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One Parameter Elliptical Planar Motion

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Abstract: In order to describe the elliptical planar motion, two moving and one fixed elliptical planes have been considered. Thus, one parameter elliptical planar motion is defined by the help of these planes. The absolute, relative and sliding velocities have been obtained and the relation between these velocities have been proven. Also, pole points of the elliptical planar motion have been derived. Finally, some results have been given regarding to the absolute, relative, sliding velocities and the pole points of elliptical planar motion.

Keywords: Elliptical planar motion, Elliptical plane, Kinematics.

1 Introduction

Motion is mathematically described as a change in the position of an object or a point with respect to time. The rate of motion in a specific direction gives us velocity. The velocities of the planar motion is always measured with respect to a coordinate systems (frames of reference). Although spherical motions are of great interest to researchers, planar motion has an important place in kinematics. Because many objects in engineering are relatively flat and thin or symmetrical. That is, the motion of these objects is considered to be approximately planar motion.
Müller initially studied one parameter planar motions and obtained the relationships between absolute, relative and sliding velocities and accelerations in the Euclidean plane E^2. Moreover, he gave the Euler-Savary formula which gives the relationship between the curvatures of the trajectory curves [1].
Blaschke and Müller have introduced one parameter planar motions in terms of complex numbers [2]. In [3], it was demonstrated that the relation between complex velocities and pole points can be obtained with the help of moving coordinate system for the one paremeter motion in the complex plane.
Pereira and Ersoy have introduced elliptical harmonic motion by using elliptical numbers. Also, they have found the relationships between the absolute, the relative and the sliding velocities and accelerations for this motion. Furthermore, the canonical relative system of the motion has been deïňĄned and EulerâĂŞSavary formula has been obtained [4].
Özdemir has given the generation of elliptical rotations by the help of the elliptic scalar product and elliptic vector product for a given ellipsoid. For this purpose, an elliptical ortogonal matrix and an elliptical skew symmetric matrix have been defined for this elliptic inner product. Thereby, he has examined the motion of a point on the ellipsoid using elliptical rotation matrices [5].
This paper is organized as follows. In the ïňĄrst part, basic concepts have been represented as if elliptic inner product, elliptical norm of a vector, elliptical rotation matrix etc. In the second part, one parameter elliptical planar motion has been introduced by the help of the elliptically orthogonal systems of $\{O; \vec{e}_1, \vec{e}_2\}$ and $\{O'; \vec{e}'_1, \vec{e}'_2\}$. These orthogonal systems represent the moving elliptical plane E and the ïňĄxed elliptical plane E', respectively. Furthermore, an elliptically orthogonal, relative system $\left\{B; \overrightarrow{h_1}, \overrightarrow{h_2}\right\}$ has been considered. Thus, the theorems and results have been given regarding to the velocities and pole points of this motion.

2 Basic Concepts

Let us consider a for an ellipse in the form

$$(E) : a_1x^2 + a_2y^2 = 1, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a_1 = \frac{1}{a^2},\ a_2 = \frac{1}{b^2}$ and $a, b \in R$ (see [5]). The elliptic inner product or B-inner product for the vectors $\vec{u} = (u_1, u_2), \vec{w} = (w_1, w_2) \in R^2$

$$B(\vec{u}, \vec{w}) = a_1\, u_1w_1 + a_2\, u_2w_2$$

where $a_1, a_2 \in R^+$. This scalar product is positive definite and also can be written as $B(\vec{u}, \vec{w}) = u^t\Omega w$ where the associated matrix Ω is defined as follows

$$\Omega = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}.$$

Thus, the real vector space R^2 equipped with the elliptic inner product will be denoted by ${R^2}_{a_1,a_2}$ and the number $\Delta = \sqrt{\det \Omega}$ will be called "constant of the scalar product" [5].
The elliptical norm of a vector $\vec{u} \in R^2$ is defined to be $\|\vec{u}\|_B = \sqrt{B(\vec{u}, \vec{u})}$. Moreover, two vectors $\vec{u}$ and $\vec{w}$ are called B-orthogonal or elliptically orthogonal vectors if $B(\vec{u}, \vec{w}) = 0$. In addition to that if their norms become 1, then these vectors are called elliptically orthonormal or B-orthonormal. The cosine of the angle between two vectors $\vec{u}$ and $\vec{w}$ is defined as,

$$\cos\theta = \frac{B(\vec{u}, \vec{w})}{\|\vec{u}\|_B \|\vec{w}\|_B}$$

where θ is compatible with the parameters of the angular parametric equations of ellipse [5]. Let T be a B-skew symmetric matrix. Then, an elliptical rotation matrix in the space $R^2_{a_1,a_2}$ is defined by

$$R^B_\theta = \begin{bmatrix} \cos\theta & \frac{\sqrt{a_2}}{\sqrt{a_1}}\sin\theta \\ -\frac{\sqrt{a_1}}{\sqrt{a_2}}\sin\theta & \cos\theta \end{bmatrix}.$$

