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Mixed-Type Functional Differential Equations: A C_0-Semigroup Approach

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Abstract
In this paper we study certain systems of mixed-type functional differential equations, from the point of view of the C_0-semigroup theory. In general, this type of equations are not well-posed as initial value problems. But there are also cases where a unique differentiable solution exists. For these cases and in order to achieve our goal, we first rewrite the system as a classical Cauchy problem in a suitable Banach space. Second, we introduce the associated semigroup and its infinitesimal generator and prove important properties of these operators. As an application, we use the results to characterize the null controllability for those systems, where the control u is constrained to lie in a non-empty compact convex subset Ω of $\mathbb{R}^n$.

Keywords: Functional differential equations, Mixed-type difference-differential equations, Strongly continuous semigroups, Exact controllability
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1. Introduction

In this paper we analyze certain systems of functional differential equations with both delayed and advanced arguments. Such equations are often referred to in the literature as mixed-type functional differential equations (MTFDE) or forward-backward equations. The study of this type of equations is less developed compared with other classes of functional equations. As a consequence, many important questions remain open. Interest in MTFDEs is motivated by problems in optimal control [1] and applications, for example, in economic dynamics [2] and travelling waves in a spatial lattice [3].

As far as we know, similar studies to the one presented here for this type of equations haven't been done. This type of equations are, in general, ill-posed as initial value problems (see for example, [1] and [4]), but there are also cases ([5], [6], [7], [8], [9] and [10]) where a unique differentiable solution exists. We make the statement of the problem in Section 2. In section 3, we give our main results. We begin with showing how the system can be rewritten as a classical Cauchy problem in a suitable Banach space, provided that the initial value problem is well-posed. Then we give the semigroup associated with the ordinary differential equation and its infinitesimal generator, and prove some important properties of these operators. In section 4 we apply the results obtained in Section 3 to characterize the null controllability for those systems, where the control u is constrained to lie in a non-empty compact convex subset Ω of $\mathbb{R}^n$, with $0 \in \Omega$.

2. Statement of the problem

Let Ω be a bounded domain in $\mathbb{R}^n$, $0 < h_1 < h_2 < \cdots < h_q$ and $A_0, A_i, C_i \in \mathscr{L}(\mathbb{R}^n)$, for $i = 1,\ldots,q$, with $A_i \neq 0$ for some i. We will consider the following mixed-type functional differential equation

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_i^q A_i x(t-h_i) + \sum_i^q C_i x(t+h_i) + Bu(t), \ \ t>0\ , \\ x(0) &= \Phi(0) = \Phi_0\ , \\ x(s) &= \Phi(s), \ \ s \in [-h_q, 2h_q], \end{aligned} \tag{2.1}$$

where $\Phi \in L_p[[-h_q,2h_q];\mathbb{R}^n]$, $1 \le p \le \infty$, is defined by

$$\Phi(s) = \left\{ \begin{array}{ll} \Phi_1(s), & s \in [-h_q, 0] \\ \Phi_2(s), & s \in [0, 2h_q] \end{array} \right. ,$$

$u : [0,\infty) \to \mathbb{R}^n$ is an essentially bounded function and $B \in \mathscr{L}(\mathbb{R}^n)$.

The function Φ is usually found in Hilbert spaces (see, for example, [11] and [12], in the case $C_i = 0$). It is noteworthy that, in the present work, we will allow it to be in a L_p -space, for any p belonging to $[1,\infty]$.

At this point, some basic facts must be recalled. It is, in fact, well known that, for X a Banach space and $f : [0,\infty) \to X$ a continuously differentiable function, the initial value problem

$$\begin{cases} x'(t) = Ax(t) + f(t), & t \ge 0 \\ x(0) = x_0, & x_0 \in D(A) \end{cases}$$

is well posed if and only if A generates a strongly continuous semigroup $(T(t))_{t\ge 0}$ on X. The unique solution can be expressed in terms of $(T(t))_{t\ge 0}$ by the following formula (usually known as mild solution):

$$x(t) = T(t)x_0 + \int_0^t T(t-s) f(s)\mathrm{d}s.$$

Working in the frame of a C_0-semigroup theory is not always possible for MTFDE, as there are cases where the problem is not well-posed. As an example, consider the equation $\dot{x(t)} = x(t+1)$. If λ is such that $\lambda - \exp(\lambda) = 0$, then it is easily seen that $x(t) = \exp(\lambda t)x_0$ is a solution. Any strongly continuous semigroup is bounded by $Me^{\omega t}$ for some M and ω, but this fails in this case. Here, we cannot assume that for initial conditions in a dense set, there exists a classical solution. It also shows that our condition "$A_i \neq 0$ for some i" in a system like (2.1) is truly essential. *In other words, we need the presence of delayed arguments*.

Another very interesting example of an ill-posed problem is the following: in [4], Harterich, Sandstede and Scheel consider the equation

$$\dot{x}(t) = x(t-1) + x(t+1)$$

with $\Phi(s) = 1$, $s \in [-m,m]$, m a natural number. The only possible solution for this initial value problem is $x(t) = (-1)^k$, for $t \in (2k-1, 2k+1]$, with k a natural number, which is not even a continuous function.

On the other hand, it is shown in [8] that this same equation has a unique differentiable solution if and only if $\Phi \in C^\infty_{[-1,1]}$ defined by

$$\Phi(s) = \left\{ \begin{array}{ll} \Phi_1(s), & s \in [-1, 0] \\ \Phi_2(s), & s \in [0, 1] \end{array} \right. ,$$

satisfies $\Phi^{(n+1)}(0) = \Phi^{(n)}(-1) + \Phi^{(n)}(1)$ for $n = 0,1,2,\ldots$. As an example, it is easy to see that $\Phi(s) = e^{\lambda s}$ satisfies this condition if $\lambda = e^{\lambda} + e^{-\lambda}$, and it is shown in [13] that there exist, in fact, complex numbers λ such that $\lambda = e^{\lambda} + e^{-\lambda}$, as they are the spectrum of a bounded linear operator on suitable Banach Space (the spectrum is always non-empty, as it is well known). It is also shown that, being $\det(\lambda I - e^{\lambda} - e^{-\lambda})$ an entire function, then, for every $\delta \in \mathbb{R}$, it has finitely many zeros in the compact set $\overline{\mathbb{C}^+_\delta} \cap \{\lambda : |\lambda| \le e^{\delta} + e^{-\delta}\}$, and in the rest of $\mathbb{C}^+_\delta$ there are none. In particular, there are finitely many λ with $Re\lambda > 0$ such that $\lambda = e^{\lambda} + e^{-\lambda}$, and we have $|\lambda| \le 2$. Thus the unique solution, given by a strongly continuous semigroup, is exponentially bounded, as it should be.

