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On Some New Paranormed Lucas Sequence Spaces and Lucas Core

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Abstract: The sequence spaces $c_0(\hat{L}), c(\hat{L}), \ell_\infty(\hat{L})$ and $\ell_p(\hat{L})$ have been recently introduced and studied by Karakaş and Karabudak. The aim of this paper is to extend the results of Karakaş and Karabudak to the paranormed case and is to work the spaces $c_0(\hat{L}, p), c(\hat{L}, p), \ell_\infty(\hat{L}, p)$ and $\ell(\hat{L}, p)$. Furthermore, Lucas core of a complex-valued sequence has been introduced, and we prove some inclusion theorems related to this new type of core.

Keywords: Lucas numbers, Lucas core, Matrix transformations, Paranormed sequence spaces.

1 Introduction

In mathematics, the Fibonacci numbers are the numbers in the following integer sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

The sequence (f_n) of Fibonacci numbers is given by the linear recurrence relations

$$f_0 = 0, f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2}, n \geq 2.$$

This sequence has many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequence of Fibonacci numbers converges to the golden ratio which is important in sciences and arts.

Similar to the Fibonacci numbers, each Lucas number is defined to be the sum of its two immediate previous terms, thereby forming a Fibonacci integer sequence. The first two Lucas numbers are $L_0 = 2$ and $L_1 = 1$ as opposed to the first two Fibonacci numbers $f_0 = 0$ and $f_1 = 1$. Though closely related in definition, Lucas and Fibonacci numbers exhibit distinct properties. The Lucas numbers may thus be defined as follows:

$$L_n = \begin{cases} 2 & , \quad n = 0, \\ 1 & , \quad n = 1, \\ L_{n-1} + L_{n-2} & , \quad n > 1. \end{cases}$$

The sequence of Lucas number is:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots.$$

The ratio of the successive both Fibonacci and Lucas numbers is as known golden ratio. There are many applications of golden ratio in many places of mathematics and physics, in general theory of high energy particle theory [1]. Also, some basic properties of Lucas numbers [1] are given as follows:

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{(Binet's formula for Lucas numbers)}$$

$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n \quad \text{and} \quad \sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2 \quad \text{(Additional identities)}$$

$$\lim_{n\to\infty} \frac{L_n}{L_{n-1}} = \frac{1+\sqrt{5}}{2} = \varphi \qquad \text{(Golden ratio)}$$

Lucas numbers was first used by Karakaş and Karabudak [2] in the theory of summability. Let L_n be the nth Lucas number for every $n \in \mathbb{N}$. Then, the infinite Lucas matrix $\widehat{L} = (\widehat{L}_{nk})$ is defined by

$$\widehat{L}_{nk} = \begin{cases} \dfrac{L_{k-1}^2}{L_n \cdot L_{n-1} + 2} & , \quad 1 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

where $n, k \in \mathbb{N}$ [2]. Recently, a lot of papers have been studying by many researchers on Lucas and Fibonacci sequences. For instance, see [3–12].

Assume here and after that (p_k) be a bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $L = \max\{1, H\}$ and by $\mathcal{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$, respectively. Then, the paranormed sequence spaces $\ell_\infty(p), c(p), c_0(p)$ and $\ell(p)$ were defined by Maddox [13] (see also Maddox [14] and Nakano [15]) as follows:

$$\begin{aligned} \ell_\infty(p) &= \{x = (x_k) \in w : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty\}, \\ c(p) &= \{x = (x_k) \in w : \lim_{k \to \infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\}, \\ c_0(p) &= \{x = (x_k) \in w : \lim_{k \to \infty} |x_k|^{p_k} = 0\}, \\ \ell(p) &= \left\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\right\}, \end{aligned}$$

which are the complete spaces paranormed by

$$g_1(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/L} \iff \inf p_k > 0 \text{ and } g_2(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/L},$$

respectively. We shall assume throughout that $p_k^{-1} + (p_k^{'})^{-1} = 1$ provided $1 < \inf p_k < H < \infty$.

It is well known that paranormed spaces have more general properties than normed spaces. Recently, there have been many studies on both normed and paranormed sequence spaces. The reader can look at the articles on this subject [16–20, 22–32].

In this work, we generalize the normed sequence spaces defined by Karakaş and Karabudak [2] to paranormed spaces. Let μ denote any of the spaces c_0, c, ℓ_∞ and ℓ_p. We prove that $\mu(\hat{L}, p)$ is linearly paranorm isomorphic to $\mu(p)$ and determine the $\alpha-, \beta-$ and $\gamma-$duals of the $\mu(\hat{L}, p)$. Furthermore, Lucas core of a complex-valued sequence has been introduced, and we prove some inclusion theorems related to this new type of core.