3 One Parameter Planar Elliptical Motion

Let E_1 and E be moving and E' be fixed elliptical planes and $\left\{B; \overrightarrow{h_1}, \overrightarrow{h_2}\right\}$, $\{O; \vec{e}_1, \vec{e}_2\}$, $\left\{O'; \vec{e}_1', \vec{e}_2'\right\}$ represent their orthogonal coordinate systems, respectively.
Therefore, the following scalar products can be written for the vectors

$$B\left(\vec{e}_i, \vec{e}_j\right) = \begin{cases} a_i, & if\, i = j \\ 0, & if\, i \neq j \end{cases}$$

and

$$B\left(\vec{e}_i', \vec{e}_j'\right) = \begin{cases} a_i, & if \quad i = j \\ 0, & if\, i \neq j \end{cases}$$

The motion of the moving relative plane E_1 with respect to other moving plane E is given by the following relation

$$\begin{aligned} \overrightarrow{h_1} &= \cos\theta\, \overrightarrow{e_1} + \frac{\sqrt{a_2}}{\sqrt{a_1}}\sin\theta\, \overrightarrow{e_2} \\ \overrightarrow{h_2} &= -\frac{\sqrt{a_1}}{\sqrt{a_2}}\sin\theta\, \overrightarrow{e_1} + \cos\theta\, \overrightarrow{e_2} \end{aligned} \tag{1}$$

and

$$\overrightarrow{OB} = \overrightarrow{b} = b_1\overrightarrow{h_1} + b_2\overrightarrow{h_2} \tag{2}$$

Here θ is elliptical rotation angle and $\overrightarrow{OB}$ represents the vector from the origin of the moving coordinate system to the origin of other moving relative coordinate system.
Similarly, the motion of the moving relative plane E_1 with respect to fixed plane E' is given by

$$\begin{aligned} \overrightarrow{h_1} &= \cos\theta\, \overrightarrow{e_1'} + \frac{\sqrt{a_2}}{\sqrt{a_1}}\sin\theta\, \overrightarrow{e_2'} \\ \overrightarrow{h_2} &= -\frac{\sqrt{a_1}}{\sqrt{a_2}}\sin\theta\, \overrightarrow{e_1'} + \cos\theta\, \overrightarrow{e_2'} \end{aligned} \tag{3}$$

and

$$\overrightarrow{O'B} = \overrightarrow{b'} = b_1'\overrightarrow{h_1} + b_2'\overrightarrow{h_2} \tag{4}$$

Here θ' is elliptical rotation angle and $\overrightarrow{O'B}$ represents the vector from the origin of the fixed coordinate system to the origin of moving relative coordinate system.
Taking the differentials of the equations (1) and (2) and rearranging for the motion of E_1/E, we obtain

$$\begin{aligned} &d\overrightarrow{h_1} = \frac{\sqrt{a_2}}{\sqrt{a_1}} d\theta\, \overrightarrow{h_2} \\ &d\overrightarrow{h_2} = -\frac{\sqrt{a_1}}{\sqrt{a_2}} d\theta\, \overrightarrow{h_1} \\ &d\,\overrightarrow{b} = \left(db_1 - \frac{\sqrt{a_1}}{\sqrt{a_2}} b_2\, d\theta\right)\overrightarrow{h_1} + \left(db_2 + \frac{\sqrt{a_2}}{\sqrt{a_1}} b_1 d\theta\right)\overrightarrow{h_2}. \end{aligned}$$

Similarly, taking the differentials of the equations (3), (4) and rearranging for the motion of E_1/E', we find

$$
\begin{aligned}
d'\overrightarrow{h_1} &= \frac{\sqrt{a_2}}{\sqrt{a_1}} d\theta' \,\overrightarrow{h_2} \\
d'\overrightarrow{h_2} &= -\frac{\sqrt{a_1}}{\sqrt{a_2}} d\theta' \,\overrightarrow{h_1} \\
d'\overrightarrow{b} &= \left(db_1{}' - \frac{\sqrt{a_1}}{\sqrt{a_2}} b_2{}' d\theta'\right) \overrightarrow{h_1} + \left(db_2{}' + \frac{\sqrt{a_2}}{\sqrt{a_1}} b_1{}' d\theta'\right) \overrightarrow{h_2}.
\end{aligned}
$$