Bearing in mind these results, it is characterized in [13] the null controllability for the associated initial value problem, where the control u is also constrained to belong to a suitable domain Ω of the control space with $0 \in \Omega$.

The present work is an attempt to see the results in [9] and [13] in a more general context. As we have indicated, there exist other examples where a unique solution can be found ([5], [6], [7] and [10]). In most of this cases, the function Φ is supposed to belong to a Banach space of sufficiently smooth functions defined on an interval $[a,b]$. Such functions are always in $L_\infty([a,b])$, and so those examples can be adapted to our model.

It should be pointed out, therefore, that we are excluding the cases where the problem is ill-posed. We attempt to give a detailed description of the associated semigroup and its infinitesimal generator for a system like (2.1) *whenever a unique differentiable solution exists*. These solutions are often found by some other independent method, as in the examples cited above.

Bearing in mind this purpose, we will show that (2.1) can be written as an ordinary differential equation in a suitable Banach space J_p, which will be defined later, as follows:

$$\begin{aligned} \dot{\omega}(t) &= A\omega(t) + \beta u(t), \ \ t > 0 \\ \omega(0) &= \Phi_0 \end{aligned}$$

where $\beta : U \to J_p$ is given by $\beta u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}$ and

$$A \begin{pmatrix} \Phi_0 \\ \Phi(s) \end{pmatrix} = \begin{pmatrix} A_0\Phi_0 + \sum_i^q A_i\Phi(-h_i) + \sum_i^q C_i\Phi(h_i) \\ \dot{\Phi}(s) \end{pmatrix}, \ \ -h_q \le s \le 2h_q,$$

is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$ defined by

$$T(t) \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} x(t) \\ x(t+\cdot) \end{pmatrix},$$

where $x(\cdot)$ is the unique solution of the system

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + L_dx(t) + L_ax(t), \ \ t > 0 \\ x(0) &= \Phi_0 \\ x(s) &= \Phi(s), \ \ s \in [-h_q, 2h_q], \end{aligned}$$

where $L_dx(t) = \sum_i^q A_ix(t-h_i)$ and $L_ax(t) = \sum_i^q C_ix(t+h_i)$.

Once achieved these results we will give necessary and sufficient conditions to ensure the exact controllability for (2.1).

3. Main results

In the following we will show an alternative representation of the given solution of (2.1), and we will also prove that $T(t)$ (as given in (**??**)) is in fact a strongly continuous semigroup with A as its infinitesimal generator.

Theorem 2.4.1 of [11], deals on delay equations and the solution $x(\cdot)$ on $[0,\infty)$ is built recursively. This same construction cannot be done in our case but, as it has been stated, we are supposing that the solution $x(\cdot)$ is previously known.

Theorem 3.1. *Suppose that the unique solution $x(\cdot)$ on $[0,\infty)$ of* (2.2) *is known. Then $x(\cdot)$ satisfies the following recursive formula*

$$x(t) = e^{A_0t}\Phi_0 + \sum_{i=1}^{q} \int_0^t e^{A_0(t-s)}(A_ix(s-h_i) + C_ix(s+h_i))ds \ \ for\ t \ge 0. \tag{3.1}$$

Proof . Notice first that for $t \in [0,h_q]$ the term $\sum_{i=1}^q A_ix(t-h_i) + C_ix(t+h_i)$ equals the function

$$v(t) := \sum_{i=1}^{q} A_i\Phi(t-h_i) + C_i\Phi(t+h_i).$$

So we may reformulate the system (2.2) on $[0,h_q]$ as

$$\dot{x}(t) = A_0x(t) + v(t), \ \ x(0) = \Phi_0. \tag{3.2}$$

It is well known that the unique solution of (3.2) is given by

$$x(t)=e^{A_0t}\Phi_0+\int_0^t e^{A_0(t-s)}v(s)ds$$

and this equals (3.1).

Let us consider now the case $t\geq h_q$. We use the hypothesis that $x(t)$ is known for every t, and so, at a given time t, the function $\sum_{i=1}^{q}A_ix(t-h_i)+C_ix(t+h_i)$ is also known. Then we can proceed in a similar way. Applying finite dimensional theory gives that the unique solution satisfies (3.1). ■

In the following we will construct the c_0-semigroup and its infinitesimal generator associated to the equation (2.2).

Lemma 3.2. *If $x(t)$ is the solution of (2.2), then the following inequalities hold:*

[**i**] $||x(t)||\leq C_t[||\Phi_0||+||\Phi(\cdot)||_{L_p([-h_q,2h_q];\mathbb{R}^n)}],\ \ 1\leq p\leq\infty$

[**ii**] $\int_{2h_q}^{2h_q+t}||x(\tau)||^pd\tau\leq D_t[||\Phi_0||^p+||\Phi(\cdot)||^p_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}],\ \ 1\leq p<\infty$

where C_t and D_t are constants depending only on t.

Proof . It is well known that for some positive constants M_0, W_0, e^{A_0t} satisfies $||e^{A_0t}||\leq M_0e^{W_0t},\ t\geq 0$.
Let us define the positive constant M by

$$M:=max(||A_1||,\cdots,||A_q||,||C_1||,\cdots,||C_q||,M_0).$$

Then, it is deduced, from the formula of the solution of equation (2.2) that

$$\begin{aligned}||x(t)|| &\leq Me^{W_0t}||\Phi_0||+\sum_{i=1}^{q}M^2\int_0^t e^{W_0(t-s)}(||x(s-h_i)||+||x(s+h_i)||)ds\\ &\leq Me^{W_0t}||\Phi_0||+SI1\\ &= Me^{W_0t}||\Phi_0||+M^2e^{W_0t}SI2,\end{aligned} \tag{3.3}$$

where

$$SI1=\sum_{i=1}^{q}M^2(\int_{-h_i}^{t-h_i}e^{W_0(t-\tau-h_i)}||x(\tau)||d\tau+\int_{h_i}^{t+h_i}e^{W_0(t-\tau+h_i)}||x(\tau)||d\tau)$$

and

$$SI2=\sum_{i=1}^{q}(\int_{-h_i}^{t-h_i}e^{-W_0(\tau+h_i)}||x(\tau)||d\tau+\int_{h_i}^{t+h_i}e^{-W_0(\tau-h_i)}||x(\tau)||d\tau).$$

