 + 8.4017s^{3.9} - 0.0025465s^3 + 0.25199s^{2.9} + 9.5492e-05s^2 \\ +0.0015429s^{1.9} - 2.1008e-06s + 1.233e-05s^{0.9} + 2.1008e-08 \end{array}} \tag{19}$$

The step responses obtained by GL method of (19) for different value of step time Δt are shown in Figure 4. It is shown that when the step time increases, the error amount increases. If the step time is small, the error amount is getting smaller. However, if step time value is 1, the toolbox cannot plot the step response since the transfer function with large time delay and fractional order is quite complicated form. Because the toolbox does not compute step response for step time values below 1, it will never be able to compute optimum unit step response using GL method for large time coefficient systems. GL method is affected by the step time changes.

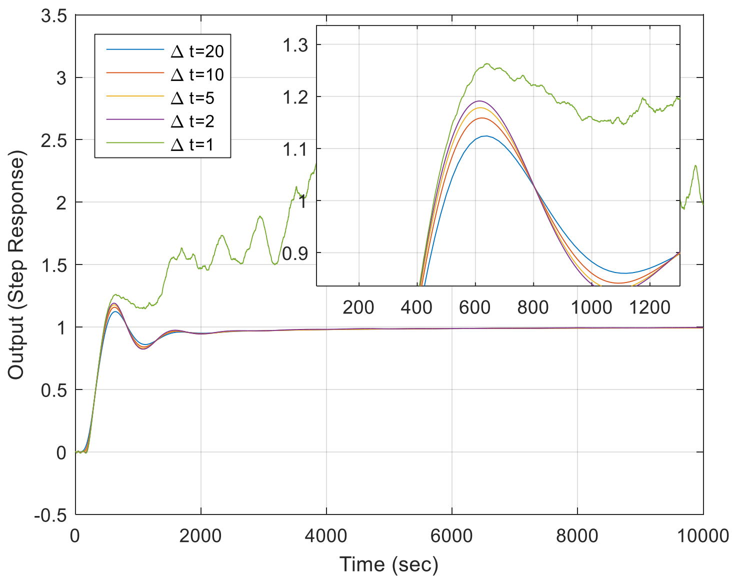

Figure 4. Step Response using GL Method for Different Δt

Considering the same control system, unit step responses computed using FSM and IFTM for different value of step time *Δt* are shown respectively in Figure 5 and Figure 6. It is shown in Figure 5 and Figure 6 that FSM and IFTM give very close results and they are not affected by step time. Namely, FSM and IFTM compute almost the same values for each step time. It is also shown that the errors between *Δt=2* and *Δt=20* is almost zero for both methods.

