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Modifications of Strongly Nodec Spaces

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Abstract
In this paper, we introduce the notion of strongly nodec spaces and study their properties. Also, we discuss strongly nodec generalized metric spaces. Furthermore, we extend these notions to T_0-strongly nodec space by using the quotient map.

Keywords: Strongly nowhere dense, Dense, Codense, Strongly nodec space
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1. Introduction

The notion of a generalized topological space was introduced by Császár in [1]. Let X be any non-empty set. A family $\mu \subset exp(X)$ is a *generalized topology* [2] in X if $\emptyset \in \mu$ and $\bigcup\limits_{i\in I} G_i \in \mu$ whenever $\{G_i : i \in I\} \subset \mu$ where $exp(X)$ is a power set of X. We call the pair (X,μ) as a *generalized topological space* (GTS) [2]. If $X \in \mu$, then the pair (X,μ) is called a *strong generalized topological space* (sGTS) [2].

The elements in μ are called the μ-open sets and the complement of a μ-open set is called the μ-closed sets.

Let (X,μ) be a GTS and $A \subset X$. The *interior of A* [2] denoted by iA, is the union of all μ-open sets contained in A and the *closure of A* [2] denoted by cA, is the intersection of all μ-closed sets containing A.

In 2013, Korczak-Kubiak et al. introduced the notations $\tilde{\mu}, \mu(x)$ defined by $\{U \in \mu : U \neq \emptyset\}, \{U \in \mu : x \in U\}$ respectively [3].

Let (X,μ) be a GTS and $Y \subseteq X$. The *subspace generalized topology* is defined by, $\mu_Y = \{U \cap Y : U \in \mu\}$. Then the pair (Y,μ_Y) is called the *subspace GTS*. Furthermore, (X,μ) is strong GTS if and only if $c(\emptyset) = \emptyset$ if and only if $\emptyset$ is closed [4].

2. Preliminaries

In this section, we remember some basic definitions and lemmas which will be useful to prove the results in the following sections.

We already familiar with nowhere dense sets in generalized topological space. In 2013, Korczak - Kubiak et al. defined a new one namely strongly nowhere dense sets and discussed their properties. Also, they gave a relation between nowhere dense and strongly nowhere dense sets in generalized topological space [3]. In [5], we analyze properties of strongly nowhere dense sets in generalized topological spaces. With this terminology, we define a new space namely strongly nodec space and study some properties of strongly nodec spaces.

Let (X,μ) be a GTS and $A \subset X$. A is said to be a α*-open* (resp. α*-closed*) set if $A \subset iciA$ (resp. $cicA \subset A$). The *interior of A* [2] denoted by $i_\alpha A$, is the union of all α-open sets contained in A and the *closure of A* [2] denoted by $c_\alpha A$, is the intersection of all α-closed sets containing A. A subset A is said to be μ-*nowhere dense* [3] (resp. μ-*dense*, μ-*codense* [2]) if $icA = \emptyset$ (resp. $cA = X$, $c(X-A) = X$).

A subset A of a GTS (X, μ) is said to be μ-*strongly nowhere dense* [3] if for any $V \in \tilde{\mu}$, there exists $U \in \tilde{\mu}$ such that $U \subset V$ and $U \cap A = \emptyset$. A is said to be a μ-*s-meager* [3] set if $A = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is a μ-strongly nowhere dense sets for all $n \in \mathbb{N}$ where $\mathbb{N}$ denote the set of all natural numbers.

Let (X, μ) be a GTS and $A \subset X$. A is said to be a μ-*s-second category* (μ-s-II category) [3] set if A is not a μ-s-meager set. A subset A of a GTS (X, μ) is said to be a μ-*s-residual* set if $X - A$ is a μ-s-meager set in X.

In GTS, every μ-strongly nowhere dense set is μ-nowhere dense and every subset of a μ-strongly nowhere dense set is μ-strongly nowhere dense [3]. Also, every subset of a μ-s-meager set is a μ-s-meager set [3]. If A is a μ-strongly nowhere dense set, then cA is a μ-strongly nowhere dense set [3].

Let (X, μ) be a GTS. X is said to be a *generalized submaximal space* [6] if every μ-dense subset of X is a μ-open set in X.

Throughout this paper μ-strongly nowhere dense, μ-nowhere dense, μ-s-meager, μ-s-residual and etc., we will write strongly nowhere dense, nowhere dense, s-meager, s-residual and etc., when no confusion can arise.

The following lemmas will be useful in the sequel.

Lemma 2.1. [7, Lemma 2.6] Let (X, μ) be a GTS and $A \subset X$. Then $i(cA - A) = \emptyset$.

Lemma 2.2. [6, Theorem 19] Let (X, μ) be a GTS. The following properties are equivalent:
(a) X is a generalized submaximal space.
(b) Each μ-codense subset A of (X, μ) is μ-closed.

Lemma 2.3. [2, Lemma 2.3] Let (X, μ) be a GTS and let $A \subset S \subset X$. Then $c_S A = cA \cap S$.

Lemma 2.4. [2, Lemma 3.2] Let (X, μ) be a GTS and let $A, U \subset X$. If $U \in \tilde{\mu}$ and $U \cap A = \emptyset$, then $U \cap cA = \emptyset$.

Lemma 2.5. [5, Theorem 2.13] Let (X, μ) be a GTS and $A \subset X$. If A is a μ-strongly nowhere dense set in X, then A is codense.

3. Strongly nodec spaces

In this section, we define strongly nodec space and give the example for the existence of this space in generalized topological spaces. Further, we discuss the properties of strongly nodec space in generalized topological spaces. Also, we prove product of two GTS is strongly nodec then each one is a strongly nodec space.

Definition 3.1. *Let (X, μ) be a GTS. A space X is said to be a strongly nodec space if every non-empty μ-strongly nowhere dense subset of X is μ-closed in X.*

Example 3.2 shows the existence of the strongly nodec space and Theorem 3.4 give the necessary condition for a sGTS to be a strongly nodec space.

Example 3.2. (a) Consider the GTS (X, μ) where $X = \{a,b,c,d,e\}$ and $\mu = \{\emptyset, \{a,d\}, \{a,e\}, \{b,e\}, \{a,d,e\}, \{a,b,e\}, \{a,b,d,e\}\}$. Here the μ-strongly nowhere dense set is $\{c\}$ which is also a μ-closed set in X. Therefore, X is a strongly nodec space.
(b) Consider the GTS (X, μ) where $X = \mathbb{R}$ and $\mu = \{\emptyset\} \cup \{A \subset X : A - \{x\} \subset A \text{ for some } x \in X\}$. Here, there is no μ-strongly nowhere dense set in X. Therefore, X is a strongly nodec space.

Lemma 3.3. *In a GTS (X, μ), every μ-strongly nowhere dense set does not contains a non-empty μ-open set.*

Proof. Let A be a μ-strongly nowhere dense set in X. Suppose there is $U \in \tilde{\mu}$ such that $U \subset A$. Then there is no $V \in \tilde{\mu}$ such that $V \subset U$ and $V \cap A = \emptyset$, which is a contradiction to A is μ-strongly nowhere dense in X. This implies $U = \emptyset$. Therefore, A does not contains a non-empty μ-open set in X. Hence every μ-strongly nowhere dense set does not contains a non-empty μ-open set. □

Theorem 3.4. *Let (X, μ) be a sGTS. Then X is a strongly nodec space if any one of the following hold.*
(a) Every α-closed set is a μ-closed set.
(b) Every α-open set is a μ-open set.
(c) For each $A \subseteq X, cA - A \subseteq cic(A)$.
(d) For each $A \subseteq X, cA = A \cup cicA$.
(e) For each $A \subseteq X, iA = A \cap iciA$.

Proof. Assume (a). Let A be a non-empty strongly nowhere dense set in X. Then A is a non-empty nowhere dense set and so $icA = \emptyset$. Since X is a sGTS, $cicA = \emptyset$. This implies $cicA \subset A$ which implies A is a α-closed set. By (a), A is a μ-closed set in X. Hence X is a strongly nodec space.
Assume (b). Similar considerations in (a), we prove X is a strongly nodec space.
Assume (c). Let A be a non-empty strongly nowhere dense set in X. Then A is a non-empty nowhere dense set and so $icA = \emptyset$. Since X is a sGTS, $cicA = \emptyset$. By (c), $cA - A = \emptyset$. Thus, $cA = A$. Therefore, A is a μ-closed set in X. Hence X is a strongly nodec space.
Assume (d). Let A a non-empty strongly nowhere dense set in X. Then by same process in (a), $cicA = \emptyset$. By (d), $cA = A \cup \emptyset = A$. Therefore, A is a μ-closed set in X. Hence X is a strongly nodec space.
Assume (e). Similar considerations in (d), we prove X is a strongly nodec space. □

Definition 3.5. *Let (X, μ) be a GTS and $A \subset X$. Then frontier of A is denoted by $Fr(A)$ and defined by $Fr(A) = cA \cap c(X - A)$. Then frontier of A is a closed set in X.*

Example 3.6 shows that the existence of $Fr(A)$ and Lemma 3.7 give some properties of $Fr(A)$ in a generalized topological space (X, μ) where $A \subset X$.

Example 3.6. (a) Consider the GTS (X, μ) where $X = [0,5]$ and $\mu = \{\emptyset, [0,2), (1,3), [2,4), [0,3), (1,4), [0,4)\}$. Let $A = [3,4)$ be a subset of X. Then $Fr(A) = [3,5]$. Also, $Fr(A)$ is a μ-closed set in X.
(b) Consider the GTS (X, μ) where $X = \{a,b,c,d\}$ and $\mu = \{\emptyset, \{a,b\}, \{b,c\},$
$\{a,b,c\}\}$. Let $A = \{a,b\}$ be a subset of X. Then $Fr(A) = \{c,d\}$. Also, $Fr(A)$ is a μ-closed set in X.

Lemma 3.7. *Let (X, μ) be a GTS. Then the following hold.*
(a) If B is a strongly nowhere dense set in X, then $Fr(B)$ is a strongly nowhere dense set in X for all $B \subset X$.
(b) If $Fr(A)$ is a strongly nowhere dense set in X, then $Fr(iA), Fr(cA)$ is a strongly nowhere dense set in X for all $A \subset X$.
(c) If A is a strongly nowhere dense set in X, then $Fr(A \cap B)$ is a strongly nowhere dense set in X for all $A, B \subset X$.