But after a standard estimation we have

$$SI2\leq e^{W_0h_q}q\int_{-h_q}^{2h_q}||\Phi(\tau)||d\tau+q\int_0^t e^{-W_0\tau}||x(\tau)||d\tau$$

$$+\sum_{i=1}^{q}\int_0^t e^{-W_0\tau}||x(\tau+h_i)||d\tau\ \ (\text{because}\ \ W_0>0)$$

$$\leq C'||\Phi(\cdot)||_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}+q\int_0^t e^{-W_0\tau}||x(\tau)||d\tau$$

$$+q\int_0^t e^{-W_0\tau}(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||))d\tau.$$

$$\leq C'||\Phi(\cdot)||_{L_p([-h_q,2h_q];\ \mathbb{R}^n)}+q\int_0^t e^{-W_0\tau}(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1)d\tau$$

$$+q\int_0^t e^{-W_0\tau}(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||))d\tau.$$

Now, if $f(\tau)$ is defined as

$$f(\tau) \quad = \quad 1+\frac{\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||)}{\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1}$$

we have that the former inequality is estimated by

$$C'||\Phi(\cdot)||_{L_p([-h_q,2h_q];\,\mathbb{R}^n)}+q\int_0^t e^{-W_0\tau}f(\tau)(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1)d\tau. \tag{3.4}$$

Combining (3.3) and (3.4) we obtain

$$||x(t)|| \leq e^{W_0 t}[M||\Phi_0||+M^2C'||\Phi||_{L_p([-h_q,2h_q];\,\mathbb{R}^n)}$$

$$+M^2q\int_0^t e^{-W_0\tau}f(\tau)(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1)d\tau]$$

.

Now, if C'' is constant (depending on t) such that $1 \leq e^{W_0 t}M^2C''||\Phi||_{L_p([-h_q,2h_q];\,\mathbb{R}^n)}$, we have

$$||x(t)||+1 \leq e^{W_0 t}[M||\Phi_0||+M^2C||\Phi||_{L_p([-h_q,2h_q];\,\mathbb{R}^n)}$$

$$+M^2q\int_0^t e^{-W_0\tau}f(\tau)(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1)d\tau]$$

where $C=C'+C''$,
or equivalently

$$z(t) \leq \beta+\int_0^t a(\tau)z(\tau)d\tau$$

where
$\beta = M||\Phi_0||+M^2C||\Phi||_{L_p([-h_q,2h_q];\,\mathbb{R}^n)}$, $a(\tau)=M^2qf(\tau)$ and $z(\tau)$ is the function defined by $z(\tau)=e^{-W_0\tau}(\max(||x(\tau+h_1)||,\cdots,||x(\tau+h_q)||,||x(\tau)||)+1)$. Then, from Gronwall Lemma (see [11], p.639) we conclude that

$$z(t) \leq \beta(\exp\int_0^t a(\tau)d\tau)$$

and so

$$\begin{aligned}||x(t)|| &\leq \beta\exp(\int_0^t a(\tau)d\tau+W_0t)\\ &\leq \exp(M^2q\int_0^t f(\tau)d\tau+W_0t)\max[M,M^2C][||\Phi_0||+||\Phi||_{L_p([-h_q,2h_q];\,\mathbb{R}^n)}].\end{aligned}$$

Now, let us note that $\int_0^t f(\tau)d\tau \leq 2t$. This shows **[i]**.

In a similar way, we obtain from the former inequality, for $p\in[1,\infty)$

$$||x(t)||^p \leq K(\exp\{p(qM^2c_t+W_0t)\}[||\Phi_0||^p+||\Phi||^p_{L_p([-h_q,2h_q];\,\mathbb{R}^n)}]),$$

where K a suitable constant. Integrating this inequality gives **[ii]**. ■

Now, we are going to construct the c_0-semigroup. Let us first recall that, for a pair X,Y of normed spaces, we can introduce a normed space $X\oplus Y$ called a *direct (topological) sum* of X and Y that consists of all ordered pairs $(x,y), x\in X, y\in Y$ together with the norm $\|(x,y)\|=\|x\|_X+\|y\|_Y$. X and Y are isometric to subspaces $\{(x,0);x\in X\}$ and $\{(0,y);y\in Y\}$ of $X\oplus Y$. If X and Y are Banach spaces, so is $X\oplus Y$. Convergence in $X\oplus Y$ means that (x_n,y_n) tends to (x,y) if and only if both $\|x_n-x\|_X$ and $\|y_n-y\|_Y$ tend to zero as n tends to infinity.

Let us also recall that the elements in $L_p([-h_q,2h_q];\,\mathbb{R}^n)$, $1\leq p\leq\infty$, are, in fact, equivalence classes of functions, with the corresponding equivalence relation $\Re$ defined by $f\Re g$ if and only if $f=g$ a.e.

Let M_p be the closure in $L_p([-h_q,2h_q];\,\mathbb{R}^n)$ of the subspace $L_p([-h_q,2h_q];\,\mathbb{R}^n)\cap C([-h_q,2h_q];\,\mathbb{R}^n)$. M_p is a Banach space with the same norm as $L_p([-h_q,2h_q];\,\mathbb{R}^n)$. Let us now consider the Banach space $\mathbb{R}^n\oplus M_p$, and let G_p be the linear subspace of all pairs $(r,f)\in\mathbb{R}^n\oplus M_p$, $1\leq p\leq\infty$, such that $r=f(0)$. If $f\in L_p([-h_q,2h_q];\,\mathbb{R}^n)$, it is well known that there exist functions

$g \in L_p([-h_q, 2h_q];\ \mathbb{R}^n)$, such that $f = g$ a.e., as we have previously indicated. But in our case the function f is supposed to be continuous on the whole interval $[-h_q, 2h_q]$ and there is no ambiguity: G_p is thus well defined.

