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# Fixed Point Sets of Multivalued Contractions and Stability Analysis

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**Abstract**
In this paper we derive a fixed point result for a multivalued generalized almost contraction which contains several rational terms through a six variables function and a four variables function. The space is assumed to satisfy some regularity conditions. In another part of the paper we establish stability results for fixed point sets of these contractions. The corresponding singlevalued case is also discussed. The results are obtained without any assumption of continuity. There are two illustrative examples.

**Keywords:** Hausdorff metric, Generalized almost contraction, Fixed point, Stability
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## 1. Introduction

In this paper we first establish the fixed point property of certain generalized multivalued mappings which are almost contractions and satisfy some admissibility conditions and then establish that these multivalued mappings have stable fixed point sets. We use rational terms in the contraction inequalities which are considered here.

Our theorems are deduced in the domain of setvalued analysis which is an extension of the ordinary mathematical analysis. Aubin et al. [1] in their book has described several aspects of this study. Banach's contraction mapping principle was extended to the domain of setvalued analysis by Nadler [2], which was followed by several other works in the same direction. Today multivalued fixed point theory has a large literature and can be regarded as a subject by itself. Some recent references from this area of study are [3]-[9].

Admissibility map was introduced in the work of Samet et al. [10]. After which in fixed point theory several other such conditions were introduced by many authors for the purpose of obtaining new fixed point results in metric spaces. The essence of such efforts is to restrict the contractive condition to appropriate subsets of $X \times X$, rather than assuming to be valid between arbitrary pairs of points from the metric space. This is the development which is parallel to the emergence of fixed point theory in partially ordered metric spaces where the introduction and use of the partial order in metric space also serves the same purpose [4], [11]-[16].

Almost contractions are generalizations of the contractive conditions by introducing an additional additive term in the contractive inequality. It was first introduced by Berinde in [17, 18] in which a generalization of the Banach's contraction mapping principle was established by using this idea. Almost contractions and its generalizations were further considered in several works like [3], [19]-[22].

The concept of stability of fixed point sets appeared first in the work of Nadler [2], i.e, in the same work though which the study of setvalued fixed point theory was initiated. There has been wide interest on these problems of stability which is related to limiting behaviors of sequence of multivalued mappings. Some of the several important works which appeared on the topic in recent times are noted in [4], [5], [12], [23]-[26].

Rational terms were used in problems of fixed point theory in a good number of papers. Such uses were initiated by Dass et al. [27] and were subsequently made in several works on fixed point theory of which some recent references are [12], [28], [29]-[32].

The purpose of this paper is to establish the existence of fixed points of multivalued cyclic $(\alpha-\beta)$- admissible mappings in metric spaces, a condition which we define here. The mappings are assumed to satisfy certain rational type generalized almost contractions which are also defined in this work. In Section 2, we describe some mathematical preliminaries which we use in our results in Sections 3 and 4. In Section 3, we prove a fixed point result for multivalued mapping satisfy certain rational type generalized almost contractions. In Section 4, we investigate the stability of fixed point sets of above mentioned setvalued contractions which is derived without continuity assumption.

## 2. Mathematical preliminaries

The following are the concepts from setvalued analysis which we use in this paper. Let $(X,\ d)$ be a metric space. Let $N(X) :=$ the collection of all nonempty subsets of $X$; $CB(X) :=$ the collection of all nonempty closed and bounded subsets of $X$; and $C(X) :=$ the collection of all nonempty compact subsets of $X$. Now for $x \in X$ and $A,\ B \in CB(X)$, the functions $D(x,\ B)$ and $H(A,\ B)$ are defined as follows:

$$D(x,\ B) = \inf\,\{d(x,\ y) : y \in B\} \ \text{ and } \ H(A,\ B) = \max\,\{\sup_{x\in A} D(x,\ B),\ \sup_{y\in B} D(y,\ A)\}.$$

$H$ is known as the Hausdorff metric induced by $d$ on $CB(X)$ [2]. Further, if $(X,\ d)$ is complete then $(CB(X),\ H)$ is also complete.

**Lemma 2.1** ([6])**.** Let $(X,\ d)$ be a metric space and $B \in C(X)$. Then for every $x \in X$ there exists $y \in B$ such that $d(x,\ y) = D(x,\ B)$.