Proof. (a) Suppose $B \subset X$ is a strongly nowhere dense set. Let $U \in \tilde{\mu}$. Then there exists $V \in \tilde{\mu}$ such that $V \subset U$ and $V \cap B = \emptyset$. By Lemma 2.4, $V \cap cB = \emptyset$. This implies $V \cap Fr(B) = \emptyset$, since $Fr(B) \subset cB$. Therefore, $Fr(B)$ is a strongly nowhere dense set in X.
(b) Suppose $Fr(A)$ is a strongly nowhere dense set in X. Now $Fr(iA) = ciA \cap c(X - iA) = ciA \cap c(X - (X - c(X - A)))$. This implies $Fr(iA) = ciA \cap c(c(X - A)) = ciA \cap c(X - A)$ which implies $Fr(iA) \subset cA \cap c(X - A) \subset Fr(A)$. Thus, $Fr(iA) \subset Fr(A)$. Since subset of a strongly nowhere dense set is strongly nowhere dense, $Fr(iA)$ is a strongly nowhere dense set in X. Now $Fr(cA) = ccA \cap c(X - cA)$. This implies $Fr(cA) \subset cA \cap c(X - A)$ which implies $Fr(cA) \subset Fr(A)$ and hence $Fr(cA)$ is strongly nowhere dense set X.
(c) Suppose A is a strongly nowhere dense set in X. Since $A \cap B \subset A$ and subset of a strongly nowhere dense set is strongly nowhere dense, $Fr(A \cap B)$ is a strongly nowhere dense set in X, by (a). □

Example 3.8 shows the reverse implication of (a) in Lemma 3.7 is not necessary.

Example 3.8. *Consider the GTS (X, μ) where $X = \{a,b,c,d,e\}$and $\mu = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. Let $B = \{a,c,d\} \subset X$. Then $Fr(B) = \{b,d,e\}$ and so $Fr(B)$ is a strongly nowhere dense set in X. But B is not strongly nowhere dense set in X.*

Proposition 3.9. *Let (X, μ) be a strongly nodec space. If A is a non-empty strongly nowhere dense set in X, then $Fr(A) \subset A$ and hence $Fr(A)$ is a strongly nowhere dense set in X.*

Proof. Suppose A is a non-empty strongly nowhere dense set in X. By hypothesis, A is a closed set in X. Now $Fr(A) = cA \cap c(X - A) = A \cap c(X - A)$. Therefore, $Fr(A) \subset A$. By Lemma 3.7 (a) and hypothesis, $Fr(A)$ is a strongly nowhere dense set in X. □

Proposition 3.10. *Let (X, μ) be a GTS. If $Fr(A)$ is strongly nowhere dense $\Rightarrow A$ is closed for all $A \subset X$, then X is a strongly nodec space.*

Proof. Let A be a non-empty strongly nowhere dense set in X. Then by Lemma 3.7 (a), $Fr(A)$ is a strongly nowhere dense set in X. By hypothesis, A is a closed set in X. Hence X is a strongly nodec space. □

Every generalized submaximal space is a strongly nodec space. This implication is not reversible as shown in the following Example 3.11.

Example 3.11. Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a,b\},\{a,d\},\{b,c\},\{a,b,d\},\{a,b,c\},X\}$. Here, every μ-strongly nowhere dense set is a μ-closed set in X. Therefore, X is a strongly nodec space. Let $A=\{a,c,d\}$. Then A is μ-dense in X but not μ-open in X. Thus, X is not a generalized submaximal space.

Theorem 3.12 gives the necessary condition for a strongly nodec space to be a generalized submaximal space and Example 3.13 shows that frontier of a dense set is strongly nowhere dense set is necessary.

Theorem 3.12. *Let (X,μ) be a sGTS. If every frontier of a dense subset of X is a strongly nowhere dense set in X and X is a strongly nodec space, then X is a generalized submaximal space.*

Proof. Suppose X is a strongly nodec space. Let A be a μ-dense subset of X. By hypothesis, $cA-iA$ is a strongly nowhere dense set in X. Then $X-iA$ is a strongly nowhere dense set in X, since $cA=X$ and so $X-A$ is a strongly nowhere dense set in X, since subset of a μ-strongly nowhere dense set is μ-strongly nowhere dense. Suppose $X-A=\emptyset$. Then $X-A$ is a closed set in X, that is $c(\emptyset)=\emptyset$, since X is a sGTS. Therefore, A is a μ-open set in X. Suppose $X-A\neq\emptyset$. Since X is a strongly nodec space, $X-A$ is a closed set in X. Therefore, A is a μ-open set in X. Hence X is a generalized submaximal space. ☐

Example 3.13. Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a,b\},\{a,c\},\{a,b,c\},X\}$. Here, every μ-strongly nowhere dense set is a μ-closed set in X. Therefore, X is a strongly nodec space. Let $A=\{a,c,d\}$. Then A is a μ-dense subset of X. But $Fr(A)=\{b,d\}$ is not a strongly nowhere dense set. For, let $U=\{a,b\}\in\tilde{\mu}$. Then there is no $V\in\tilde{\mu}$ such that $V\subset U$ and $V\cap Fr(A)$. Let $B=\{a,d\}$. Then B is μ-dense in X but not μ-open in X. Thus, X is not a generalized submaximal space.

Next Example 3.14 shows the existence of a non-generalized submaximal space satisfying the necessary condition in Theorem 3.12 which is not a strongly nodec space.

Example 3.14. Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$. Here, $A=\{c\}$ is a μ-strongly nowhere dense set in X but not μ-closed in X. Thus, X is not a strongly nodec space. Clearly, every frontier of A is a strongly nowhere dense set where A is a dense subset of X. Let $B=\{a,b,d\}$. Then B is μ-dense in X but not μ-open in X. Thus, X is not a generalized submaximal space.

Theorem 3.15 characterizes strongly nodec space in strong generalized topological space and Theorem 3.16 give one property of s-meager, s-residual set in strongly nodec space, the essay proof is omitted.

Theorem 3.15. *Let (X,μ) be a sGTS and $A\subseteq X$. If frontier of A is a strongly nowhere dense set, then the following are equivalent.*
(a) X is a strongly nodec space.
(b) For each $A\subseteq X, cA-A\subseteq cicA$.
(c) For each $A\subseteq X, cA=A\cup cicA$.
(d) For each $A\subseteq X, iA=A\cap iciA$.

Proof. (a) $\Rightarrow$ (b) Suppose X is a strongly nodec space. Let $A\subseteq X$. By hypothesis, frontier of A is a strongly nowhere dense set and so $A-icA$ is strongly nowhere dense. Suppose $A-icA=\emptyset$. Since (X,μ) is a sGTS, $c(\emptyset)=\emptyset$ and so $A-icA$ is a closed set in X. Then $d(A-icA)=\emptyset$ where the notation $d(A-icA)$ is the derived set of $A-icA\subset X$. Now $d(icA)\subseteq cicA$ and so $d(A)-cicA\subseteq d(A)-d(icA)\subseteq d(A-icA)=\emptyset$. Thus, $d(A)\subseteq cicA$. Therefore, $cA-A\subseteq cicA$. Suppose $A-icA\neq\emptyset$. By (a), $A-icA$ is a closed set and so $d(A-icA)\subset A-icA$. Let $x\in A-icA$. Since $A-icA$ is strongly nowhere dense, $i(A-icA)=\emptyset$, by Lemma 2.5 and so $i(A-icA-\{x\})=\emptyset$. Take $B=A-icA$. Then B is a codense set and so $B-\{x\}$ is a codense set in X. By hypothesis and Theorem 3.12, X is a generalized submaximal space. Therefore, $B-\{x\}$ is a closed set in X. Hence $\{x\}\cup(X-B)$ is a non-empty open set in X. Let $U_x=\{x\}\cup(X-B)$. Thus, there is a neighbourhood U_x of x such that $U_x\cap(B-\{x\})=\emptyset$ and so $x\notin d(B)$. Therefore, $d(A-icA)=\emptyset$. Now $d(icA)\subseteq cicA$ and so $d(A)-cicA\subseteq d(A)-d(icA)\subseteq d(A-icA)=\emptyset$. Thus, $d(A)\subseteq cicA$. Therefore, $cA-A\subseteq cicA$.
(b) $\Rightarrow$ (c) Let $A\subseteq X$. By (b), $cA\subseteq A\cup cicA$. Now $cicA\subseteq cA$. This implies $A\cup cicA\subseteq A\cup cA=cA$ which implies $A\cup cicA\subseteq cA$. Therefore, $cA=A\cup cicA$.
(c) $\Leftrightarrow$ (d) it is obvious.
(c) $\Rightarrow$ (a) Let $\emptyset\neq A\subset X$ be a strongly nowhere dense set. Then A is a nowhere dense set in X, and so $icA=\emptyset$. By (c) and hypothesis, $cA=A\cup\emptyset=A$. Thus, A is a closed set in X. Therefore, X is a strongly nodec space. ☐

Theorem 3.16. *Let (X,μ) be a GTS. If X is a strongly nodec space, then the following hold.*
(a) Every s-meager set is a F_σ-set.
(b) Every s-residual set is a G_δ-set.

In GTS, a subspace of a strongly nodec space need not be a strongly nodec space even if the subspace is either closed or dense-open as shown by the following Example 3.17.

Example 3.17. (a) Consider the GTS (X,μ) where $X=\{a,b,c,d,e\}$ and $\mu=\{\emptyset,\{a\},\{a,b\},\{a,d\},\{a,b,d\},\{b,d,e\},\{a,b,d,e\}\}$. Here, every μ-strongly nowhere dense set is a μ-closed set in X. Therefore, X is a strongly nodec space. Let $Y=\{b,c,d,e\}$ be a closed subset of X. Then $\mu_Y=\{\emptyset,\{b\},\{d\},\{b,d\},\{b,d,e\}\}$. Let $A=\{e\}\subset Y$. Then A is a μ_Y-strongly nowhere dense set in Y. But A is not a μ_Y-closed set. Thus, Y is not a strongly nodec space.
(b) Consider the GTS (X,μ) where $X=\{a,b,c,d,e,f\}$ and $\mu=\{\emptyset,\{a,b\},\{b,c\},\{d,f\},\{a,b,c\},\{c,d,f\},\{a,b,d,f\},\{b,c,d,f\},\{a,b,c,d,f\},\{b,c,d,e,f\},\{a,b,c,d,e\},X\}$. Here, every μ-strongly nowhere dense set is a μ-closed set in X. Therefore, X is a strongly nodec space. Let $Y=\{b,c,d,e,f\}$ be a dense-open subspace of X. Then $\mu_Y=\{\emptyset,\{b\},\{b,c\},\{d,f\},\{b,d,f\},\{c,d,f\},\{b,c,d,f\},\{b,c,d,e\},Y\}$. Let $A=\{c\}\subset Y$. Then A is a μ_Y-strongly nowhere dense set in Y. But A is not a μ_Y-closed set. Thus, Y is not a strongly nodec space.