Finally, let J_p be the closure in $\mathbb{R}^n \oplus M_p$ of the linear subspace G_p. For $1 \le p \le \infty$, J_p is thus a Banach space.

Since $(x, f) = (f(0), f) + (x - f(0), 0)$, it is easily seen that J_∞ is a topologically complemented subspace of $\mathbb{R}^n \oplus M_\infty$.

On the other hand, for $h \in L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$, there exist a continuous function $f \in L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$ such that $h = f$ a.e. Bearing in mind that $(r, h) = (r, f) + (0, h - f)$, we have that $\mathbb{R}^n \oplus M_\infty$ is also topologically complemented in $\mathbb{R}^n \oplus L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$, and thus so is J_∞.

Theorem 3.3. *The operator $T(t)$ defined for each $t \ge 0$ by (**??**) satisfies*
(1) $T(t) \in L(J_p)$ for every $t \ge 0$
(2) $T(t)$ is a C_0-semigroup in J_p

Proof . (1) First, we suppose $p \in [1, \infty)$. Note that

$$\begin{aligned}\left(\int_{-h_q}^{2h_q} ||x(t+\tau)||^p d\tau\right)^{1/p} &= \left(\int_{-h_q+t}^{2h_q+t} ||x(\tau)||^p d\tau\right)^{1/p} \le \left(\int_{-h_q}^{2h_q+t} ||x(\tau)||^p d\tau\right)^{1/p} \\ &\le K\left((\int_{-h_q}^{2h_q} ||\Phi(\tau)||^p d\tau)^{1/p} + (\int_{2h_q}^{2h_q+t} ||x(\tau)||^p d\tau)^{1/p}\right).\end{aligned}$$

Then, using Lemma 3.2, we have for $\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \in J_p$,

$$\begin{aligned}||T(t)\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}|| &= ||x(t)|| + \left(\int_{-h_q}^{2h_q} ||x(t+\tau)||^p d\tau\right)^{1/p} \\ &\le R_t[||\Phi_0|| + ||\Phi(\cdot)||_{L_p([-h_q, 2h_q];\mathbb{R}^n)}].\end{aligned}$$

In the case $p = \infty$, we can suppose $x(t) \neq 0$ (otherwise the result is trivial), and let us choose t_0 such that $x(t_0) \neq 0$. Then, bearing in mind that for each t, $||x(t+\tau)||_\infty$ is a positive real number whose value only depends on t and using Lemma 3.2, we have

$$||x(t+\tau)||_\infty = \frac{||x(t+\tau)||_\infty}{||x(t_0)||} \cdot ||x(t_0)|| \le C_t[||\Phi_0|| + ||\Phi(\cdot)||_{L_p([-h_q, 2h_q];\mathbb{R}^n)}],$$

where $C_t = \frac{||x(t+\tau)||_\infty}{||x(t_0)||} \cdot C_{t_0}$.

Now, we will prove (2). The semigroup property can be proven similarly as in Theorem 2.4.4 of [11]. We only have to note that, in this case, it is considered the function $g(t) = x(t+s)$, where $x(\cdot)$ is the solution of system (2.2). Then $g(t)$ satisfies

$$\begin{aligned}\dot{g}(t) &= A_0 g(t) + \sum_{i=1}^{q} (A_i g(t-h_i) + C_i g(t+h_i)),\ \ t \ge 0 \\ g(0) &= x(s) \\ g(\theta) &= x(s+\theta),\ \ \theta \in [-h_q, 2h_q].\end{aligned}$$

To prove the strong continuity, we begin with the case $p \in [1, \infty)$. For $t < h_1$ we have

$$\begin{gathered}||T(t)\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} - \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}|| = \\ ||e^{A_0 t}\Phi_0 + \textstyle\sum_{i=1}^{q} \int_0^t e^{A_0(t-s)}(A_i\Phi(s-h_i) + C_i\Phi(s+h_i))ds - \Phi_0|| + \\ \left(\textstyle\int_{-h_q}^{-t} ||\Phi(t+\tau) - \Phi(\tau)||^p d\tau + \int_{-t}^{2h_q} ||x(t+\tau) - \Phi(\tau)||^p d\tau\right)^{1/p}.\end{gathered}$$

The first term converges to zero as $t \to 0$, because

$$e^{A_0 t} + \sum_{i=1}^{q} \int_0^t e^{A_0(t-s)}(A_i\Phi(s-h_i) + C_i\Phi(s+h_i))ds$$

is continuous. On the other side, using the triangle inequality and Lemma 3.2, the integral terms tend to zero by Lebesgue's Dominated Convergence Theorem.

The case $p = \infty$ is similar. We only have to note that $||x(t+\tau) - x(\tau)||_\infty \to 0$ as $t \to 0$. ∎

Lemma 3.4. *Consider the c_0-semigroup $T(t)$ defined above and let A be its infinitesimal generator. For sufficiently large $\alpha\in\mathbb{R}$, the resolvent is given by*

$$(\alpha I-A)^{-1}\begin{pmatrix}\Phi_0\\ \Phi(\cdot)\end{pmatrix}=\begin{pmatrix}g(0)\\ g(\cdot)\end{pmatrix}$$

where

$$g(\theta)=e^{\alpha\theta}g(0)-\int_0^{\theta}e^{\alpha(\theta-s)}\Phi(s)ds,\quad \theta\in[-h_q,2h_q] \tag{3.5}$$

and

$$g(0)=(\Delta(\alpha))^{-1}\left(\Phi_0+\sum_{i=1}^{q}\int_{-h_i}^{0}e^{-\alpha(\theta+h_i)}A_i\Phi(\theta)d\theta\right). \tag{3.6}$$

where

$$\Delta(\lambda)=\lambda I-A_0-\left(\sum_{i=1}^{q}e^{-\lambda h_i}A_i+e^{\lambda h_i}C_i\right),\quad \lambda\in\mathbb{C}.$$

Furthermore, g satisfies the following relation

$$\alpha g(0)=\Phi_0+A_0g(0)+\sum_{i=1}^{q}A_ig(-h_i)+C_ig(h_i). \tag{3.7}$$

Proof . According to Lemma 2.1.11 in [11], we have for $\alpha>\omega_0$ that

$$(\alpha I-A)^{-1}\begin{pmatrix}\Phi_0\\ \Phi(\cdot)\end{pmatrix}=\int_0^{\infty}e^{-\alpha t}T(t)\begin{pmatrix}\Phi_0\\ \Phi(\cdot)\end{pmatrix}dt=\int_0^{\infty}e^{-\alpha t}\begin{pmatrix}x(t)\\ x(t+\cdot)\end{pmatrix}dt.$$