**Definition 2.2.** Let $X$ be a nonempty set, $f : X \to X$ be a singlevalued mapping and $T : X \to N(X)$ be a multivalued mapping. A point $x \in X$ is called a fixed point of $f$ (resp. $T$ ) if and only if $x = fx$ (resp. $x \in Tx$).

The set of all fixed points of $f$ and $T$ are denoted respectively by $F(f)$ and $F(T)$.

In [10] Samet et al. introduced the concept of $\alpha$- admissible mappings and utilized these mappings to prove some fixed point results in metric spaces.

**Definition 2.3** ([10])**.** Let $X$ be a nonempty set, $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$. $T$ is said to be an $\alpha$-admissible mapping if for $x,\ y \in X$, $\alpha(x,\ y) \geq 1 \Longrightarrow \alpha(Tx,\ Ty) \geq 1$.

In the following we define cyclic $(\alpha-\beta)$ admissibility for multivalued mappings.

**Definition 2.4.** Let $X$ be a nonempty set, $T : X \longrightarrow N(X)$ be a multivalued mapping and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Then $T$ is said to be a cyclic $(\alpha,\ \beta)$- admissible mapping if for $x,\ y \in X$,

$(i)$ $\alpha(x) \geq 1 \ \Longrightarrow \beta(u) \geq 1$ for all $u \in Tx$,

$(ii)$ $\beta(y) \geq 1 \ \Longrightarrow \alpha(v) \geq 1$ for all $v \in Ty$.

**Definition 2.5.** Let $(X,\ d)$ be a metric space and $\gamma : X \longrightarrow [0,\ \infty)$. Then $X$ is said to have $\gamma$- regular property if $\{x_n\}$ is a sequence in $X$ with $\gamma(x_n) \geq 1$ for all $n$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then $\gamma(x) \geq 1$.

Let $\Theta$ be the collection of all mappings $\theta : [0,\ \infty)^6 \longrightarrow [0,\ \infty)$ such that $(i)$ $\theta$ is continuous and nondecreasing in each coordinate; (ii) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ and $\psi(t) < t$ for each $t > 0$, where $\psi(t) = \theta(t,\ t,\ t,\ t,\ t,\ t)$.

It is to be noted that the properties of $\theta$ imply that $\theta(0,\ 0,\ 0,\ 0,\ 0,\ 0) = 0$.

Let $\Omega$ be the collection of all mappings $\varphi : [0,\ \infty)^4 \longrightarrow [0,\ \infty)$ such that $(i)$ $\varphi$ is continuous and nondecreasing in each coordinate; $(ii)$ $\varphi(t_1,\ t_2,\ t_3,\ t_4) = 0$ if $t_1t_2t_3t_4 = 0$.

**Definition 2.6.** Let $(X,\ d)$ be a metric space, $T : X \longrightarrow X$ and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Let $\mu,\ \nu \geq 0$, $\theta \in \Theta$ and $\varphi \in \Omega$. We say that $T$ is generalized almost contraction if for $x,\ y \in X$ with $\alpha(x)\ \beta(y) \geq 1$ or $\alpha(y)\ \beta(x) \geq 1$,

$$d(Tx,\ Ty) \leq M(x,\ y) + N(x,\ y),$$

where

$$M(x,\ y) = \theta\Big(d(x,\ y),\ d(x,\ Tx),\ d(y,\ Ty),\ \frac{1}{2}[d(y,\ Tx)+d(x,\ Ty)],\frac{d(y,\ Ty)\ [1+d(x,\ Tx)^{\mu}]}{1+d(x,\ y)^{\mu}},\ \frac{d(y,\ Tx)\ [1+d(x,\ Ty)^{\nu}]}{1+d(x,\ y)^{\nu}}\Big)$$

and

$$N(x,\ y) = \varphi\Big(d(x,\ Tx),\ d(y,\ Ty),\ d(x,\ Ty),\ d(y,\ Tx)\Big).$$

**Definition 2.7.** Let $(X,\ d)$ be a metric space, $T : X \longrightarrow C(X)$ be a multivalued mapping and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Let $\mu,\ \nu \geq 0$, $\theta \in \Theta$ and $\varphi \in \Omega$. We say that $T$ is generalized almost contraction if for $x,\ y \in X$ with $\alpha(x)\ \beta(y) \geq 1$ or $\alpha(y)\ \beta(x) \geq 1$,

$$H(Tx,\ Ty) \leq M(x,\ y)+N(x,\ y), \tag{2.1}$$

where

$$M(x,\ y) = \theta\Big(d(x,\ y),\ D(x,\ Tx),\ D(y,\ Ty),\ \frac{1}{2}[D(y,\ Tx)+D(x,\ Ty)],\frac{D(y,\ Ty)\ [1+D(x,\ Tx)^{\mu}]}{1+d(x,\ y)^{\mu}},\ \frac{D(y,\ Tx)\ [1+D(x,\ Ty)^{\nu}]}{1+d(x,\ y)^{\nu}}\Big)$$

and

$$N(x,y) = \varphi\Big(D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\Big).$$