Next one is the definition for a subspace of a space is strongly nodec with respect to the space and Example 3.19 shows the existence of this space.

Definition 3.18. *Let (X,μ) be a GTS. A subspace Y of X is said to be strongly nodec with respect to X if every non-empty μ_Y-strongly nowhere dense set is a μ-closed set. Y is said to be a strongly nodec space if Y is strongly nodec as a subspace.*

Example 3.19. Consider the GTS (X,μ) where $X=\{a,b,c,d,e\}$ and $\mu=\{\emptyset,\{a,d,e\},\{b,d,e\},\{a,b,d,e\},\{a,c,d,e\},X\}$. Let $Y=\{a,b,c,e\}$. Then $\mu_Y=\{\emptyset,\{a,e\},\{b,e\},\{a,b,e\},\{a,c,e\},Y\}$. Here, every μ_Y-strongly nowhere dense set is μ-closed. Thus, Y is a strongly nodec space with respect to X.

In GTS, every subspace strongly nodec with respect to X is a strongly nodec space as a subspace. This implication is not reversible as shown in Example 3.20.

Example 3.20. Consider the GTS (X,μ) where $X=\{a,b,c,d,e,f\}$ and $\mu=\{\emptyset,\{a,b,c\},\{b,c,d\},\{a,b,c,d\},\{a,b,c,e\},\{b,c,e,f\},\{a,b,c,d,e\},\{a,b,c,e,f\},X\}$. Let $Y=\{a,b,c,e\}$. Then $\mu_Y=\{\emptyset,\{b,c\},\{a,b,c\},\{b,c,e\},Y\}$. Here, every μ_Y-strongly nowhere dense set is a μ_Y-closed set in Y. Therefore, Y is a strongly nodec space. But Y is not strongly nodec with respect to X. For, let $A=\{a\}$. Then A is μ_Y-strongly nowhere dense set but not μ-closed.

Theorem 3.21. *Let (X,μ) be a GTS and Y be a dense subspace of X. If Y is a strongly nodec with respect to X, then X is a strongly nodec space.*

Proof. Suppose X is a strongly nodec space. Let A be a non-empty μ-strongly nowhere dense set in X. Suppose $A\cap Y=\emptyset$. Then A is a non-empty μ_Y-strongly nowhere dense set. By hypothesis, A is a μ-closed set. Suppose $A\cap Y\neq\emptyset$. Let $U\in\tilde{\mu}_Y$. Then $U=U_1\cap Y$ where $U_1\in\tilde{\mu}$. Since A is μ-strongly nowhere dense set, there exists $V_1\in\tilde{\mu}$ such that $V_1\subset U_1$ and $V_1\cap A=\emptyset$. This implies that $V_1\cap Y\in\tilde{\mu}_Y$, since Y is a dense subspace of X. Take $V=V_1\cap Y$. Thus, there exists $V\in\tilde{\mu}_Y$ such that $V\subset U$ and $V\cap A=\emptyset$. Therefore, A is a non-empty μ_Y-strongly nowhere dense set. By hypothesis, A is μ-closed. Hence X is a strongly nodec space. □

Theorem 3.22. *Let (X,μ) be a generalized submaximal space. Then every subset of X is a strongly nodec with respect to X and hence a strongly nodec space.*

Proof. Let Y be a subset of X and A be a non-empty μ_Y-strongly nowhere dense subset of Y. Then A is a μ_Y-codense set and so $c_{\mu_Y}(Y-A)=Y$, by Lemma 2.5. Now $c_{\mu_Y}(Y-A)=c_\mu(Y-A)\cap Y$, by Lemma 2.3. This implies $Y=c_\mu(Y-A)\cap Y$ which implies $Y\subseteq c_\mu(Y-A)\subseteq c_\mu(X-A)$. Thus, $Y\subseteq X-i_\mu A$. Therefore, $i_\mu A=\emptyset$ and so A is a μ-codense set in X. By hypothesis, A is a μ-closed set. Hence Y is a strongly nodec with respect to X. By Lemma 2.3, $c_Y A=cA\cap Y=A\cap Y=A$, since $A\subset Y$ is a μ-closed set. Therefore, A is a μ_Y-closed set. Hence Y is a strongly nodec space. □

Theorem 3.23. *Let (X,μ) be a strongly nodec space. Then every non-empty μ-strongly nowhere dense subspace of X is a strongly nodec with respect to X and hence a strongly nodec space.*

Proof. Let Y be a non-empty μ-strongly nowhere dense subspace of X. Let A be a non-empty μ_Y-strongly nowhere dense subset of Y. Then A is a non-empty μ-strongly nowhere dense set, since subset of a μ-strongly nowhere dense set is a μ-strongly nowhere dense set. By hypothesis, A is a μ-closed set. Therefore, Y is a strongly nodec with respect to X. By Lemma 2.3, $c_Y A=cA\cap Y=A\cap Y=A$, since $A\subset Y$ is a μ-closed set. Therefore, A is a μ_Y-closed set. Hence Y is a strongly nodec space. □

Theorem 3.24. *Let (X,μ) be a strongly nodec space. Then every non-empty frontier of a μ-strongly nowhere dense subspace of X is a strongly nodec with respect to X and hence a strongly nodec space.*

Proof. Let A be a non-empty μ-strongly nowhere dense set in X. By Lemma 3.7, $Fr(A)$ is a non-empty μ-strongly nowhere dense set in X. Then by Theorem 3.23, $Fr(A)$ is a strongly nodec with respect to X and hence a strongly nodec space. Hence every non-empty frontier of a μ-strongly nowhere dense subspace of X is a strongly nodec with respect to X and hence a strongly nodec space. □

Lemma 3.25. *Let $(X,\mu_X),(Y,\mu_Y)$ be a two GTSs. Then the following hold.*
(a) If A and B are strongly nowhere dense sets in X,Y respectively, then $A\times B$ is strongly nowhere dense in $X\times Y$.
(b) If $C\times D$ is strongly nowhere dense in $X\times Y$, then C or D or C and D is strongly nowhere dense set in X or Y respectively.

Proof. (a) Suppose A and B are strongly nowhere dense sets in X,Y respectively. Let $U_1\times U_2\in\tilde{\mu}_{X\times Y}$. Then $U_1\in\tilde{\mu}_X$ and $U_2\in\tilde{\mu}_Y$. By hypothesis, there exists $V_1\in\tilde{\mu}_X, V_2\in\tilde{\mu}_Y$ such that $V_1\subset U_1, V_2\subset U_2$ and $V_1\cap A=\emptyset, V_2\cap B=\emptyset$. Thus, there exists $V_1\times V_2\in\tilde{\mu}_{X\times Y}$ such that $V_1\times V_2\subset U_1\times U_2$ and $V_1\times V_2\cap A\times B=\emptyset$. Therefore, $A\times B$ is strongly nowhere dense in $X\times Y$.
(b) Suppose $C\times D$ is strongly nowhere dense in $X\times Y$. Let $G_1\in\tilde{\mu}_X, G_2\in\tilde{\mu}_Y$. Then $G_1\times G_2\in\tilde{\mu}_{X\times Y}$. By hypothesis, there exists $H_1\times H_2\in\tilde{\mu}_{X\times Y}$ such that $H_1\times H_2\subset G_1\times G_2$ and $H_1\times H_2\cap C\times D=\emptyset$. Since $H_1\times H_2\neq\emptyset$ and $H_1\times H_2\subset G_1\times G_2$, $H_1\subset G_1$ and $H_2\subset G_2$. Now $H_1\times H_2\cap C\times D=\emptyset, H_1\cap C\times H_2\cap D=\emptyset$. This implies $H_1\cap C=\emptyset$ or $H_2\cap D$ or $H_1\cap C=\emptyset$ and $H_2\cap D=\emptyset$. Thus, C is strongly nowhere dense in X or D is strongly nowhere dense in Y or C and D are strongly nowhere dense sets in X,Y respectively. □

Theorem 3.26. *Product of two GTS is strongly nodec, then each one is strongly nodec.*

Proof. Let $(X,\mu_X),(Y,\mu_Y)$ be a two GTSs. Suppose $X\times Y$ is a strongly nodec space. Let A and B are non-empty strongly nowhere dense sets in X,Y respectively. Then by Lemma 3.25, $A\times B$ is a non-empty strongly nowhere dense set in $X\times Y$. By hypothesis, $A\times B$ is a closed set in $X\times Y$. This implies A is a closed set in X and B is a closed set in Y. Hence X and Y are strongly nodec space. □

4. On T_0-strongly nodec spaces

In this section, we define T_0-strongly nodec space and give the example for the existence of this space in generalized topological spaces. Further, we discuss the properties of T_0-strongly nodec space in generalized topological spaces by using a quotient maps. Also, we introduce and give some results for T_0-generalized submaximal space in generalized topological space.

Let (X,μ) be a GTS. We define the binary relation $\sim$ on X by $x\sim y$ if and only if $c\{x\}=c\{y\}$. Then $\sim$ is an equivalence relation on X and the resulting quotient space $T_0(X)=X/\sim$ is the T_0*-reflection of* X and the generalized quotient topology on $T_0(X)$ is defined to be $\mu_q=\{G\subset T_0(X): f^{-1}(G)\in\mu\}$ where q is a *canonical or quotient map* from X into $T_0(X)$ by setting $x\in X$ to its equivalence class $[x]$ in $T_0(X)$. Then the pair $(T_0(X),\mu_q)$ is called the *generalized quotient space of* X.

Let (X,μ) and (Y,λ) be two generalized topological spaces. A function $f:X\to Y$ is called (μ,λ)*-continuous* if $f^{-1}(V)\in\mu$ for each $V\in\lambda$ [2]. A function $f:X\to Y$ is called (μ,λ)*-open* [2] if $f(V)\in\lambda$ for each $V\in\mu$. A function $f:X\to Y$ is called (μ,λ)*-closed* if $f(U)$ is a λ-closed set for each U is a μ-closed set.

A (μ,λ)-continuous map $f:(X,\mu)\to(Y,\lambda)$ is said to be a *quasi-homeomorphism* if $U\to f^{-1}(U)$ (resp. $C\to f^{-1}(C)$) defines a bijection $O(Y)\to O(X)$ (resp. $F(Y)\to F(X)$) where $O(X)$ (resp. $F(X)$) is the collection of all μ-open (resp. μ-closed) sets of X.

Equivalently, (μ,λ)-continuous map $f:X\to Y$ is a *quasi-homeomorphism* if for each μ-open subset U of X, there exists a unique λ-open subset V of Y such that $U=f^{-1}(V)$ (equivalently, for each μ-closed subset F of X, there exists a unique λ-closed subset G of Y such that $F=f^{-1}(G)$).