We define

$$g(\theta)=\int_0^{\infty}e^{-\alpha t}x(t+\theta)dt,\ \text{ for }\ \theta\in[-h_q,2h_q].$$

Rewriting this function as $g(\theta)=\int_{\theta}^{\infty}e^{-\alpha(s-\theta)}x(s)ds$ it is easy to see that $g(\cdot)$ is a solution of

$$\frac{\partial g(\theta)}{\partial\theta}=\alpha g(\theta)-x(\theta),\quad \theta\in[-h_q,2h_q].$$

In $[-h_q,2h_q]$, the variation of constants formula for this ordinary differential equation shows that $g(\cdot)$ equals (3.5). It only remains to prove (3.6).

Bearing in mind that, according to Lemma 3.2, $C_t\left[||\Phi_0||+||\Phi(\cdot)||_{L_p([-h_q,2h_q];\mathbb{R}^n)}\right]$ is an upper bound for $x(t)$, we have

$$\begin{aligned}\alpha g(0) &= \alpha\int_0^{\infty}e^{-\alpha t}x(t)dt=-[x(t)e^{-\alpha t}]_0^{\infty}+\int_0^{\infty}e^{-\alpha t}\dot{x}(t)dt\\ &= \Phi_0+\int_0^{\infty}e^{-\alpha t}[A_0x(t)+\sum_{i=1}^{q}A_ix(t-h_i)+C_ix(t+h_i)]dt\\ &= \Phi_0+A_0\int_0^{\infty}e^{-\alpha t}x(t)dt+\sum_{i=1}^{q}\int_0^{\infty}e^{-\alpha t}(A_ix(t-h_i)+C_ix(t+h_i))dt\\ &= \Phi_0+A_0g(0)+\sum_{i=1}^{q}A_ig(-h_i)+C_ig(h_i).\end{aligned}$$

This proves equation (3.7). On the other hand, if we split the integrals in the former equation, we obtain

$$\begin{aligned}\alpha g(0) &= \Phi_0+A_0g(0)+\sum_{i=1}^{q}\int_{h_i}^{\infty}e^{-\alpha t}(A_ix(t-h_i)+C_ix(t+h_i))dt\\ &+ \sum_{i=1}^{q}\int_0^{h_i}e^{-\alpha t}(A_i\Phi(t-h_i)+C_i\Phi(t+h_i))dt\\ &= \Phi_0+A_0g(0)+\sum_{i=1}^{q}e^{-\alpha h_i}A_ig(0)+\sum_{i=1}^{q}e^{-\alpha h_i}C_ig(0)\\ &- \sum_{i=1}^{q}e^{-\alpha h_i}\int_{h_i}^{2h_i}e^{-\alpha\theta}C_i\Phi(\theta)d\theta+\sum_{i=1}^{q}\int_0^{h_i}e^{-\alpha t}(A_i\Phi(t-h_i)+C_i\Phi(t+h_i))dt\end{aligned}$$

and so

$$\left[\alpha I - A_0 - \left(\sum_{i=1}^{q} e^{-\alpha h_i} A_i + e^{\alpha h_i} C_i)\right)\right] g(0) = \Phi_0 - \sum_{i=1}^{q} e^{\alpha h_i} \int_{h_i}^{2h_i} e^{-\alpha\theta} C_i \Phi(\theta) d\theta$$
$$+ \sum_{i=1}^{q} \left(\int_{-h_i}^{0} e^{-\alpha(\theta+h_i)} A_i \Phi(\theta) d\theta + \int_{h_i}^{2h_i} e^{-\alpha(\theta-h_i)} C_i \Phi(\theta) d\theta \right)$$
$$= \Phi_0 + \sum_{i=1}^{q} \int_{-h_i}^{0} e^{-\alpha(\theta+h_i)} A_i \Phi(\theta) d\theta$$

which proves (3.6) for sufficiently large α. ■

In the following theorem, we first give an explicit formula for the infinitesimal generator A. The second part of the theorem deals with the spectral properties. Some of the main conclusions of this second part are not true for MTFDE if the problem is ill-posed ([4]), but for well-posed problems, as in our case, they remain valid.

Theorem 3.5. *Consider the c_0-semigroup defined as before. Its infinitesimal generator is given by*

$$A \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} A_0\Phi_0 + \sum_{i=1}^{q} A_i \Phi(-h_i) + C_i \Phi(h_i) \\ \dfrac{\partial \Phi(\cdot)}{\partial \theta} \end{pmatrix}$$

with domain

$$D(A) = \left\{ \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \in J_p : \Phi \text{ is absolutely continuous}, \frac{\partial \Phi}{\partial \theta} \in L_p([-h_q, 2h_q]; \mathbb{R}^n) \right\}.$$

Furthermore, the spectrum of A is discrete and is given by

$$\sigma(A) = \sigma_p(A) = \{\lambda \in \mathbb{C} : \det(\Delta(\lambda)) = 0\},$$

where $\Delta(\lambda)$ was defined in Lemma 3.4 and the multiplicity of each eigenvalue is finite for $p = 2$.

For every $\delta \in \mathbb{R}$, there are only finitely many eigenvalues in $\mathbb{C}_\delta^+$. If $\lambda \in \sigma_p(A)$, then $\begin{pmatrix} r \\ e^{\lambda \cdot} r \end{pmatrix}$, where $r \neq 0$ satisfies $\Delta(\lambda) r = 0$, is an eigenvector of A with eigenvalue λ. On the other hand, if ζ is an eigenvector of A with eigenvalue λ, then $\zeta = \begin{pmatrix} r \\ e^{\lambda \cdot} r \end{pmatrix}$ with $\Delta(\lambda) r = 0$.