For the sake of shortness let us use

$$
\begin{aligned}
&\lambda = d\theta, \quad \sigma_1 = db_1 - \frac{\sqrt{a_1}}{\sqrt{a_2}} b_2 \, d\theta, \quad \sigma_2 = db_2 + \frac{\sqrt{a_2}}{\sqrt{a_1}} b_1 d\theta \\
&\lambda' = d\theta', \quad \sigma_1{}' = db_1{}' - \frac{\sqrt{a_1}}{\sqrt{a_2}} b_2{}' d\theta', \quad \sigma_2{}' = db_2{}' + \frac{\sqrt{a_2}}{\sqrt{a_1}} b_1{}' d\theta'
\end{aligned}
$$

The derivative equations of the motion E_1/E become

$$
\begin{aligned}
d\overrightarrow{h_1} &= \frac{\sqrt{a_2}}{\sqrt{a_1}} \lambda \,\overrightarrow{h_2} \\
d\overrightarrow{h_2} &= -\frac{\sqrt{a_1}}{\sqrt{a_2}} \lambda \,\overrightarrow{h_1} \\
d\,\overrightarrow{b} &= \sigma_1 \overrightarrow{h_1} + \sigma_2 \overrightarrow{h_2}.
\end{aligned}
$$

Similarly, the derivative equations of the motion E_1/E' become

$$
\begin{aligned}
d'\overrightarrow{h_1} &= \frac{\sqrt{a_2}}{\sqrt{a_1}} \lambda' \,\overrightarrow{h_2} \\
d'\overrightarrow{h_2} &= -\frac{\sqrt{a_1}}{\sqrt{a_2}} \lambda' \,\overrightarrow{h_1} \\
d'\,\overrightarrow{b} &= \sigma_1{}' \overrightarrow{h_1} + \sigma_2{}' \overrightarrow{h_2}.
\end{aligned}
$$

Here $\sigma_1, \sigma_2, \sigma_1{}', \sigma_2{}'$ are Pfaffian forms of the motion.
Let we us a point $X = (x_1, x_2)$ according to the relative moving coordinate system to analyze the elliptical motions on the elliptical plane. Since the vector equations

$$
\begin{aligned}
\overrightarrow{OB} &= \overrightarrow{b} = b_1 \overrightarrow{h_1} + b_2 \overrightarrow{h_2} \\
\overrightarrow{BX} &= \overrightarrow{x} = x_1 \overrightarrow{h_1} + x_2 \overrightarrow{h_2}
\end{aligned}
$$

and

$$
\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX}
$$

can be written as above, we have

$$
\overrightarrow{x} = (b_1 + x_1)\,\overrightarrow{h_1} + (b_2 + x_2)\,\overrightarrow{h_2}.
$$

So differential of X with respect to E is

$$
\overrightarrow{OB} = \overrightarrow{b} = b_1 \overrightarrow{h_1} + b_2 \overrightarrow{h_2} \tag{5}
$$

Therefore the relative velocity vector of X with respect to E is

$$
\overrightarrow{V_r} = \frac{d\,\overrightarrow{x}}{dt}.
$$

and also differential of X with respect to E' is

$$
d'\,\overrightarrow{x} = \left(dx_1 + \sigma_1{}' - \frac{\sqrt{a_1}}{\sqrt{a_2}} x_2 \,\lambda'\right) \overrightarrow{h_1} + \left(dx_2 + \sigma_2{}' + \frac{\sqrt{a_2}}{\sqrt{a_1}} x_1 \;\lambda'\right) \overrightarrow{h_2}. \tag{6}
$$

Thus, the absolute velocity vector of X with respect to E' is

$$
\overrightarrow{V_a} = \frac{d'\,\overrightarrow{x}}{dt}.
$$

If $\overrightarrow{V_r} = 0$ or $\overrightarrow{V_a} = 0$ then the point X is fixed in the planes E and E', respectively. Thus, the conditions that the points to be fixed in elliptical planes E and E' become

$$
dx_1 = -\sigma_1 + \frac{\sqrt{a_1}}{\sqrt{a_2}} x_2 \,\lambda, \quad dx_2 = -\sigma_2 - \frac{\sqrt{a_2}}{\sqrt{a_1}} x_1 \,\lambda \tag{7}
$$

and

$$
dx_1 = -\sigma_1{}' + \frac{\sqrt{a_1}}{\sqrt{a_2}} x_2 \,\lambda', \quad dx_2 = -\sigma_2{}' - \frac{\sqrt{a_2}}{\sqrt{a_1}} x_1 \,\lambda'.
$$