Lemma 4.1. [2, Lemma 7.3] Let (X,μ) and (Y,λ) be two generalized topological spaces. A mapping $f:(X,\mu)\to(Y,\lambda)$ is (μ,λ)-open if and only if $f^{-1}(cB)\subset c(f^{-1}(B))$ for any $B\subset Y$.

Proposition 4.2. *Let (X,μ) and (Y,λ) be two generalized topological spaces. If $f:X\to Y$ is a surjective, quasi-homeomorphism map, then f is a (μ,λ)-open map.*

Proof. Let A be a μ-open set in X. Since f is quasi-homeomorphism, there exists a unique λ-open subset V of Y such that $A=f^{-1}(V)$. Now $f(A)=f(f^{-1}(V))=V$, since f is a surjective map. Therefore, $f(A)$ is a λ-open set in Y. Hence f is a (μ,λ)-open map. □

Example 4.3. *Consider two GTSs (X,μ) and (Y,λ) where $X=\{a,b,c,d\}$, $\mu=\{\emptyset,\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$ and $Y=\{a,b,c,d,e\}$, $\lambda=\{\emptyset,\{a,b\},\{a,b,e\},\{a,c,e\},\{b,c,e\},\{a,b,c,e\}\}$. Define a map $f:X\to Y$ by $f(a)=a, f(b)=b, f(c)=c, f(d)=d$. Then f is a quasi-homeomorphism but not a surjective map. Let $U=\{a,c\}$. Now $f(U)=\{a,c\}$. But $f(U)$ is not a λ-open set. Thus, f is not a (μ,λ)-open map.*

Notations 4.4. *Let (X,μ) be a GTS, $a\in X$ and $A\subseteq X$. We use the following notations:*
(1) $d_0(a)=\{x\in X : c\{x\}=c\{a\}\}$.
(2) $d_0(A)=\bigcup[d_0(a): a\in A]$.

Example 4.5 shows the existence of $d_0(A)$ in generalized topological space (X,μ) where $A\subset X$ and the next Lemma 4.6 give some properties of $d_0(A)$ and the canonical surjective map in generalized topological space.

Example 4.5. (a). Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a\},\{a,c\},\{b,c\},\{a,b,c\}\}$. Let $A=\{a,c,d\}$. Now $d_0(a)=\{a\}, d_0(c)=\{c\}$ and $d_0(d)=\{d\}$. Therefore, $d_0(A)=A$.
(b). Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a\},\{b,c\},\{a,b,c\}\}$. Let $A=\{a,c,d\}$. Now $d_0(a)=\{a\}, d_0(c)=\{b,c\}$ and $d_0(d)=\{d\}$. Therefore, $d_0(A)=X$.

Lemma 4.6. *Let (X,μ) be a GTS, $A\subset X$ and $q:(X,\mu)\to(T_0(X),\mu_q)$ be a canonical surjective map. Then the following hold.*
(a) The map q is a quasi-homeomorphism.
(b) The map q is (μ,μ_q)-open, (μ,μ_q)-closed map.
(c) $A\subseteq d_0(A)\subseteq cA$ and consequently $c(d_0(A))=cA$.
(d) If A is a closed set, then $d_0(A)=A$.
(e) $d_0(A)=q^{-1}(q(A))$.
(f) If $\{A_n\}_{n\in\mathbb{N}}$ is a collection of subsets of X, then $d_0(\bigcup_{n\in\mathbb{N}} A_n)=\bigcup_{n\in\mathbb{N}} d_0(A_n)$.

Proof. (a) Define a map $f:O(T_0(X))\to O(X)$ by $f(U)=q^{-1}(U)$. It is enough to prove, f is bijective between μ_q-open sets and μ-open sets. Let $U_1,U_2\in\mu_q$ such that $U_1\neq U_2$. Suppose $f(U_1)=f(U_2)$. Then $q^{-1}(U_1)=q^{-1}(U_2)$ and so $q^{-1}(U_1-U_2)=\emptyset$. This implies $U_1-U_2=\emptyset$. Therefore, $U_1=U_2$, which is not possible. Therefore, f is injective between μ_q-open sets and μ-open sets. Let U be a μ-open set in X. Then $U=q^{-1}(V)$ where V is a μ_q-open set in $T_0(X)$. Now $f(V)=q^{-1}(V)=U$. Therefore, f is surjective between μ_q-open sets and μ-open sets. Hence q is a quasi-homeomorphism.
(b) By Proposition 4.2, q is a (μ,μ_q)-open map. Similar considerations in Proposition 4.2, we get every canonical surjective map q is a (μ,μ_q)-closed map.
(c) Obviously, $A\subseteq d_0(A)$. Let $s\in d_0(A)$. Then $s\in d_0(a)$ and so $c(\{s\})=c(\{a\})$ for some $a\in A$. This implies $s\in c(\{a\})\subseteq cA$ which implies $s\in cA$. Therefore, $d_0(A)\subseteq cA$. Thus, $A\subseteq d_0(A)\subseteq cA$ and hence $c(d_0(A))=cA$.
(d) follows from (c).
(e) follows from the definition of $d_0(A)$ and a canonical map q.
(f) Let $t\in d_0(\bigcup_{n\in\mathbb{N}} A_n)$. Then $t\in\cup d_0(a)$ for all $a\in\bigcup_{n\in\mathbb{N}} A_n$. This implies $c(\{t\})=c(\{a\})$ for some $a\in A_1$ or ...... or $a\in A_n$ or .... which implies $t\in\bigcup_{n\in\mathbb{N}}(d_0(A_n))$. Therefore, $d_0(\bigcup_{n\in\mathbb{N}} A_n)\subset\bigcup_{n\in\mathbb{N}}(d_0(A_n))$. Conversely, let $s\in\bigcup_{n\in\mathbb{N}}(d_0(A_n))$. Then $s\in d_0(A_1)$ or $s\in d_0(A_2)$ or .... or $s\in d_0(A_n)$ or .... and so $c(\{s\})=c(\{a\})$ for some $a\in A_1$ or $c(\{s\})=c(\{b\})$ for some $b\in A_2$ or .... or $c(\{s\})=c(\{c\})$ for some $c\in A_n$ or .... . Therefore, $s\in d_0(\bigcup_{n\in\mathbb{N}} A_n)$. Thus, $\bigcup_{n\in\mathbb{N}}(d_0(A_n))\subset d_0(\bigcup_{n\in\mathbb{N}} A_n)$. Hence $d_0(\bigcup_{n\in\mathbb{N}} A_n)=\bigcup_{n\in\mathbb{N}} d_0(A_n)$. □

Definition 4.7. *Let (X,μ) be a GTS. X is called a T_0-strongly nodec space if its T_0-reflection is a strongly nodec space, that is $T_0(X)$ is a strongly nodec space.*

Example 4.8 shows the existence of a T_0-strongly nodec space and Theorem 4.9 is a characterization theorem for a T_0-strongly nodec space in generalized topological space.

Example 4.8. Consider the GTS (X,μ) where $X=\{a,b,c,d\}$ and $\mu=\{\emptyset,\{a\},\{b,c\},\{a,b\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X\}$. Define a map $q:X\to T_0(X)$ by $x\in X$ to its equivalence class $[x]$ in $T_0(X)$, where $T_0(X)$ is the T_0-reflection of X. This implies $\mu_q=\{\emptyset,\{a\}\}$. Here, every strongly nowhere dense set is a closed set in $T_0(X)$. Therefore, $T_0(X)$ is a strongly nodec space.

Theorem 4.9. *Let (X,μ) be a GTS and $q:(X,\mu)\to(T_0(X),\mu_q)$ be a canonical surjective map. Then the following are equivalent.*
(a) X is a T_0-strongly nodec space.
(b) For any non-empty strongly nowhere dense subset A of X, $d_0(A)$ is closed.

Proof. (a) $\Rightarrow$ (b). Suppose X is a T_0-strongly nodec space. Let A be a non-empty strongly nowhere dense subset of X. Then cA is a non-empty strongly nowhere dense set in X. Suppose $c(q(A))$ is not a strongly nowhere dense set in $T_0(X)$. Then by Lemma 3.3, there is a non-empty open set U of $T_0(X)$ such that $U \subset c(q(A))$. Since q is a (μ, μ_q)-continuous map, $q^{-1}(U)$ is a non-empty open set and $q^{-1}(U) \subseteq q^{-1}(c(q(A)))$. Since q is a (μ, μ_q)-closed map, $c(q(A)) \subseteq c(q(cA)) = q(cA)$. Then $q^{-1}(U) \subseteq q^{-1}(q(cA)) = d_0(cA) = cA$, which contradict the fact that cA is a strongly nowhere dense subset of X. Therefore, $c(q(A))$ is a non-empty strongly nowhere dense set in $T_0(X)$. Hence $q(A)$ is a non-empty strongly nowhere dense set in $T_0(X)$, since subset of a strongly nowhere dense set is strongly nowhere dense. Since $T_0(X)$ is a strongly nodec space, $q(A)$ is a closed set in $T_0(X)$ and so $q^{-1}(q(A))$ is a closed set in X, since q is a (μ, μ_q)-continuous map. By Lemma 4.6(e), $d_0(A) = q^{-1}(q(A))$ and hence $d_0(A)$ is a closed set in X.
(b) $\Rightarrow$ (a). Let B be a non-empty strongly nowhere dense subset of $T_0(X)$ and $A = q^{-1}(B)$. Then $q(A) = q(q^{-1}(B)) = B$, since q is surjective. Thus, $q(A)$ is a strongly nowhere dense set in $T_0(X)$. Since q is a surjective map, $q^{-1}(q(A)) = q^{-1}(q(q^{-1}(B))) = A$. By Lemma 4.6(e), $d_0(A) = A$. Suppose A is not a strongly nowhere dense set in X. Then there exists $V \in \tilde{\mu}$ such that $V \subset A$. Since q is an (μ, λ)-open map, $q(V)$ is a non-empty open set in $T_0(X)$. This implies $q(V) \subset q(A)$. By Lemma 3.3, $q(A)$ is not a strongly nowhere dense set in X, which is not possible. Therefore, A is a non-empty strongly nowhere dense subset of X. By (b), A is a closed set in X, since $d_0(A) = A$. Since q is a (μ, μ_q)-closed map, $q(A)$ is closed in X. Hence B is a closed set in $T_0(X)$, since $q(A) = B$. Hence X is a T_0-strongly nodec space. ☐

The following Corollary 4.10 and Corollary 4.11 follows from Lemma 4.6 and Theorem 4.9.

Corollary 4.10. *Let (X, μ) be a sGTS and $q : (X, \mu) \to (T_0(X), \mu_q)$ be a canonical surjective map. Then X is a T_0-strongly nodec space if any one of the following hold.*
(a) For every $A \subseteq X$, if $cicA \subseteq d_0(A)$, then $d_0(A) = cA$.
(b) For every $A \subseteq X$, $cA - d_0(A) \subseteq cicA$.
(c) For every $A \subseteq X$, $cA = d_0(A) \cup cicA$.