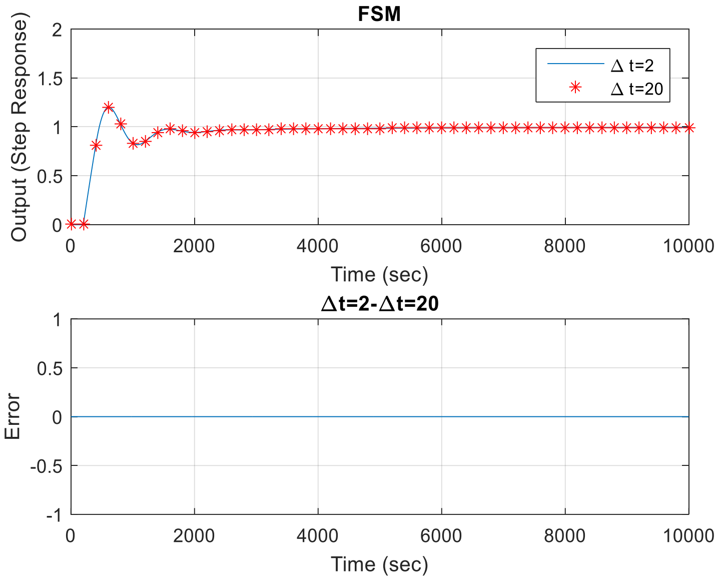

Figure 5. Step Response using FSM for Different *Δt*

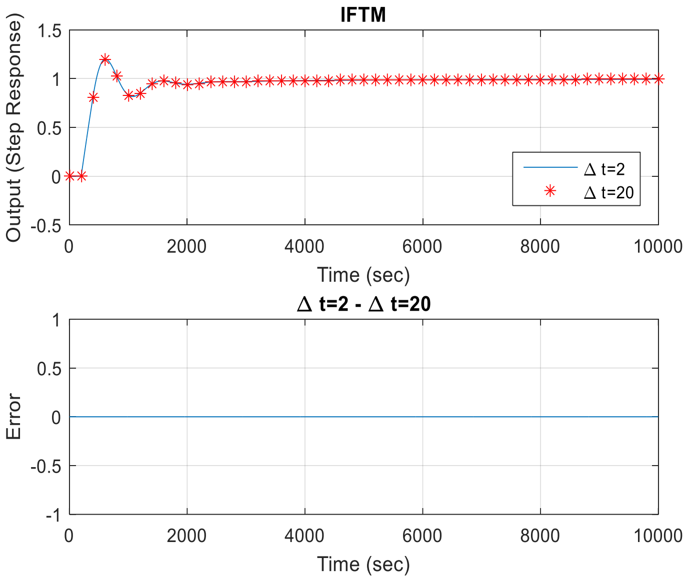

Figure 6. Step Response using IFTM for Different *Δt*

The simulation times for each of the 3 methods are given in Table 1. Calculations are made by a computer with 2.2 GHz i5 processor and 4 GB Ram memory. It is shown in Table 1 that GL method performs faster than other methods. However, the results of the GL method contain errors according to the other methods. The errors of GL method for *Δt=20* and

Δt=2 are shown respectively in Figure 7 and Figure 8. If desired, the step time can be increased thus, the results can be calculated more quickly since IFTM and FSM are not affected by step time changes.

Table 1. Simulation Times for Different *Δt* and Methods

	$\Delta t = 20$	$\Delta t = 10$	$\Delta t = 5$	$\Delta t = 2$	$\Delta t = 1$
Method	**Simulation Time (sec)**	**Simulation Time (sec)**	**Simulation Time (sec)**	**Simulation Time (sec)**	**Simulation Time (sec)**
GL	0.28	0.36	0.49	1.32	4.69
FSM	35.70	69.93	142.02	357.55	703.05
IFTM	56.62	109.70	220.13	559.41	1099.19

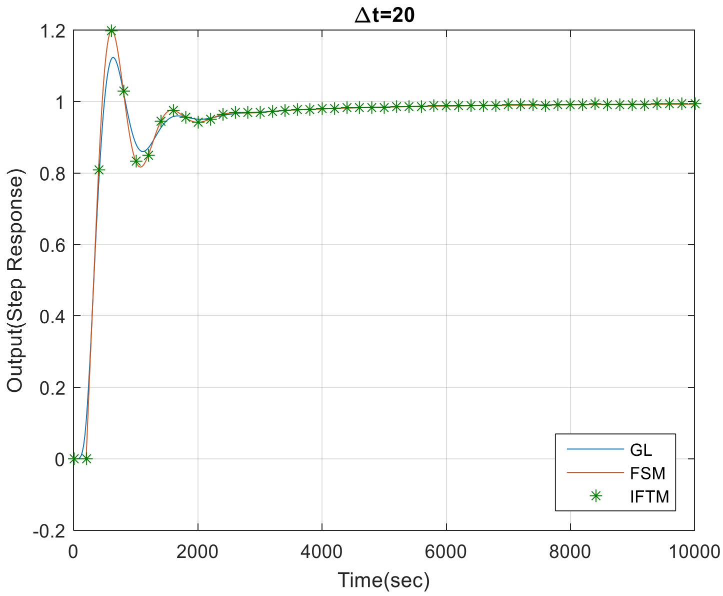

Figure 7. GL, FSM and IFTM Comparisons for $\Delta t = 20$

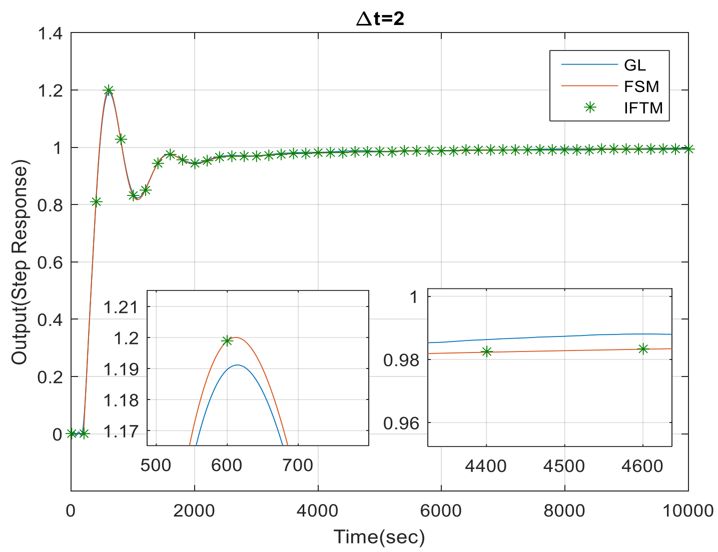

Figure 8. GL, FSM and IFTM Comparisons for $\Delta t = 2$

5. Conclusions

In this study, the computational advantages and disadvantages of Grünwald-Letnikov, IFTM and FSM are presented by analysing the time response of the fractional order system model with a large time coefficient. The results of numerical approximation methods have been compared with each other. Numerical example showed the advantages of FSM and IFTM. It can be seen in example that IFTM and FSM are not affected by step time changes, however GL method is affected by step time despite of quick step response computation. As a result, IFTM and FSM give the most reliable results for control system with fractional order and large time coefficient.

Acknowledgments

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