Proof . We denote by $\tilde{A}$ the operator

$$\tilde{A} \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} A_0\Phi_0 + \sum_{i=1}^{q} A_i \Phi(-h_i) + C_i \Phi(h_i) \\ \dfrac{\partial \Phi(\cdot)}{\partial \theta} \end{pmatrix}$$

with domain

$$D(\tilde{A}) = \left\{ \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \in J_p : \Phi \text{ is absolutely continuous}, \frac{\partial \Phi}{\partial \theta} \in L_p([-h_q, 2h_q]; \mathbb{R}^n) \right\}.$$

We have to show that the infinitesimal generator A equals $\tilde{A}$. Let α_0 be a sufficiently large real number such that the results of Lemma 3.4 hold. We will show that the inverse of $(\alpha_0 I - \tilde{A})$ equals $(\alpha_0 I - A)^{-1}$. This is enough to show that $A = \tilde{A}$. To this end, we calculate

$$(\alpha_0 I - \tilde{A})(\alpha_0 I - A)^{-1} \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = (\alpha_0 I - \tilde{A}) \begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix} \text{(where g is as in Lemma 3.4)}$$

$$= \begin{pmatrix} \alpha_0 g(0) - A_0 g(0) - (\sum_{i=1}^{q} A_i g(-h_i) + C_i g(h_i)) \\ \alpha_0 g(\cdot) - \dfrac{\partial g(\cdot)}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix},$$

where the last equality holds by differentiating (3.5) from Lemma 3.4. Then, for $\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \in J_p$, we have shown that

$$(\alpha_0 I - \tilde{A})(\alpha_0 I - A)^{-1} \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}. \tag{3.8}$$

It remains to show that

$$(\alpha_0 I - A)^{-1}(\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \text{ in } D(A).$$

For $\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} \in D(A)$ we define $\begin{pmatrix} \Phi_1 \\ \Phi_1(\cdot) \end{pmatrix} := (\alpha_0 I - A)^{-1}(\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}$.

Then according to (3.8), we have $(\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_1 \\ \Phi_1(\cdot) \end{pmatrix} = (\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix}$. Then $\begin{pmatrix} \Phi_0 \\ \Phi(\cdot) \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \Phi_1(\cdot) \end{pmatrix}$ if and only if $(\alpha_0 I - \tilde{A})$ is injective. Let us suppose, on the contrary, that there exists $\begin{pmatrix} \Phi_2 \\ \Phi_2(\cdot) \end{pmatrix} \in D(A)$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (\alpha_0 I - \tilde{A}) \begin{pmatrix} \Phi_2 \\ \Phi_2(\cdot) \end{pmatrix} = \begin{pmatrix} \alpha_0 \Phi_2(0) - A_0 \Phi_2(0) - L_d \Phi_2(0) - L_a \Phi_2(0) \\ \alpha_0 \Phi_2(\cdot) - \dfrac{\partial \Phi_2(\cdot)}{\partial \theta} \end{pmatrix},$$

where we have used the definition of $\tilde{A}$ and $D(\tilde{A})$ in the last two steps. Then

$$\Phi_2(\theta) = \Phi_2(0) e^{\alpha_0 \theta} \text{ and } \alpha_0 \Phi_2(0) - A_0 \Phi_2(0) - (\sum_{i=1}^{q} A_i \Phi_2(-h_i) + C_i \Phi_2(h_i))$$
$$= \alpha_0 \Phi_2(0) - A_0 \Phi_2(0) - (\sum_{i=1}^{q} A_i \Phi_2(0) e^{-\alpha_0 h_i} + C_i \Phi_2(0) e^{\alpha_0 h_i}) = 0.$$

However, since

$$\alpha_0 I - A_0 - (\sum_{i=1}^{q} A_i e^{-\alpha_0 h_i} + C_i e^{\alpha_0 h_i})$$

is invertible, this implies that $\Phi_2(0) = 0$ and thus $\Phi_2(\cdot) = \Phi_2(0) e^{-\alpha_0 \cdot} = 0$. This contradiction implies that $(\alpha_0 I - \tilde{A})$ is injective. This proves the assertion that A equals $\tilde{A}$.

Now, we calculate the spectrum of A. In Lemma 3.4 we obtained an expression for the resolvent operator for $\alpha \in \mathbb{R}$ large enough, in terms of g given by (3.5) and (3.6). Let us denote by Q_λ the extension of the resolvent operator to $\mathbb{C}$:

$$Q_\lambda \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} := \begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix}.$$

A simple calculation shows that if $\lambda \in \mathbb{C}$ satisfies

$$\det(\lambda I - A_0 - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i})) \neq 0,$$

then Q_λ is a bounded linear operator from $J_{p_\mathbb{C}}$ to $J_{p_\mathbb{C}}$, where $J_{p_\mathbb{C}}$ is the closed linear subspace of pairs $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix}$ in $\mathbb{C}^n \oplus L_p([-h_q, 2h_q]; \mathbb{C}^n)$ such that $r = f(0)$. Furthermore, for these λ we have $(\lambda I - A) Q_\lambda = I$ and $(\lambda I - A)$ is injective. As in the first part of the proof, we conclude that $Q_\lambda = (\lambda I - A)^{-1}$, the resolvent operator of A. We have that

$$\left\{ \lambda \in \mathbb{C} : \det(\lambda I - A_0 - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i})) \neq 0 \right\} \subset \rho(A).$$

On the other hand, if $\det(\Delta(\lambda)) = 0$, there exists $z \in \mathbb{C}^n$ such that

$$(\lambda I - A_0 - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i})) z = 0.$$

The following element of $J_{p_{\mathbb{C}}}$

$$z_0 = \begin{pmatrix} z \\ e^{\lambda \cdot} z \end{pmatrix} \text{ is in } D(A)$$

and

$$(\lambda I - A)z_0 = \begin{pmatrix} \lambda z - A_0 z - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i}) z \\ \lambda e^{\lambda \theta} z - \frac{\partial}{\partial \theta} e^{\lambda \theta} z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then

$$\sigma_p(A) \supset \left\{ \lambda \in \mathbb{C} : \det(\lambda I - A_0 - (\sum_{i=1}^{q} A_i e^{-\lambda h_i} + C_i e^{\lambda h_i})) = 0 \right\}.$$

The remaining of the proof can be done, mutatis mutandis, as in Theorem 2.4.6 of [11]. ■

4. An application: Controllability

In this section we will apply some of the the results obtained in section 3 to study the null controllability for the system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{q} A_i x(t-h_i) + \sum_{i=1}^{q} C_i x(t+h_i) + Bu(t), \quad t > 0 \\ x(0) &= \Phi_0 \\ x(s) &= \Phi(s), \quad s \in [-h_q, 2h_q], \end{aligned} \tag{4.1}$$

where as before $0 < h_1 < h_2 < \cdots < h_q$, $A_i, C_i \in \mathscr{L}(\mathbb{R}^n)$, $i = 1, \cdots, q$, $A_i \neq 0$ for some i, $\Phi_0 \in \mathbb{R}^n$, $\Phi \in L_p([-h_q, 2h_q]; \mathbb{R}^n)$, $1 \leq p \leq \infty$.