Corollary 4.11. *Let (X, μ) be a GTS and $q : (X, \mu) \to (T_0(X), \mu_q)$ be a canonical surjective map. If X is a T_0-strongly nodec space, then the following hold.*
(a) If $A \subset X$ is a s-meager set, then $d_0(A)$ is a F_σ-set.
(b) If $A \subset X$ is a s-residual set, then $d_0(A)$ is a G_δ-set.

Corollary 4.12. *Let (X, μ) be a GTS, $A \subset X$ and $q : (X, \mu) \to (T_0(X), \mu_q)$ be a canonical surjective map. If A is a strongly nowhere dense set in X, then $Fr(q(A))$ is a strongly nowhere dense set in $T_0(X)$.*

Proof. Suppose A is a strongly nowhere dense set in X. Then cA is a non-empty strongly nowhere dense set in X. Suppose $q(A)$ is not a strongly nowhere dense set in $T_0(X)$. By similar considerations in Theorem 4.9 (a) $\Rightarrow$ (b), we get a contradiction. Hence $q(A)$ is a non-empty strongly nowhere dense set in $T_0(X)$. ☐

Lemma 4.13 shows inverse of a canonical surjective map preserve closure and interior of a subset of a codomain set.

Lemma 4.13. *Let (X, μ) be a GTS and $q : (X, \mu) \to (T_0(X), \mu_q)$ be a canonical surjective map. Then the following hold.*
(a) For every subset A of $T_0(X)$, $q^{-1}(cA) = c(q^{-1}(A))$.
(b) For every subset A of $T_0(X)$, $q^{-1}(iA) = i(q^{-1}(A))$.
(c) For every subset A of $T_0(X)$, $q^{-1}(cicA) = cic(q^{-1}(A))$.

Proof. (a) Let $A \subset T_0(X)$. By Lemma 4.6(b), q is an (μ, μ_q)-open map. Then $q^{-1}(cA) \subset c(q^{-1}(A))$ where $A \subset T_0(X)$, by Lemma 4.1. Let $a \in c(q^{-1}(A))$. Then $U_a \cap q^{-1}(A) \neq \emptyset$ for every $U_a \in \mu(a)$. Since q is a quasi-homeomorphism, by Lemma 4.6(a), there exists a unique open set $V_{q(a)}$ in $T_0(X)$ such that $U_a = q^{-1}(V_{q(a)})$. This implies $q^{-1}(V_{q(a)}) \cap q^{-1}(A) \neq \emptyset$ which implies $q^{-1}(V_{q(a)} \cap A) \neq \emptyset$. Thus, $V_{q(a)} \cap A \neq \emptyset$. Therefore, $q(a) \in cA$ and so $a \in q^{-1}(cA)$. Thus, $c(q^{-1}(A)) \subset q^{-1}(cA)$. Hence $q^{-1}(cA) = c(q^{-1}(A))$.
(b) Since q is a (μ, μ_q)-continuous map, q^{-1} is a (μ, μ_q)-open map. Then $q^{-1}(iB) = i(q^{-1}(iB)) \subset i(q^{-1}(B))$ where $B \subset T_0(X)$. Therefore, $q^{-1}(iA) \subset i(q^{-1}(A))$. Let $b \in i(q^{-1}(A))$. Then there exists $U_b \in \mu(b)$ such that $U_b \subset q^{-1}(A)$. Since q is a quasi-homeomorphism, by Lemma 4.6(a), there exists a unique open set $V_{q(b)}$ in $T_0(X)$ such that $U_b = q^{-1}(V_{q(b)})$. This implies $q^{-1}(V_{q(b)}) \subset q^{-1}(A)$ which implies $V_{q(b)} \subset q(q^{-1}(A)) = A$, since q is a surjective map. Thus, $q(b) \in iA$ and so $b \in q^{-1}(iA)$. Therefore, $i(q^{-1}(A)) \subset q^{-1}(iA)$. Hence $q^{-1}(iA) = i(q^{-1}(A))$.
(c) Now $q^{-1}(cicA) = c(q^{-1}(icA))$, by (a). Then $q^{-1}(cicA) = c(q^{-1}(icA)) = ci(q^{-1}(cA))$, by (b) and so $q^{-1}(cicA) = cic(q^{-1}(A))$, by (a). ☐

Lemma 4.14. *Let (X, μ) be a GTS, $A \subset X$ and $q : X \to (T_0(X), \mu_q)$ be a canonical bijective map. If $Fr(A)$ is a strongly nowhere dense set in X, then $Fr(q(A))$ is a strongly nowhere dense set in $T_0(X)$.*

Proof. Let G be a non-empty μ_q-open set in $T_0(X)$. Since q is a (μ, μ_q)-continuous map, $q^{-1}(G)$ is a non-empty μ-open set in X. Since $Fr(A)$ is a strongly nowhere dense set in X, there exists $V \in \tilde{\mu}$ such that $V \subset q^{-1}(G)$ and $V \cap Fr(A) = \emptyset$. Since q is an (μ, μ_q)-open map, $q(V)$ is a non-empty μ_q-open set. Thus, there exists a non-empty μ_q-open set $q(V)$ such that $q(V) \subset q(q^{-1}(G)) \subseteq G$ and $q(V \cap Fr(A)) = \emptyset$. Suppose there is an element $t \in q(V) \cap Fr(q(A))$. Then $t \in q(V)$ and $t \in Fr(q(A))$. This implies $q^{-1}(t) \in V$, since q is injective. Now, $t \in c(q(A)) \cap c(T_0(X) - q(A))$. Consider, $t \in c(q(A))$. Since q is a (μ, μ_q)-closed map, $q(cB) = c(q(cB))$, where $B \subset X$. Now $B \subset cB$. This implies $c(q(B)) \subset c(q(cB)) = q(cB)$. Thus, $c(q(B)) \subset q(cB)$. Therefore, $t \in q(c(A))$. Then $q^{-1}(t) \in cA$, since q is injective. Now, $t \in c(T_0(X) - q(A)) = c(q(X) - q(A))$, since q is a surjective map. Then $t \in c(q(X - A))$, since q is injective. Since q is a (μ, μ_q)-closed map and by same process, we get $t \in q(c(X - A))$. Then $q^{-1}(t) \in c(X - A)$, since q is injective. Therefore, $q^{-1}(t) \in cA \cap c(X - A) = Fr(A)$. Thus, $q^{-1}(t) \in V \cap Fr(A)$, which is not possible. Therefore, $q(V) \cap Fr(q(A)) = \emptyset$. Hence $Fr(q(A))$ is a strongly nowhere dense set in $T_0(X)$. □

Next Theorem 4.15 is another charecderaization theorem for a T_0-strongly nodec space in genearlized topological space.

Theorem 4.15. *Let (X, μ) be a sGTS, $q : X \to T_0(X)$ be a canonical bijective map and $A \subset X$. If frontier of A is a strongly nowhere dense set, then the following are equivalent.*
(a) X is T_0-strongly nodec space.
(b) $cA - d_0(A) \subseteq cic(A)$.
(c) $cA = d_0(A) \cup cic(A)$.

Proof. (a) ⇒ (b) Let $A \subset X$. Since X is T_0-strongly nodec, $c(q(A)) - q(A) \subseteq cic(q(A))$, by Theorem 3.15. Now $q^{-1}(c(q(A)) - q(A)) = q^{-1}(c(q(A))) - q^{-1}(q(A)) = c(q^{-1}(q(A))) - q^{-1}(q(A)) = c(d_0(A)) - d_0(A)$, by Lemma 4.13 and Lemma 4.6(e). By Lemma 4.6(c), $q^{-1}(c(q(A)) - q(A)) = cA - d_0(A)$. This implies $cA - d_0(A) \subseteq q^{-1}(cic(q(A))) = cic(q^{-1}(q(A)))$, by Lemma 4.13 which implies $cA - d_0(A) \subseteq cic(d_0(A)) = cic(A)$, by Lemma 4.6(c) and (e). Thus, $cA - d_0(A) \subseteq cic(A)$.
(b) ⇒ (c) Let A be a subset of X. Then $cic(A) \subseteq cA$ and $d_0(A) \subseteq cA$, by Lemma 4.6(c). Therefore, $d_0(A) \cup cic(A) \subseteq cA$. Conversely, $cA = d_0(A) \cup (cA - d_0(A)) \subseteq d_0(A) \cup cic(A)$, by (b). Thus, $cA = d_0(A) \cup cic(A)$.
(c) ⇒ (a) Let A be a non-empty strongly nowhere dense subset of X. Then A is a nowhere dense set in X and so $ic(A) = \emptyset$. By (c) and hypothesis, $cA = d_0(A)$. Therefore, $d_0(A)$ is a closed set in X. Hence X is a T_0-strongly nodec space, by Theorem 4.9. □

Definition 4.16. *Let (X, μ) be a GTS. X is called a T_0-generalized submaximal space if its T_0-reflection is a generalized submaximal space, that is $T_0(X)$ is a generalized submaximal space.*

Next Theorem 4.17 is the characterization theorem for a T_0-generalized submaximal space in generalized topological space.

Theorem 4.17. *Let (X, μ) be a GTS, $q : X \to T_0(X)$ be a canonical surjective map and $A \subset X$. Then the following are equivalent.*
(a) X is T_0-generalized submaximal.
(b) A is dense in X, then $d_0(A)$ is an open set in X.
(c) $c(d_0(A)) - d_0(A)$ is a closed set in X.