respectively. Substituting equation (7) into equation (6) and considering that the sliding velocity of the point X is $\overrightarrow{V_f} = \frac{d_f \overrightarrow{x}}{dt}$, we have

$$d_f \overrightarrow{x} = \left[\left({\sigma_1}' - \sigma_1 \right) - x_2 \frac{\sqrt{a_1}}{\sqrt{a_2}} \left(\lambda' - \lambda \right) \right] \overrightarrow{h_1} + \left[\left({\sigma_2}' - \sigma_2 \right) + x_1 \frac{\sqrt{a_2}}{\sqrt{a_1}} \left(\lambda' - \lambda \right) \right] \overrightarrow{h_2}. \tag{8}$$

Thus, the following theorem can be given.

Theorem 1. *Let X be a moving point on the plane E_1 and $\overrightarrow{V_r}$, $\overrightarrow{V_a}$, $\overrightarrow{V_f}$ be the relative, absolute and sliding velocities of X under the one-parameter planar motions, respectively. Then, the relation between the velocities is given as below:*

$$\overrightarrow{V_a} = \overrightarrow{V_r} + \overrightarrow{V_f}.$$

Proof: The proof can be easily seen by considering the equations (5), (6) and (8). □

Result 1. *In the case of $a_1 = 1, \ a_2 = 1$, the relative, absolute and sliding velocities are found as*

$$\begin{aligned} d\vec{x} &= \left(dx_1 + \sigma_1 - x_2 \lambda \right) \overrightarrow{h_1} + \left(dx_2 + \sigma_2 + x_1 \lambda \right) \overrightarrow{h_2} \\ d' \overrightarrow{x} &= \left(x_1 + dx_1 + {\sigma_1}' - x_2 \lambda' \right) \overrightarrow{h_1} + \left(x_2 + dx_2 + {\sigma_2}' + x_1 \lambda' \right) \overrightarrow{h_2} \\ d_f \overrightarrow{x} &= \left[\left({\sigma_1}' - \sigma_1 \right) - x_2 \left(\lambda' - \lambda \right) \right] \overrightarrow{h_1} + \left[\left({\sigma_2}' - \sigma_2 \right) + x_1 \left(\lambda' - \lambda \right) \right] \overrightarrow{h_2} \end{aligned}$$

respectively [1].

Theorem 2. *The pole point P of the one parameter elliptical planar motion E/E' is obtained by*

$$p_1 = -\frac{\sqrt{a_1}}{\sqrt{a_2}} . \frac{\left({\sigma_2}' - \sigma_2 \right)}{\left(\lambda' - \lambda \right)}, \qquad p_2 = \frac{\sqrt{a_2}}{\sqrt{a_1}} \cdot \frac{\left({\sigma_1}' - \sigma_1 \right)}{\left(\lambda' - \lambda \right)}$$

where $\overrightarrow{BP} = \overrightarrow{p} = p_1 \overrightarrow{h_1} + p_2 \overrightarrow{h_2}$.

Proof: In a one parameter elliptical motion, pole points of the motion are characterized for cases that the sliding velocity vector becomes zero. Namely, $d_f \overrightarrow{x} = 0$. It will be taken into account $\theta \neq 0$ and $\theta' \neq 0$ in order to avoid the pure rotation motion. Then, considering that the equality of (8) equals zero

$$\left[\left({\sigma_1}' - \sigma_1 \right) - x_2 \frac{\sqrt{a_1}}{\sqrt{a_2}} \left(\lambda' - \lambda \right) \right] \overrightarrow{h_1} + \left[\left({\sigma_2}' - \sigma_2 \right) + x_1 \frac{\sqrt{a_2}}{\sqrt{a_1}} \left(\lambda' - \lambda \right) \right] \overrightarrow{h_2} = 0$$

is found. From this last equation, coordinates of the pole point P for the one parameter elliptical motion E/E' are obtained by

$$\begin{aligned} x_1 = p_1 &= -\tfrac{\sqrt{a_1}}{\sqrt{a_2}} . \tfrac{\left({\sigma_2}' - \sigma_2 \right)}{\left(\lambda' - \lambda \right)}, \\ x_2 = p_2 &= \tfrac{\sqrt{a_2}}{\sqrt{a_1}} \cdot \tfrac{\left({\sigma_1}' - \sigma_1 \right)}{\left(\lambda' - \lambda \right)}. \end{aligned}$$

□

Result 2. *In the case of $a_1 = 1, \ a_2 = 1$, pole points of the one parameter elliptical planar motion E/E' correspond to pole points of the one parameter planar motion on the Euclidean plane [1].*

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