Also for this case, we will consider $B \in \mathscr{L}(\mathbb{R}^n)$ and $u : [0, \infty) \to \mathbb{R}^n$ an essentially bounded function.

We have already shown that, if the problem is well-posed, (4.1) can be written equivalently as the following system of ordinary differential equations in J_p

$$\begin{aligned} \dot{w}(t) &= Aw(t) + \bar{B}u(t), \quad t > 0 \\ w(0) &= w_0 = (\Phi_0, \Phi(\cdot)), \end{aligned} \tag{4.2}$$

where A is the infinitesimal generator of the semigroup $\{T(t)\}_{t \geq 0}$ and $\bar{B} : \mathbb{R}^n \to J_p$ is given by $\bar{B}u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}$.

The mild solution of (4.2) is thus given by

$$w(t) = T(t)w_0 + \int_0^t T(t-s)\bar{B}u(s)ds.$$

Let Ω be a non-empty compact convex subset of $\mathbb{R}^n$. The set

$$\tilde{\Omega}_r = \{u \in L^{\infty}_{\mathbb{R}^n}[0, r] : u \in \Omega \text{ a.e}\}$$

is called the set of *admissible controls* of (4.2) (or equivalently (4.1)), while the set

$$A_r(w_0) = \left\{ T(r)w_0 + \int_0^r T(r-s)\bar{B}u(s)ds : u \in \tilde{\Omega}_r \right\}$$

is the set of *accesible points* of (4.2). The system (4.2) is *controllable* if $0 \in A_r(w_0)$.

In a more general context, we have a system similar to (4.2), with X and U Banach spaces, $A : X \to X$ the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$, $B : U \to X$ a bounded linear operator and $u : [0, \infty) \to U$ a strongly measurable, essentially bounded function. We suppose that Ω is a non-empty separable, weakly compact subset of U. The formula for the mild solution is completely similar, $\tilde{\Omega}_r = \{u \in L^{\infty}_U[0, r] : u \in \Omega \text{ a.e}\}$ is the set of admissible controls, while $A_r(w_0) = \left\{ S(r)w_0 + \int_0^r S(r-s)Bu(s)ds : u \in \tilde{\Omega}_r \right\}$ is the set of accesible points. Analogously, the system is controllable if $0 \in A_r(w_0)$.

The controllability map on $[0, r]$ for some $r \geq 0$ is the linear map

$$B^r : L_\infty([0, r]; U) \to X$$

defined by

$$B^r u = \int_0^r S(r-s)Bu(s)\mathrm{d}s$$

Now, one says that the system is exactly controllable on $[0,r]$ if every point in X can be reached from the origin at r, i.e., if $ran\,(B^r) = X$.

If $ran\,(B^r) = X$, then $0 \in A_r(0)$. On the other hand, one can prove, using the Open Mapping Theorem, the following: if $0 \in interior(A_r(0))$, then $ran\,(B^r) = X$. See ([14])

Next, we recall two results that we will use to characterize the null controllability. The Theorem of Peichl and Schappacher ([15]) is as follows:

Theorem 4.1. *Let X and U be reflexive Banach spaces with U separable. Let $B: U \to X$ be a bounded linear operator, A be the infinitesimal generator of a c_0-semigroup $\{S(s)\}_{s\geq 0}$ of operators on X and Ω be a weakly compact convex subset of U that contains 0. Then for each $T > 0$, $0 \in A_T(x_o)$ if and only if for each $x^* \in X^*$*

$$< x^*, S(T)x_0 > + \int_0^T \max_{v\in\Omega} < x^*, S(t)Bv > dt \;\geq 0.$$

Additionally, we have the Bárcenas-Diestel ([16]) extension

Theorem 4.2. *Let X and U be Banach spaces, let $B: U \to X$ be a bounded linear operator, and $A: X \to X$ be the infinitesimal generator of a c_0-semigroup $\{S(t)\}_{t\geq 0}$ on X whose dual semigroup is strongly continuous on $(0,\infty)$. Suppose Ω is a non-empty separable weakly compact convex subset of U containing 0. Then for each $T > 0$, $0 \in A_T(x_o)$ if and only if for each $x^* \in X^*$*

$$< x^*, S(T)x_0 > + \int_0^T \max_{v\in\Omega} < x^*, S(t)Bv > dt \;\geq 0.$$

Theorems 4 and 5 show how to set the control problems in a Banach Space context, focusing on the question of accessibility of controls. For separable reflexive spaces, the elegant result of Peichl-Schappacher proves to be very useful.

The Bárcenas-Diestel Theorem is, on the other hand, an important and recent achievement on exact controllability. Throughout the literature, hypotheses like "separable and reflexive" are frequently encountered. By employing techniques from Banach space theory and the theory of vector measures, the authors show how to remove the hypothesis of reflexivity (thus giving considerably greater generality to the resulting conclusions) and translate the question of accessibility of controls to a problem in semigroups of operators, namely, given a c_0-semigroup $(S(t))_{t\geq 0}$ of operators on a Banach space X, under what conditions is the dual semigroup strongly continuous on $(0,\infty)$? This is the question we will try to answer for the non-reflexive cases $p = 1$ and $p = \infty$

We recall that a Banach space is a *Grothendieck space* if every weakly*-convergent sequence in X^* is also weakly convergent. Equivalently, X is a Grothendieck space if every linear bounded operator from X to any separable Banach space is weakly compact. Among Grothendieck spaces, we will list all reflexive Banach spaces and $L^\infty(\Omega,\Sigma,\mu)$, where (Ω,Σ,μ) is a positive measure space. A Banach space isomorphic to a complemented subspace of a Grothendieck space is also a Grothendieck space. The direct sum of two Grothendieck spaces is also a Grothendieck space. Several characterizations of Grothendieck spaces are found in [17].