## 3. Main results

**Theorem 3.1.** *Let $(X,\ d)$ be a complete metric space, $T : X \longrightarrow C(X)$ be a multivalued mapping and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Suppose that (i) $X$ is regular with respect to $\alpha$ and $\beta$; (ii) $T$ is a cyclic $(\alpha,\ \beta)$- admissible mapping; (iii) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ or $\beta(x_0) \geq 1$ and (iv) there exist $\mu,\ \nu \geq 0$, $\theta \in \Theta$ and $\varphi \in \Omega$ such that $T$ is a generalized almost contraction. Then $T$ has a fixed point in $X$.*

**Proof.** By the assumption (iii), suppose there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ (the proof is similar if $\beta(x_0) \geq 1$). Let $x_1 \in Tx_0$. By the assumption (ii), $\beta(x_1) \geq 1$. Now by Lemma 2.1, there exists $x_2 \in Tx_1$ such that $d(x_1,\ x_2) = D(x_1,\ Tx_1)$. As $\beta(x_1) \geq 1$ and $x_2 \in Tx_1$, by the assumption (ii), we have $\alpha(x_2) \geq 1$. Also by Lemma 2.1, there exists $x_3 \in Tx_2$ such that $d(x_2,\ x_3) = D(x_2,\ Tx_2)$. Since $x_3 \in Tx_2$ and $\alpha(x_2) \geq 1$, by the assumption (ii), $\beta(x_3) \geq 1$. Continuing this process, we construct a sequence $\{x_n\}$ such that for all $n \geq 0$,

$$x_{n+1} \in Tx_n,\ \ d(x_n,\ x_{n+1}) = D(x_n,\ Tx_n)\ \text{ and }\ \alpha(x_{2n}) \geq 1,\ \beta(x_{2n+1}) \geq 1. \tag{3.1}$$

By (3.1) either $\alpha(x_n)\beta(x_{n+1}) \geq 1$ or $\alpha(x_{n+1})\beta(x_n) \geq 1$. Applying the assumption (iv), we have

$$d(x_{n+1},\ x_{n+2}) = D(x_{n+1},\ Tx_{n+1}) \leq H(Tx_n,\ Tx_{n+1}) \leq M(x_n,\ x_{n+1})+N(x_n,\ x_{n+1}). \tag{3.2}$$

Now,

$$\begin{aligned}
M(x_n,x_{n+1}) = \theta\Big(&d(x_n,x_{n+1}),\ D(x_n,Tx_n),\ D(x_{n+1},Tx_{n+1}),\\
&\frac{D(x_{n+1},Tx_n)+D(x_n,Tx_{n+1})}{2},\frac{D(x_{n+1},\ Tx_{n+1})\ [1+D(x_n,\ Tx_n)^{\mu}]}{1+d(x_n,\ x_{n+1})^{\mu}},\ \frac{D(x_{n+1},\ Tx_n)\ [1+D(x_n,\ Tx_{n+1})^{\nu}]}{1+d(x_n,\ x_{n+1})^{\nu}}\Big)\\
\leq \theta\Big(&d(x_n,\ x_{n+1}),\ d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}),\\
&\frac{d(x_{n+1},\ x_{n+1})+d(x_n,\ x_{n+2})}{2},\frac{d(x_{n+1},\ x_{n+2})\ [1+d(x_n,\ x_{n+1})^{\mu}]}{1+d(x_n,\ x_{n+1})^{\mu}},\ \frac{d(x_{n+1},\ x_{n+1})\ [1+d(x_n,\ x_{n+2})^{\nu}]}{1+d(x_n,\ x_{n+1})^{\nu}}\Big)\\
\leq \theta\Big(&d(x_n,\ x_{n+1}),\ d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2}),\ \frac{d(x_n,\ x_{n+2})}{2},\ d(x_{n+1},\ x_{n+2}),\ 0\Big)
\end{aligned}$$