Proof. (a) ⇒ (b) Let $A \subset X$ be a dense set in X. Then $cA = X$ and so $c(d_0(A)) = X$, by Lemma 4.6 (c). By Lemma 4.6 (e), $c(q^{-1}(q(A))) = X$. Since q is a canonical surjective map, $q^{-1}(c(q(A))) = X$, by Lemma 4.13 (a). Then $c(q(A)) = T_0(X)$, since q is surjective. By (a), $q(A)$ is an open set in $T_0(X)$. This implies $q^{-1}(q(A))$ is an open set in X, since q is (μ, μ_q)-continuous. By Lemma 4.6 (e), $d_0(A)$ is an open set in X.
(b) ⇒ (a) Let $A \subset X$ such that $q(A)$ is a dense subset of $T_0(X)$. Then $c(q(A)) = T_0(X)$ and so $q^{-1}(c(q(A))) = X$. By Lemma 4.13, $c(q^{-1}(q(A))) = X$. This implies $c(d_0(A)) = X$ which implies $cA = X$, by Lemma 4.6 (e) and (c). By (b), $d_0(A)$ is an open set in X. Then $q^{-1}(q(A))$ is an open set in X and so $q(q^{-1}(q(A)))$ is an open set in $T_0(X)$, by Lemma 4.6 (e) and q is a (μ, μ_q)-open map. Since q is surjective, $q(A)$ is an open set in $T_0(X)$. Therefore, $T_0(X)$ is a generalized submaximal space. Hence X is a T_0-generalized submaximal space.
(a) ⇒ (c) Let A be a subset of X, then $c(d_0(A)) - d_0(A) = c(q^{-1}(q(A))) - q^{-1}(q(A)) = q^{-1}(c(q(A))) - q^{-1}(q(A)) = q^{-1}(c(q(A)) - q(A))$, by Lemma 4.6 (e) and Lemma 4.13 (a). By Lemma 2.1, $i(c(q(A)) - q(A)) = \emptyset$. Thus, $c(q(A)) - q(A)$ is a codense set in $T_0(X)$. Since X is a T_0-generalized submaximal space, then $c(q(A)) - q(A)$ is a closed subset of $T_0(X)$. Then $q^{-1}(c(q(A)) - q(A))$ is a closed subset of X, since q is a (μ, μ_q)-continuous map. Therefore, $c(d_0(A)) - d_0(A)$ is closed in X.
(c) ⇒ (a) Let B be a subset of $T_0(X)$ such that $B = c(q(A)) - q(A)$. Then $iB = \emptyset$, by Lemma 2.1. Let A be a subset of X such that $c(d_0(A)) - d_0(A)$ is closed in X. Then by above process $q^{-1}(c(q(A)) - q(A))$ is a closed subset of X and so $q(q^{-1}(B))$ is a closed subset of $T_0(X)$, since q is (μ, μ_q)-closed map. This implies B is a closed set in $T_0(X)$, since q is a surjective map. Therefore, X is a T_0-generalized submaximal space. □

Theorem 4.18. *Let (X,μ) be a GTS and $q: X \to T_0(X)$ be a canonical surjective map. If X is a generalized submaximal space, then X is a T_0-generalized submaximal space.*

Proof. Let A be a dense subset of X. Then A is an open set in X, by hypothesis. Since q is a (μ,μ_q)-open map, $q(A)$ is an open set in $T_0(X)$. This implies $q^{-1}(q(A))$ is an open set in X, since q is a (μ,μ_q)-continuous map. By Lemma 4.6 (e), $d_0(A)$ is an open set in X. Therefore, X is a T_0-generalized submaximal space, by Theorem 4.17. □

Theorem 4.19. *Let (X,μ) be a GTS and $q: X \to T_0(X)$ be a canonical bijective map. If X is a T_0-generalized submaximal space, then X is a generalized submaximal space.*

Proof. Let A be a dense subset of X. Then $d_0(A)$ is an open set in X, by hypothesis and Theorem 4.17. This implies $q^{-1}(q(A))$ is an open set in X, by Lemma 4.6 (e). Since q is injective, A is an open set in X. Therefore, X is a X generalized submaximal space. □

Next Theorem 4.20 shows every T_0-generalized submaximal space is a T_0-strongly nodec space.

Theorem 4.20. *Let (X,μ) be a GTS and $q: X \to T_0(X)$ be a canonical surjective map. If X is T_0-generalized submaximal space, then X is a T_0-strongly nodec space.*

Proof. Let $q(A)$ be a non-empty strongly nowhere dense set in $T_0(X)$. Then $q(A)$ is a codense set in $T_0(X)$, by Lemma 2.5. Since X is T_0-generalized submaximal space, $q(A)$ is a closed set in $T_0(X)$. Thus, $T_0(X)$ is a strongly nodec space. Therefore, X is a T_0-strongly nodec space. □

5. Strongly nodec, T_0-strongly nodec spaces by functions

In this section, we discuss the properties of images of a strongly nodec, T_0-strongly nodec spaces by a quasi-homeomorphism function.

Lemma 5.1. *Let $(X,\mu),(Y,\lambda)$ be two GTSs and $f: X \to Y$ be a quasi-homeomorphism map. Then the following hold.*
(a) If f is a bijective map and A is strongly nowhere dense in X, then $f(A)$ is strongly nowhere dense in Y.
(b) If B is strongly nowhere dense in Y, then $f^{-1}(B)$ is strongly nowhere dense in X.
(c) If A is of s-II category in X, then $f(A)$ is of s-II category in Y.
(d) If f is a bijective map and B is of s-II category set in Y, then $f^{-1}(B)$ is of a s-II category set in X.

Proof. (a) Suppose A is strongly nowhere dense in X. Let $W \in \tilde{\lambda}$. Since f is (μ,λ)-continuous, $f^{-1}(W) \in \tilde{\mu}$. By hypothesis, there exists $V \in \tilde{\mu}$ such that $V \subset f^{-1}(W)$ and $V \cap A = \emptyset$. Since f is a quasi-homeomorphism, there exists a unique $V_1 \in \lambda$ such that $V = f^{-1}(V_1)$. Thus, $V_1 \in \tilde{\lambda}$ and $f(V) = V_1$, since f is a surjective map. Now $V \subset f^{-1}(W)$ implies $f(V) \subset W$ and so $V_1 \subset W$. Since f is injective, $f(V \cap A) = f(V) \cap f(A) = \emptyset$. Therefore, $V_1 \cap f(A) = \emptyset$. Hence $f(A)$ is strongly nowhere dense in Y.
(b) Suppose B is strongly nowhere dense in Y. Let $W \in \tilde{\mu}$. Since f is a quasi-homeomorphism, there exists a unique $V \in \lambda$ such that $W = f^{-1}(V)$. Since $V \in \tilde{\lambda}$ and by hypothesis, there exists $V_1 \in \tilde{\lambda}$ such that $V_1 \subset V$ and $V_1 \cap B = \emptyset$. Since f is a (μ,λ)-continuous map, there exists $f^{-1}(V_1) \in \tilde{\mu}$ such that $f^{-1}(V_1) \subset W$ and $f^{-1}(V_1) \cap f^{-1}(B) = \emptyset$. Therefore, $f^{-1}(B)$ is strongly nowhere dense in X.
(c) Assume that, A is of a s-II category set in X. Suppose $f(A)$ is of a s-meager set in Y. Then $f(A) = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is a strongly nowhere dense set in Y. By (b), each $f^{-1}(A_n)$ is a strongly nowhere dense set in X. Now $f^{-1}(f(A)) = f^{-1}(\bigcup_{n\in\mathbb{N}} A_n)$. Then $f^{-1}(f(A)) = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n)$. This implies $f^{-1}(f(A))$ is a s-meager set in X which implies A is a s-meager set in X, since subset of a strongly nowhere dense set is strongly nowhere dense set, which is not possible. Therefore, A is of a s-II category set in Y.
(d) Suppose f is a bijective map and A is of s-II category set in Y. Suppose $f^{-1}(A)$ is of a s-meager set in X. Then $f^{-1}(A) = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is a strongly nowhere dense set in X. By (a), each $f(A_n)$ is a strongly nowhere dense set in Y. Now $f(f^{-1}(A)) = f(\bigcup_{n\in\mathbb{N}} A_n)$. Since f is a bijective map, $A = \bigcup_{n\in\mathbb{N}} f(A_n)$. This implies A is a s-meager set in Y, which is not possible. Therefore, $f^{-1}(A)$ is of a s-II category set in X. □

Example 5.2 shows the condition that surjective on f cannot be dropped in Lemma 5.1 (a). Next Theorem 5.3 shows that an image and inverse-image of a strongly nodec is strongly nodec under a bijective, quasi homeomorphism function in generalized topological space.

Example 5.2. (a) Consider the two GTSs (X,μ) and (Y,λ) where $X=\{a,b,c,d\}, \mu=\{\emptyset,\{b\},\{c\},\{b,c\},\{a,c\},\{a,b,c\}\}, Y=\{a,b,c,d,e\}$ and $\lambda=\{\emptyset,\{e\},\{a,c\},\{b,e\},\{c,e\},\{a,c,e\},\{b,c,e\},\{a,b,c,e\}\}$. Define a function $f:X\to Y$ by $f(a)=a, f(b)=b, f(c)=c, f(d)=d$. Clearly, f is a (μ,λ)-continuous and for each μ-open subset U of X, there exists a unique λ-open subset V of Y such that $U=f^{-1}(V)$. Therefore, f is a quasi-homeomorphism but not a surjective map, since $f(X)\neq Y$. Let $A=\{a\}\subset X$. Then A is a strongly nowhere dense set in X. Now $f(A)=\{a\}\subset Y$. Let $G=\{a,c\}\in\tilde{\lambda}$. Then there is no $H\in\tilde{\lambda}$ such that $H\subset G$ and $H\cap f(A)=\emptyset$. Therefore, $f(A)$ is not a strongly nowhere dense set in Y.

Theorem 5.3. *Let $(X,\mu),(Y,\lambda)$ be two GTSs and $f:X\to Y$ be a bijective, quasi-homeomorphism map. Then X is a strongly nodec space if and only if Y is a strongly nodec space.*

Proof. Suppose X is a strongly nodec space. Let A be a non-empty strongly nowhere dense set in Y. By hypothesis and Lemma 5.1, $f^{-1}(A)$ is a strongly nowhere dense set in X. By hypothesis, $f^{-1}(A)$ is a closed set in X. Since f is a quasi-homeomorphism, there exists a unique closed set V in Y such that $f^{-1}(A)=f^{-1}(V)$. Thus, A is a closed set in Y, since f is a surjective map. Therefore, Y is a strongly nodec space. Conversely, assume that Y is a strongly nodec space. Let B be a strongly nowhere dense set in X. By hypothesis and Lemma 5.1, $f(B)$ is a strongly nowhere dense set in Y. By hypothesis, $f(B)$ is a closed set in Y. Since f is a (μ,λ)-continuous and bijective map, B is a closed set in X. Therefore, X is a strongly nodec space. □

Theorem 5.4. *Let (X,μ) be a sGTS and $q:X\to T_0(X)$ be a canonical surjective map. Then the following hold.*
(a) Every strongly nodec space is a T_0-strongly nodec space.
(b) Every generalized submaximal space is a T_0-strongly nodec space.

Proof. We will present the proof only for (a). Let X be a strongly nodec space and $q(A)$ be a strongly nowhere dense set in $T_0(X)$. By Lemma 5.1, $q^{-1}(q(A))$ is a strongly nowhere dense set in X. Since $A\subset q^{-1}(q(A))$ and subset of a strongly nowhere dense set is strongly nowhere dense, A is a strongly nowhere dense set in X. By hypothesis, A is a closed set in X. Since q is a closed map, by Lemma 4.6, $q(A)$ is a closed set in $T_0(X)$. Therefore, X is a T_0-strongly nodec space. □

Next Example 5.5 shows that the converse of Theorem 5.4 (a) is not necessary and Theorem 5.6 is the reverse implication of Theorem 5.4(a).