2 The Paranormed Sequence Spaces $c_0(\widehat{L}, p)$, $c(\widehat{L}, p)$, $\ell_\infty(\widehat{L}, p)$ and $\ell(\widehat{L}, p)$

In this section, we define the new sequence spaces $c_0(\widehat{L}, p)$, $c(\widehat{L}, p)$, $\ell_\infty(\widehat{L}, p)$ and $\ell(\widehat{L}, p)$ by using the sequences of Lucas numbers, and prove that these sequence spaces are the complete paranormed linear metric spaces and compute their $\alpha-, \beta-$ and $\gamma-$ duals.

We define the sequence spaces $c_0(\widehat{L}, p)$, $c(\widehat{L}, p)$, $\ell_\infty(\widehat{L}, p)$ and $\ell(\widehat{L}, p)$ by

$$\begin{aligned} c_0(\widehat{L}, p) &= \left\{x = (x_k) \in w : \lim_{n \to \infty} \left|\frac{1}{L_n L_{n+1} + 2} \sum_{i=1}^n L_{i-1}^2 x_i\right|^{p_n} = 0\right\}, \\ c(\widehat{L}, p) &= \left\{x = (x_k) \in w : \exists l \in \mathbb{C} \ni \lim_{n \to \infty} \left|\frac{1}{L_n L_{n+1} + 2} \sum_{i=1}^n L_{i-1}^2 x_i - l\right|^{p_n} = 0\right\}, \\ \ell_\infty(\widehat{L}, p) &= \left\{x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left|\frac{1}{L_n L_{n+1} + 2} \sum_{i=1}^n L_{i-1}^2 x_i\right|^{p_n} < \infty\right\}, \\ \ell(\widehat{L}, p) &= \left\{x = (x_k) \in w : \sum_n \left|\frac{1}{L_n L_{n+1} + 2} \sum_{i=1}^n L_{i-1}^2 x_i\right|^{p_n} < \infty\right\}. \end{aligned}$$

In the case $(p_n) = e = (1, 1, 1, ...)$, the sequence spaces $c_0(\widehat{L}, p)$, $c(\widehat{L}, p)$, $\ell_\infty(\widehat{L}, p)$ and $\ell(\widehat{L}, p)$ are, respectively, reduced to the sequence spaces $c_0(\widehat{L})$, $c(\widehat{L})$, $\ell_\infty(\widehat{L})$ and $\ell(\widehat{L})$ which are introduced by Karakaş and Karabudak [2].

Define the sequence $y = (y_k)$, which will be frequently used as the $\widehat{L}-$transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \widehat{L}_k(x) = \frac{1}{L_k . L_{k-1} + 2} \sum_{i=1}^k L_{i-1}^2 . x_i \ ; \quad (k \in \mathbb{N}_0).$$

Theorem 1. *The following statements hold:*

(i) The sequence spaces $c_0(\widehat{L},p), c(\widehat{L},p)$ and $\ell_\infty(\widehat{L},p)$ are the complete linear metric spaces paranormed by g, defined by

$$g(x)=\sup_{k\in\mathbb{N}}\left|\frac{1}{L_k.L_{k-1}+2}\sum_{i=1}^{k}L_{i-1}^2.x_i\right|^{p_k/L}.$$

(ii) $\ell(\widehat{L},p)$ is a complete linear metric space paranormed by

$$g^*(x)=\left(\sum_k\left|\frac{1}{L_k.L_{k-1}+2}\sum_{i=1}^{k}L_{i-1}^2.x_i\right|^{p_k}\right)^{1/L}.$$

Therefore, one can easily check that the absolute property does not hold on the spaces $c_0(\widehat{L},p), c(\widehat{L},p)$, $\ell_\infty(\widehat{L},p)$ and $\ell(\widehat{L},p)$ that is $h(x)\neq h(|x|)$ for at least one sequence in those spaces, and this says that $c_0(\widehat{L},p), c(\widehat{L},p)$, $\ell_\infty(\widehat{L},p)$ and $\ell(\widehat{L},p)$ are the sequence spaces of non-absolute type; where $|x|=(|x_k|)$.

Theorem 2. *The sequence spaces $c_0(\widehat{L},p), c(\widehat{L},p)$,$\ell_\infty(\widehat{L},p)$ and $\ell(\widehat{L},p)$ are linearly isomorphic to the spaces $c_0(p), c(p), \ell_\infty(p)$ and $\ell(p)$, respectively, where $0<p_k\leq H<\infty$.*