Since $\dfrac{d(x_n,\ x_{n+2})}{2} \leq \dfrac{d(x_n,\ x_{n+1})+d(x_{n+1},\ x_{n+2})}{2} \leq \max\ \{d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2})\}$, it follows from the property of $\theta$ that

$$\begin{aligned}
M(x_n,\ x_{n+1}) \leq \theta\Big(&\max\ \{d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2})\},\ \max\ \{d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2})\},\\
&\max\ \{d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2})\},\ \max\ \{d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2})\},\\
&\max\ \{d(x_n,x_{n+1}),\ d(x_{n+1},x_{n+2})\},\ \max\ \{d(x_n,x_{n+1}),\ d(x_{n+1},x_{n+2})\}\Big)\\
= \psi\Big(&\max\ \{d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2})\}\Big).
\end{aligned} \tag{3.3}$$

Also,

$$\begin{aligned}N(x_n,\, x_{n+1}) &= \varphi\Big(D(x_n,Tx_n),\, D(x_{n+1},Tx_{n+1}),\, D(x_n,Tx_{n+1}),\, D(x_{n+1},Tx_n)\Big)\\ &\le \varphi\Big(d(x_n,\, x_{n+1}),\, d(x_{n+1},\, x_{n+2}),\, d(x_n,\, x_{n+2}),\, d(x_{n+1},\, x_{n+1})\Big)\\ &= 0. \end{aligned} \tag{3.4}$$

Suppose that $d(x_n,\, x_{n+1}) < d(x_{n+1},\, x_{n+2})$. Then $d(x_{n+1},\, x_{n+2}) > 0$ and it follows from (3.2), (3.3), (3.4) and a property of $\theta$ that

$$d(x_{n+1},\, x_{n+2}) \le \psi(d(x_{n+1},\, x_{n+2})) < d(x_{n+1},\, x_{n+2}),$$

which is a contradiction. Hence $d(x_{n+1},\, x_{n+2}) \le d(x_n,\, x_{n+1})$. Then by (3.2), (3.3) and (3.4), we have

$$d(x_{n+1},\, x_{n+2}) \le \psi(d(x_n,\, x_{n+1})). \tag{3.5}$$

By repeated application of (3.5) and the monotone property of $\theta$, we have

$$d(x_{n+1},\, x_{n+2}) \le \psi(d(x_n,\, x_{n+1})) \le \psi^2(d(x_{n-1},\, x_n)) \le \ldots \le \psi^{n+1}(d(x_0,\, x_1)).$$

By a property of $\theta$, we have

$$\sum_n d(x_n,\, x_{n+1}) \le \sum_n \psi^n(d(x_0,\, x_1)) < \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence. From the completeness of $X$, there exists $z \in X$ such that

$$x_n \longrightarrow z \;\text{ as }\; n \longrightarrow \infty. \tag{3.6}$$

Now $\{x_{2n+1}\}$ is a subsequence of $\{x_n\}$ which, by (3.1) and (3.6), satisfies $\beta(x_{2n+1}) \ge 1$ for all $n$ and $x_{2n+1} \longrightarrow z$ as $n \longrightarrow \infty$. By $\beta$- regular property of $X$, we have $\beta(z) \ge 1$. Also by (3.1), $\alpha(x_{2n}) \ge 1$ for all $n \ge 0$. Applying the assumption (iv), we have

$$D(x_{2n+1},\, Tz) \le H(Tx_{2n},\, Tz) \le M(x_{2n},\, z) + N(x_{2n},\, z). \tag{3.7}$$

Now,

$$\begin{aligned}M(x_{2n},z) &= \theta\Big(d(x_{2n},z),\, D(x_{2n},Tx_{2n}),\, D(z,Tz),\, \frac{D(z,Tx_{2n})+D(x_{2n},Tz)}{2},\\ &\qquad \frac{D(z,\, Tz)\,[1+D(x_{2n},\, Tx_{2n})^{\mu}]}{1+d(x_{2n},\, z)^{\mu}},\, \frac{D(z,\, Tx_{2n})\,[1+D(x_{2n},\, Tz)^{\nu}]}{1+d(x_{2n},\, z)^{\nu}}\Big)\\ &\le \theta\Big(d(x_{2n},\, z),\, d(x_{2n},\, x_{2n+1}),\, D(z,\, Tz),\, \frac{d(z,\, x_{2n+1})+D(x_{2n},\, Tz)}{2},\\ &\qquad \frac{D(z,\, Tz)\,[1+d(x_{2n},\, x_{2n+1})^{\mu}]}{1+d(x_{2n},\, z)^{\mu}},\, \frac{d(z,\, x_{2n+1})\,[1+D(x_{2n},\, Tz)^{\nu}]}{1+d(x_{2n},\, z)^{\nu}}\Big).\end{aligned}$$