Example 5.5. Consider the GTS (X,μ) where $X=\{a,b,c\}$ and $\mu=\{\emptyset,\{a\},X\}$. Define a map $f:X\to T_0(X)$ by $x\in X$ to its equivalence class $[x]$ in $T_0(X)$, where $T_0(X)$ is the T_0-reflection of X. This implies $\mu_f=\{\emptyset,\{a\}\}$. Let $A=[b]=\{b,c\}$. Then A is strongly nowhere dense set in $T_0(X)$. Now $f^{-1}(A)=f^{-1}([b])=\{b,c\}$ is a closed set in X. Then A is a closed set in $T_0(X)$. Therefore, $T_0(X)$ is a strongly nodec space. Let $A=\{b\}$. Then A is a strongly nowhere dense set in X but not closed in X. Thus, X is not a strongly nodec space.

Theorem 5.6. *Let (X,μ) be a sGTS and $q:X\to T_0(X)$ be a canonical surjective map. If q is injective and X is a T_0-strongly nodec space, then it is a strongly nodec space.*

Proof. Let A be a non-empty strongly nowhere dense subset of X. By Lemma 5.1, $q(A)$ is a non-empty strongly nowhere dense set in $T_0(X)$. By hypothesis, $q(A)$ is a closed set in $T_0(X)$. Since q is a (μ,λ)-continuous map, $q^{-1}(q(A))$ is a closed set in X. Since q is injective map, A is a closed set in X. Therefore, X is a strongly nodec space. □

Example 5.7 shows the condition injective on q is necessary in Theorem 5.6 and Theorem 5.8 shows a (μ,λ)-open map from a GTS (X,μ) into a GTS (Y,λ) preserve the frontier of B where $B\subset Y$.

Example 5.7. Consider the GTS (X,μ) where $X=\{a,b,c,d,e\}$ and $\mu=\{\emptyset,\{a\},\{a,b\},\{b,c\},\{a,b,c\},\{a,b,d,e\},\{a,c,d,e\},X\}$. Define a map $f:X\to T_0(X)$ by $x\in X$ to its equivalence class $[x]$ in $T_0(X)$, where $T_0(X)$ is the T_0-reflection of X. Then f is a canonical surjective but not an injective map and so $\mu_f=\{\emptyset,\{a\}\}$. Let $A=[d]=\{d,e\}$. Then A is strongly nowhere dense set in $T_0(X)$. Now $f^{-1}(A)=f^{-1}([d])=\{d,e\}$ is a closed set in X. Then A is a closed set in $T_0(X)$. Similarly, every strongly nowhere dense set in $T_0(X)$ is closed set in $T_0(X)$. Therefore, $T_0(X)$ is a strongly nodec space. Let $A=\{e\}$. Then A is a strongly nowhere dense set in X but not closed in X. Thus, X is not a strongly nodec space.

Theorem 5.8. *Let $(X,\mu),(Y,\lambda)$ be two GTSs and $f:X\to Y$ be a map. If f is (μ,λ)-open, then $f^{-1}(Fr(B))=Fr(f^{-1}(B))$ for all $B\subset Y$.*

Proof. Suppose f is a (μ,λ)-open map. Let $t\in f^{-1}(Fr(B))$. Then $t\in f^{-1}(cB\cap c(Y-B))$. This implies $t\in f^{-1}(cB)$ and $t\in f^{-1}(c(Y-B))$. Since f is (μ,λ)-open and by Lemma 4.1, $t\in c(f^{-1}(B))$ and $t\in c(X-f^{-1}(B))$. Therefore, $t\in Fr(f^{-1}(B))$. Hence $f^{-1}(Fr(B))\subseteq Fr(f^{-1}(B))$. Let $s\in Fr(f^{-1}(B))$. Then $s\in c(f^{-1}(B))\cap c(X-f^{-1}(B))$. Now $s\in c(f^{-1}(B))$. Then $U\cap f^{-1}(B)\neq\emptyset$ for all $U\in\mu(s)$. By hypothesis, $f(U)\in\tilde{\lambda}$. This implies $f(U\cap f^{-1}(B))\neq\emptyset$ for all $f(U)\in\lambda(f(s))$. Now $f(U\cap f^{-1}(B))\subset f(U)\cap f(f^{-1}(B))\subset f(U)\cap B$, since $f(f^{-1}(B))\subset B$. Then $f(U)\cap B\neq\emptyset$ and so $f(s)\in cB$. Consider, $s\in$

$c(f^{-1}(Y-B))$. Then $V \cap f^{-1}(Y-B) \neq \emptyset$ for all $V \in \mu(s)$. By hypothesis, $f(V) \in \lambda(f(s))$. This implies $f(V \cap f^{-1}(Y-B)) \neq \emptyset$ for all $f(V) \in \lambda(f(s))$. Now $f(V \cap f^{-1}(Y-B)) \subset f(V) \cap f(f^{-1}(Y-B)) \subset f(V) \cap (Y-B)$. Then $f(V) \cap (Y-B) \neq \emptyset$ and so $f(s) \in c(Y-B)$. Therefore, $f(s) \in Fr(B)$. Thus, $s \in f^{-1}(Fr(B))$. Hence $Fr(f^{-1}(B)) \subset f^{-1}(Fr(B))$. □

Theorem 5.9. *Let $(X,\mu),(Y,\lambda)$ be two GTSs and $f : X \to Y$ be a surjective, quasi-homeomorphism map. If B is a strongly nowhere dense set in Y, then $f^{-1}(Fr(B))$ is a strongly nowhere dense set in X for all $B \subset Y$.*

Proof. Suppose B is a strongly nowhere dense set in Y. By Lemma 5.1, $f^{-1}(B)$ is a strongly nowhere dense set in X. Then $Fr(f^{-1}(B))$ is a strongly nowhere dense set in X, by Lemma 3.7 (a). By hypothesis and Proposition 4.2, f is a (μ,λ)-open map. Therefore, $f^{-1}(Fr(B)) = Fr(f^{-1}(B))$, by Theorem 5.8. Hence $f^{-1}(Fr(B))$ is a strongly nowhere dense set in X. □

Let (X,μ) be a GTS. A space X is said to be a *weak Baire space (wBS)* if every non-empty μ-open set in X is of μ-s-II category in X [3].

Theorem 5.10. *In a GTS, every wBS is of s-II category.*

Proof. Let (X,μ) be a GTS and X is a wBS. Suppose X is a s-meager. Then $X = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is a strongly nowhere dense set in X. Then each A_n is a nowhere dense set in X for $n \in \mathbb{N}$. This implies cA_n has no interior points so any non-empty open set in X must intersect $G_n = X - cA_n$ for all $n \in \mathbb{N}$. Take $\{G_n\}_{n\in\mathbb{N}}$ is a collection of non-empty open-dense sets in X. This implies $cG_n = X$ for all $n \in \mathbb{N}$ which implies cG_n is a s-meager set in X. Therefore, G_n is a s-meager set in X for all $n \in \mathbb{N}$, since subset of a s-meager set is s-meager. Thus, a non-empty open set G_n is not s-II category, which is a contradiction to X is a wBS. Hence X is of s-II category. □

Theorem 5.11 and Theorem 5.12 shows the behaviour of wBS under the quasi-homeomorphism and canonical surjective map in generalized topological space.

Theorem 5.11. *Let (X,μ), (Y,λ) be two GTSs and $f : X \to Y$ be a surjective, quasi-homeomorphism map. Then the following hold.*
(a) If X is a wBS, then Y is of s-II category.
(b) If f is a injective map and Y is a wBS, then X is of s-II category.

Proof. (a) Suppose X is a wBS. It is enough to prove, Y is a wBS, by Theorem 5.10. Let $A \in \tilde{\lambda}$. Since f is a (μ,λ)-continuous map, $f^{-1}(A) \in \tilde{\mu}$. By hypothesis, $f^{-1}(A)$ is of s-II category in X. By Lemma 5.1, $f(f^{-1}(A))$ is of s-II category in Y. Since f is a surjective map, A is of s-II category in Y. Therefore, Y is a wBS. Hence Y is of s-II category.
(b) Suppose f is a injective map and Y is a wBS. It is enough to prove, X is a wBS, by Theorem 5.10. Let $A \in \tilde{\mu}$. Since f is a quasi-homeomorphism map, there exists a set $A_1 \in \tilde{\lambda}$ such that $A = f^{-1}(A_1)$. Since f is a surjective map, $f(A) = A_1$. By hypothesis, A_1 is of s-II category in Y. Thus, $f(A)$ is of s-II category in Y. By Lemma 5.1, A is of s-II category in X. Therefore, X is a wBS. Hence X is of s-II category. □

Theorem 5.12. *Let (X,μ) be a GTS and $q : (X,\mu) \to (T_0(X),\mu_q)$ be a canonical surjective map. Then the following hold.*
(a) If X is a wBS, then $T_0(X)$ is a wBS and hence a s-II category space.
(b) If $T_0(X)$ is a wBS and q is a injective map, then X is a wBS and hence a s-II category space.

Proof. (a) Suppose X is a wBS. Let $q(A)$ be a non-empty set in $T_0(X)$. Since q is a (μ,μ_q)-continuous map, $q^{-1}(q(A)) \in \tilde{\mu}$. By hypothesis, $q^{-1}(q(A))$ is of s-II category in X. By Lemma 5.1, $q(A)$ is of s-II category in $T_0(X)$. Therefore, $T_0(X)$ is a wBS and hence a s-II category space, by Theorem 5.10.
(b) Suppose $T_0(X)$ is a wBS and q is an injective map. Let $A \in \tilde{\mu}$. By Lemma 4.6, q is a (μ,μ_q)-open map, $q(A)$ is a non-empty open set in $T_0(X)$. By hypothesis, $q(A)$ is of s-II category in $T_0(X)$. By Lemma 5.1 and q is a injective map, A is of s-II category in X. Therefore, X is a wBS. Hence X is of s-II category space, by Theorem 5.10. □

6. Strongly nodec space in GMS

In this section, we discuss the behaviour of μ-strongly nowhere dense set and strongly nodec space in generalized metric spaces.

In 2013, Korczak-Kubiak et al. introduced the notion of a generalized metric space [3]. They define the notions kernel and perfect kernel in GMS and discuss some properties of kernel, perfect kernel and three types of a Baire space in generalized metric space in [3].

Here, we focus only the properties of strongly nowhere dense sets and give some results for strongly nodec space in generalized metric space by using kernel and perfect kernel. Also, we give one result for wBS in generalized metric space. First we see the definitions and notation defined in generalized metric space.