A Banach space is said to have the *Dunford-Pettis property* if every weakly compact operator in $L(X)$ applies relatively weakly compact sets onto norm compact sets. The most common examples of Banach spaces with this property are $L^1(\mu)$ and $C(K)$. Complemented subspaces and the direct sum of any two of such spaces also have the property. For more details, see [18].

If X is a Grothendieck space with the Dunford-Pettis property, Lotz ([19]) has shown that every strongly continuous semigroup is uniformly continuous, and therefore also is the adjoint semigroup.

We also recall that a bounded linear operator $T: X \to Y$ (where X and Y are Banach Spaces) *factors through a Banach space* Z if there are bounded linear operators $u: X \to Z$ and $v: Z \to Y$ such that $T = vu$

It is proven in [20] that if X is a Banach space and $\{T(t)\}_{t\geq 0}$ a c_0-semigroup defined on X such that for every $a > 0$ there exists a Grothendieck space Y_a such that $T(a)$ factors through Y_a, then $\{T^*(t)\}_{t\geq 0}$ is strongly continuous on $(0,\infty)$. This will prove useful to establish our main result for the case $p = 1$.

Factoring through Grothendieck spaces is, in general, not easy to verify, but among semigroups satisfying those assumptions (and, hence, having adjoints which are strongly continuous on $(0,\infty)$) we mention weakly compact semigroups, i.e, semigroups such that $T(t)$ is weakly compact for each t (see [20] for more details). There are many examples of weakly compact semigroups, a category that includes all compact semigroups. Moreover, for $p = 1$ the terms "weakly compact" and "compact" are equivalent, due to the classical Schur theorem.

It is true that those assumptions cannot be verified without any analysis of the semigroup, which is here presented in an abstract, general form. But provided that $x(t)$ and $\Phi(\cdot)$ are known, one can manage to get more precise information about it.

Finally, one should remember that all those considerations are relevant only for the case $p = 1$. For all other cases, no additional assumptions are needed.

Now, we can state the result concerning (4.2)

Theorem 4.3. *For each $r > 0$, $0 \in A_r(w_0)$ if and only if for each $x^* \in J_p^*$, $1 < p \leq \infty$,*

$$< x^*, T(r)w_0 > + \int_0^r \max_{v\in\Omega} < x^*, T(t)\bar{B}v(t) > dt \ \geq 0.$$

If additionally, we suppose that the associated semigroup satisfies that, for every $a > 0$ there exists a Grothendieck space Y_a such that $T(a)$ factors through Y_a, (in particular, if it is compact) then the same holds for $p = 1$.

Proof . The case $p \in (1, \infty)$ is an inmediate consequence of Theorem 4.1. We only have to remember that the direct sum of any two reflexive Banach spaces and every subspace of a reflexive Banach space are also reflexive.

Semigroups which factor through Grothendieck spaces have adjoints $\{T^*(t)\}_{t\geq 0}$ which are strongly continuous on $(0,\infty)$. Then Theorem 4.2 can be applied for the case $p = 1$.

Now, let us suppose $p = \infty$. Note that $\mathbb{R}^n$ and $L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$ are Grothendieck spaces with the Dunford-Pettis property (remember that $L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$ is isomorphic to $C(K)$ for some suitable compact Hausdorff space K, see [21]). Consequently, $\mathbb{R}^n \oplus L_\infty([-h_q, 2h_q];\ \mathbb{R}^n)$ is also a Grothendieck space with the Dunford-Pettis property, and so is the complemented subspace J_∞. Therefore, the associated semigroup $\{T(t)\}_{t\geq 0}$ is uniformly continuous, according to the Lotz Theorem [19]. In particular, the adjoint semigroup $\{T^*(t)\}_{t\geq 0}$ is uniformly continuous, and we can apply Theorem 4.2 again. ■

As a conclusion, let us indicate that the results obtained in this work can be applied to certain mixed-type systems of partial differential equations like the following

$$\begin{aligned}
\frac{\partial x(t,y)}{\partial t} &= D\Delta x(t,y) + \sum_i^q A_i x(t-h_i,y) + \sum_i^q C_i x(t+h_i,y) + Bu(t,y), \\
\frac{\partial x}{\partial \eta} &= 0, \ \ y \in \partial\Omega, \\
x(0,y) &= \Phi_0(y), \ \ y \in \Omega \\
x(s,y) &= \Phi(s,y),
\end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $t \in (0, r]$, $0 < h_1 < h_2 < \cdots < h_q$, D is an $n \times n$ nondiagonal matrix whose eigenvalues are semi-simple with nonnegative real part, $B, A_i, C_i \in \mathscr{L}(\mathbb{R}^n)$, $i = 1, 2, \ldots, q, A_i \neq 0$ for some i, $\Phi_0 \in \mathbb{R}^n$, the control $u : [0,\infty) \to \mathbb{R}^n$ is essentially bounded and $\Phi \in L_p[[-h_q, 2h_q]; \mathbb{R}^n]$, $1 \leq p \leq \infty$, is defined by

$$\Phi(s,y) = \begin{cases} \Phi_1(s,y), & s \in [-h_q, 0], \ \ y \in \Omega \\ \Phi_2(s,y), & s \in [0, 2h_q], \ \ y \in \Omega \end{cases}$$

The symbol η denotes the normal to $\partial\Omega$, and $\frac{\partial x}{\partial \eta}$ is the normal derivative, which is defined as the inner product of the gradient ∇x with the (unit) normal vector η. The condition $\frac{\partial x}{\partial \eta} = 0$ for $y \in \partial\Omega$ and $t \in (0, r]$ is thus an homogeneous Neumann condition.

We would like to finish with a brief note about the particular case of delay equations. Several interesting examples of this type are found in the literature. Among them we have systems of parabolic equations with delay (including particular cases of the nD heat equation and systems without diffusion coefficients), and in general a broad class of functional reaction-diffusion equations (see, for example, [12]). But there is now an important difference: in all those examples the function Φ is supposed to lie in a Hilbert space, while here it is allowed to belong to a L_p-space, $1 \leq p \leq \infty$. This in turn allows to study these classical equations (and, in particular, their null controllability) in a considerably more general context.

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