Theorem 3. *The matrix $D=(d_{nk})$ is defined by*

$$d_{nk}=\begin{cases}\tilde{\Delta}\left[\dfrac{a_k}{L_{k-1}^2}\right](L_kL_{k-1}+2) & , \quad (0\leq k\leq n-1)\\[2ex] \dfrac{L_nL_{n-1}+2}{L_{n-1}^2}a_n & , \quad (k=n)\\[2ex] 0 & , \quad (k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$ and $M\in\mathbb{N}_2$. Let $K^*=\{k\in\mathbb{N}:0\leq k\leq n\}\cap K$ for $K\in\mathcal{F}$ and $M\in\mathbb{N}_2$. Define the sets $\widehat{L}_6(p)$, $\widehat{L}_7$, $\widehat{L}_8(p)$, $\widehat{L}_9$, $\widehat{L}_{10}(p)$, $\widehat{L}_{11}(p)$, $\widehat{L}_{12}(p)$, $\widehat{L}_{13}(p)$ as follows:

$$\begin{aligned}
\widehat{L}_6(p) &= \bigcup_{M>1}\left\{a=(a_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|d_{nk}|\,M^{-1/p_k}<\infty\right\},\\
\widehat{L}_7 &= \left\{a=(a_k)\in w:\lim_{n\to\infty}|d_{nk}| \text{ exists for each } k\in\mathbb{N}\right\},\\
\widehat{L}_8(p) &= \bigcup_{M>1}\left\{a=(a_k)\in w:\exists(\alpha_k)\in\mathbb{R}\ni\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|d_{nk}-\alpha_k|\,M^{-1/p_k}<\infty\right\},\\
\widehat{L}_9 &= \left\{a=(a_k)\in w:\exists\alpha\in\mathbb{R}\ni\lim_{n\to\infty}\left|\sum_{k=0}^{n}d_{nk}-\alpha\right|=0\right\},\\
\widehat{L}_{10}(p) &= \bigcap_{M>1}\left\{a=(a_k)\in w:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}|d_{nk}|\,M^{-1/p_k}<\infty\right\},\\
\widehat{L}_{11}(p) &= \bigcap_{M>1}\left\{a=(a_k)\in w:\exists(\alpha_k)\in\mathbb{R}\ni\lim_{n\to\infty}\sum_{k=0}^{n}|d_{nk}-\alpha_k|\,M^{1/p_k}=0\right\},\\
\widehat{L}_{12}(p) &= \bigcup_{M>1}\left\{a=(a_k)\in w:\sup_{n}\sup_{k\in K^*}|d_{nk}M^{-1}|^{p_k}<\infty\right\},\\
\widehat{L}_{13}(p) &= \bigcup_{M>1}\left\{a=(a_k)\in w:\sup_{n}\sum_{k\in K^*}|d_{nk}M^{-1}|^{p_k'}<\infty\right\}.
\end{aligned}$$

Then,

(i) $\{c_0(\widehat{L},p)\}^\beta=\widehat{L}_6(p)\cap\widehat{L}_7\cap\widehat{L}_8(p)$,

(ii) $\{c(\widehat{L},p\}^\beta=\{c_0(\widehat{L},p)\}^\beta\cap\widehat{L}_9$,

(iii) $\{\ell_\infty(\widehat{L},p)\}^\beta=\widehat{L}_{10}(p)\cap\widehat{L}_{11}(p)$,

(iv) $\{\ell(\widehat{L},p)\}^\beta=\widehat{L}_{12}(p)\cap\widehat{L}_{13}(p)$.

3 Lucas Core

Following Knopp, a core theorem is characterized a class of matrices for which the core of the transformed sequence is included by the core of the original sequence. For example Knopp Core Theorem [[33], p.138] states that $K-core(Ax) \subseteq K-core(x)$ for all real valued sequences x whenever A is a positive matrix in the class $(c:c)_{reg}$.

Now, let us write

$$y_n(x) = \widehat{L}_n(x) = \frac{1}{L_nL_{n-1}+2}\sum_{k=1}^{n} L_{k-1}^2 x_k; \quad (k \in \mathbb{N}_0).$$

Then we can define $\widehat{L}-$ core of a complex sequence as follows:

Let H_n be the least closed convex hull containing $y_n(x), y_{n+1}(x), , \ldots$. Then, $\hat{L}-$ core of x is the intersection of all H_n, i.e.,

$$\hat{L} - core(x) = \bigcap_{n=1}^{\infty} H_n.$$

Now, we may give some inclusion theorems. For brevity, in what follows we write $\tilde{e}_{nk}$ in place of

$$\frac{1}{L_nL_{n-1}+2}\sum_{k=1}^{n} L_{k-1}^2 x_k.$$

Theorem 4. *Let $B \in (c : c(\hat{L}))_{reg}$. Then, $\hat{L} - core(Bx) \subseteq K - core(x)$ for all $x \in \ell_\infty$ if and only if*

$$\lim_n \sum_k |\tilde{e}_{nk}| = 1. \tag{1}$$

Theorem 5. *Let $B \in (st \cap \ell_\infty : c(\hat{L}))_{reg}$. Then, $\hat{L} - core(Bx) \subseteq st - core(x)$ for all $x \in \ell_\infty$ if and only if (1) holds.*

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