Taking limit supremum on both sides of the above inequality, using (3.6) and the continuity of $\theta$, we have

$$\begin{aligned}\overline{\lim}\, M(x_{2n},\, z) &\le \theta\Big(0,\, 0,\, D(z,\, Tz),\, \frac{D(z,\, Tz)}{2},\, D(z,\, Tz),\, 0\Big)\\ &\le \theta\Big(D(z,Tz),\, D(z,Tz),\, D(z,Tz),\, D(z,Tz),\, D(z,Tz),\, D(z,Tz)\Big)\\ &= \psi(D(z,\, Tz)).\end{aligned} \tag{3.8}$$

Also

$$\begin{aligned}N(x_{2n},\, z) &= \varphi\Big(D(x_{2n},\, Tx_{2n}),\, D(z,\, Tz),\, D(x_{2n},\, Tz),\, D(z,\, Tx_{2n})\Big)\\ &\le \varphi\Big(d(x_{2n},\, x_{2n+1}),\, D(z,\, Tz),\, D(x_{2n},\, Tz),\, d(z,\, x_{2n+1})\Big).\end{aligned}$$

Taking limit supremum and using the property of $\varphi$, we have

$$\overline{\lim}\, N(x_{2n},\ z) \le \varphi\Big(0,\ D(z,\ Tz),\ D(z,\ Tz),\ 0\Big) = 0. \tag{3.9}$$

Taking limit supremum on both sides of (3.7) and using (3.8) and (3.9), we have

$$D(z,\ Tz) \le \psi(D(z,\ Tz)).$$

Suppose that $D(z,\ Tz) \ne 0$. From the above inequality and using a property of $\theta$, we have

$$D(z,\ Tz) \le \psi(D(z,\ Tz)) < D(z,\ Tz),$$

which is a contradiction. Hence $D(z,\ Tz) = 0$. Since $Tz \in C(X)$, $Tz$ is compact and hence $Tz$ is closed, that is, $Tz = \overline{Tz}$, where $\overline{Tz}$ denotes the closure of $Tz$. Now, $D(z,\ Tz) = 0$ implies that $z \in \overline{Tz} = Tz$, that is, $z$ is a fixed point of $T$.

**Note.** The conclusion of the above theorem is still valid if in its assumptions the condition that the space $X$ is regular with respect to $\alpha$ and $\beta$ is replaced by the continuity of $T$. The proof remains the same except for minor modifications which is not separately shown here.

**Example 3.2.** Let $X = [0,\ \infty)$ and "$d$" be the usual metric on $X$. Then $(X, d)$ is a complete metric space. Let $T : X \longrightarrow C(X)$ be defined as $Tx = [0,\ \frac{x}{256}]$, for $x \in X$ and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$ be defined as

$$\alpha(x) = \begin{cases} e^x, & \text{if} \quad x \in [0,\ 1], \\ \frac{1}{10}, & \text{otherwise}, \end{cases} \qquad \beta(x) = \begin{cases} x+1, & \text{if} \quad x \in [0,\ 1], \\ \frac{1}{100}, & \text{otherwise}. \end{cases}$$

Let $\theta : [0,\ \infty)^6 \longrightarrow [0,\ \infty)$ and $\varphi : [0,\ \infty)^4 \longrightarrow [0,\ \infty)$ be defined respectively as follows:

$$\theta(t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6) = \frac{1}{4} \max\ \{t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6\}$$

and

$$\varphi(t_1,\ t_2,\ t_3,\ t_4) = \log\Big(1 + t_1\ t_2\ t_3\ t_4\Big).$$

Take $\mu,\ \nu \ge 0$ be any real numbers.