Let $X \neq \emptyset$. The symbol π to denote the family consisting of metrics defined on subsets of X, that is $\rho \in \pi$ then there exists a non-empty set $A_\rho \subset X$ such that ρ is a metric on A_ρ where A_ρ is a domain space of ρ and it will be denoted by $dom(\rho)$. The pair (X,π) is called a *generalized metric space* (GMS) [3].

Denote μ_π is the family of π-*open sets* in a GMS (X,π), more precisely, $V \in \mu_\pi$ if and only if for each $x \in V$, there exists $\rho \in \pi$ and $\varepsilon > 0$ such that $B_\rho(x,\varepsilon) \subset V$ where $B_\rho(x,\varepsilon) = \{y \in dom(\rho) : \rho(x,y) < \varepsilon\}$ [3].

Let (X,π) be a GMS. A finite family $\pi_0 \subset \pi$ is called a *perfect kernel* (resp. *kernel*) [3] of the space X if for any $V_1, V_2, ..., V_m \in \mu_\pi$ such that $V_1 \cap V_2 \cap ... \cap V_m \neq \emptyset$ (resp. $V \in \tilde{\mu}_\pi$), there exists $\rho \in \pi_0$ such that $i_\rho(V_1 \cap V_2 \cap ... \cap V_m) \neq \emptyset$ (resp. $i_\rho(V) \neq \emptyset$). The set of all perfect kernels (resp. kernels) of the space (X,π) will be denoted by $Ker_p(X,\pi)$ (resp. $Ker(X,\pi)$). Obviously, if π_0 is a perfect kernel of the space (X,π), then it is a kernel of the space too [3].

Lemma 6.1. *If the GMS (X,π) has a kernel $\pi_0 \subset \pi$ and A is a dense subset of X, then $\pi_0|_A$ is a kernel of the GMS $(A,\pi|_A)$.*

Proof. Suppose A is dense subset of X. Let $V \in \tilde{\mu}_{\pi|_A}$ and $x \in V$. Then there exists $\rho \in \pi$ and $\varepsilon > 0$ such that $B_{\rho|_A}(x,\varepsilon) \subset V$. Since $B_\rho(x,\varepsilon) \neq \emptyset$ and π_0 is a kernel, there exists $\rho_0 \in \pi_0$ such that $i_{\rho_0}(B_\rho(x,\varepsilon)) \neq \emptyset$. Choose $y \in i_{\rho_0}(B_\rho(x,\varepsilon))$ and $\varepsilon_0 > 0$. Then $B_{\rho_0}(y,\varepsilon_0) \subset B_\rho(x,\varepsilon)$ and so $B_{\rho_0}(y,\varepsilon_0) \cap A \subset B_\rho(x,\varepsilon) \cap A$. That is, $B_{\rho_0|_A}(y,\varepsilon_0) \subset B_{\rho|_A}(x,\varepsilon)$. Also, $B_{\rho_0|_A}(y,\varepsilon_0) \in \tilde{\mu}_{\pi|_A}$, since A is dense and $B_{\rho_0|_A}(y,\varepsilon_0) \subset V$. Thus, there exists $\rho_0|_A \in \pi_0|_A$ such that $i_{\rho_0|_A}(V) \neq \emptyset$. Hence $\pi_0|_A$ is a kernel of the GMS $(A,\pi|_A)$. □

Lemma 6.2 shows the properties of strongly noowhere dense sets in subspace generalized metric space.

Lemma 6.2. *Let (X,π) be a GMS with a kernel $\pi_0 \subset \pi$, U be a dense, μ_{π_0}-open subset of X and $A \subset U \subset X$. Then the following hold.*
(a) If A is a strongly nowhere dense set in $(U,\pi|_U)$, then A is a strongly nowhere dense set in (X,π).
(b) If A is a s-meager set in $(U,\pi|_U)$, then A is a s-meager set in (X,π).
(c) If B is of s-II category in (X,π), then B is of s-II category in $(U,\pi|_U)$ where $B \subset X$.

Proof. (a) Let $W \in \tilde{\mu}_\pi$. Then $U \cap W \in \tilde{\mu}_{\pi|_U}$. Since A is a strongly nowhere dense set in U, there exists $V \in \tilde{\mu}_{\pi|_U}$ such that $V \subset U \cap W$ and $V \cap A = \emptyset$. By Lemma 6.1, $\pi_0|_U$ is a kernel. Then there exists $\rho_0|_U \in \pi|_U$ such that $i_{\rho_0|_U}(V) \neq \emptyset$. Let $x \in i_{\rho_0|_U}(V)$. Then there is $\varepsilon > 0$ such that $B_{\rho_0|_U}(x,\varepsilon) \subset V$. This implies $B_{\rho_0|_U}(x,\varepsilon) \subset W$ and $B_{\rho_0|_U}(x,\varepsilon) \cap A = \emptyset$. Now $x \in i_{\rho_0|_U} U = U = i_{\rho_0} U$, since U is a μ_{π_0}-open set in X. Let $\varepsilon_1 > 0$ such that $\varepsilon_1 > \varepsilon$. Then $B_{\rho_0}(x,\varepsilon_1) \subset U$ and so $B_{\rho_0}(x,\varepsilon) \subset U$. Therefore, $B_{\rho_0|_U}(x,\varepsilon) = B_{\rho_0}(x,\varepsilon)$. Thus, there is $B_{\rho_0}(x,\varepsilon) \in \tilde{\mu}_\pi$ such that $B_{\rho_0}(x,\varepsilon) \subset W$ and $B_{\rho_0}(x,\varepsilon) \cap A = \emptyset$. Hence A is a strongly nowhere dense set in X.
(b) and (c) follows from (a). □

Theorem 6.3 shows every dense-μ_{π_0}-open subspace of a strongly nodec space having kernel is a strongly nodec space.

Theorem 6.3. *If GMS (X,π) has a kernel $\pi_0 \subset \pi$, U be a dense, μ_{π_0}-open subset of X and if (X,μ_π) is strongly nodec, then $(U,\mu_{\pi|_U})$ is strongly nodec.*

Proof. Suppose X is a strongly nodec space. Let A be a non-empty strongly nowhere dense subset of U. By hypothesis and Lemma 6.2, A is a non-empty strongly nowhere dense subset of X. Then A is closed in X. By Lemma 2.3, $c_U(A) = A$. Hence U is a strongly nodec space. □

Theorem 6.4 shows every dense-μ_{π_0}-open subspace of a wBS having perfect kernel is a wBS. Next Lemma 6.5 shows the properties of strongly nowhere dense sets in generalized metric space.

Theorem 6.4. *If GMS (X,π) has a perfect kernel $\pi_0 \subset \pi$, U be a dense, μ_{π_0}-open subset of X and if (X,μ_π) is wBS, then $(U,\mu_{\pi|_U})$ is wBS and hence a s-II category.*

Proof. Suppose X is a wBS. Let V be a non-empty open set in U. Then $V = V_1 \cap U$ where V_1 is a non-empty μ_π-open set in X. Since V_1, U are μ_π-open sets and $V_1 \cap U \neq \emptyset$, there is $\rho_0 \in \pi_0$ such that $i_{\rho_0}(V_1 \cap U) \neq \emptyset$. Take $G = i_{\rho_0}(V_1 \cap U)$. Then G is a non-empty μ_π-open set in X. By hypothesis, G is of s-II category in X. By Lemma 6.2, G is of s-II category in U. Since $G \subset V$, V is of s-II category in U. For, if V is a s-meager in U. Since subset of a s-meager set is s-meager, G is a s-meager set in U. Hence U is a wBS. By Theorem 5.10, U is of s-II category. □

Lemma 6.5. *Let (X,π) be a GMS, U be a closed subset of X and $A \subset X$. Then the following hold.*
(a) If A is a strongly nowhere dense set in (X,π), then A is a strongly nowhere dense in $(U,\pi|_U)$.
(b) If A is a s-meager set in (X,π), then A is a s-meager set in $(U,\pi|_U)$.
(c) If A is a s-residual set in (X,π), then A is a s-residual set in $(U,\pi|_U)$.
(d) If B is of s-II category in (U,π), then B is of s-II category in (X,π) where $B \subset U$.

Proof. (a) Let A be a strongly nowhere dense subset of X. Suppose $A \cap U = \emptyset$. Then A is a strongly nowhere dense set in U, by definition of strongly nowhere dense. Assume, $A \cap U \neq \emptyset$. Let $W \in \tilde{\mu}_{\pi|_U}$ and $x \in W$. Then there is $\rho|_U \in \pi|_U$ and $\varepsilon > 0$ such that $B_{\rho|_U}(x,\varepsilon) \subset W$. Since A is a strongly nowhere dense set in X and $B_\rho(x,\varepsilon) \in \tilde{\mu}_\pi$, there exists $V \in \tilde{\mu}_\pi$ such that $V \subset B_\rho(x,\varepsilon)$ and $V \cap A = \emptyset$. Choose $V = B_\rho(x,\varepsilon_1)$ where $\varepsilon_1 < \varepsilon$. Since U is closed and $x \in U$, $V \cap U \neq \emptyset$. Thus, there is $B_{\rho|_U}(x,\varepsilon_1) \in \tilde{\mu}_{\pi|_U}$ such that $B_{\rho|_U}(x,\varepsilon_1) \subset B_{\rho|_U}(x,\varepsilon) \subset W$ and $B_{\rho|_U}(x,\varepsilon_1) \cap A = \emptyset$. Therefore, A is a strongly nowhere dense in $(U,\pi|_U)$.
(b), (c) and (d) follows from (a). □

Theorem 6.6. *Let (X,π) be a GMS and U be a closed subset of X. If $(U,\mu_{\pi|_U})$ is a strongly nodec space, then (X,μ_π) is a strongly nodec space.*

Proof. Suppose $(U,\mu_{\pi|_U})$ is a strongly nodec space. Let A be a non-empty strongly nowhere dense subset of X. Then by Lemma 6.5, A is a non-empty strongly nowhere dense set in U. By hypothesis, A is a closed set in U. Then $c_U A = A$ and so $A = U \cap cA$, by Lemma 2.1. Since U is a closed subset in X, A is a closed set in X. Therefore, (X,μ_π) is a strongly nodec space. □

Theorem 6.7. *Let (X,π) be a GMS. If frontier of a subspace of X is strongly nodec, then (X,μ_π) is a strongly nodec space.*

Proof. Let Y be a subspace of X. Suppose $Fr(Y)$ is a strongly nodec space. Since every frontier of a subset of X is a closed set in X, $Fr(Y)$ is a closed subset of X. Hence X is a strongly nodec space, by Theorem 6.6. □

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