(i) Suppose that $\{x_n\}$ is a sequence in $X$ converging to $x \in X$ such that $\alpha(x_n) \ge 1$ and $\beta(x_n) \ge 1$ for all $n$. Then $\{x_n\}$ is a sequence in $[0,\ 1]$ and also $x \in [0,\ 1]$. Then it follows that $\alpha(x) \ge 1$ and $\beta(x) \ge 1$. Therefore, $X$ is regular with respect to $\alpha$ and $\beta$.

(ii) Suppose that $x \in X$ and $\alpha(x) \ge 1$. Then $x \in [0,\ 1]$ and $Tx = [0,\ \frac{x}{256}] \subseteq [0,\ 1]$. It follows that $\beta(u) \ge 1$ for all $u \in Tx$. Similarly, if $y \in X$ and $\beta(y) \ge 1$, it can be shown that $\alpha(v) \ge 1$ for all $v \in Ty$. Therefore, $T$ is a cyclic $(\alpha,\ \beta)$- admissible mapping.

(iii) $\alpha(x) \ge 1$ and $\beta(x) \ge 1$ for every $x \in [0,\ 1]$.

(iv) Here $\theta \in \Theta$ and $\varphi \in \Omega$. Let $x,\ y \in X$. Now, $\alpha(x)\ \beta(y) \ge 1$ (or $\alpha(y)\ \beta(x) \ge 1$) implies that $x,\ y \in [0,\ 1]$. So we require to check the validity of the inequality (2.1) for $x,\ y \in [0,\ 1]$. Now $H(Tx,\ Ty) = \dfrac{|\,x-y\,|}{256}$ and $M(x,\ y) \ge \dfrac{|\,x-y\,|}{4}$ for $x,\ y \in [0,\ 1]$. Then (2.1) is satisfied for all $x,\ y \in X$ with $\alpha(x)\ \beta(y) \ge 1$ or $\alpha(y)\ \beta(x) \ge 1$. Therefore, $T$ is a generalized almost contraction.

Hence all the conditions of Theorem 3.1 are satisfied and 0 is a fixed point of $T$.

In Theorem 3.1, considering $\psi(x_1,\ x_2,\ x_3,\ x_4, x_5,\ x_6) = k \max\ \{x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6\}$, where $k \in [0,\ 1)$ and $\varphi(t_1,\ t_2,\ t_3,\ t_4) = L \min\ \{t_1,\ t_2,\ t_3,\ t_4\}$, where $L \ge 0$ be any real number, we have the following corollary.

**Corollary 3.3.** *Let $(X,\ d)$ be a complete metric space, $T : X \longrightarrow C(X)$ be a multivalued mapping and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Suppose that (i) $X$ is regular with respect to $\alpha$ and $\beta$; (ii) $T$ is a cyclic $(\alpha,\ \beta)$- admissible mapping; (iii) there exists $x_0 \in X$ such that $\alpha(x_0) \ge 1$ or $\beta(x_0) \ge 1$ and (iv) there exist $\mu,\ \nu \ge 0$, $L \ge 0$ and $k \in [0,\ 1)$ such that for $x,\ y \in X$ with $\alpha(x)\ \beta(y) \ge 1$ or $\alpha(y)\ \beta(x) \ge 1$,*

$$H(Tx,\ Ty) \le k\, M(x,\ y) + L\, N(x,\ y),$$

*where*

$$M(x,\ y) = max\left\{d(x,\ y),\ D(x,\ Tx),\ D(y,\ Ty),\ \frac{1}{2}[D(y,\ Tx) + D(x,\ Ty)], \frac{D(y,\ Ty)\ [1 + D(x,\ Tx)^{\mu}]}{1 + d(x,\ y)^{\mu}},\ \frac{D(y,\ Tx)\ [1 + D(x,\ Ty)^{\nu}]}{1 + d(x,\ y)^{\nu}}\right\}$$

*and $N(x,\ y) = min\Big\{D(x,\ Tx),\ D(y,\ Ty),\ D(y,\ Tx),\ D(x,\ Ty)\Big\}$. Then $T$ has a fixed point.*

The following theorem is the special case of Theorem 3.1 when we treat $T : X \to X$ as a multivalued mapping in which case $Tx$ can be treated as a singleton set for every $x \in X$.

**Theorem 3.4.** *Let $(X, d)$ be a complete metric space, $T : X \to X$ and $\alpha, \beta : X \to [0, \infty)$. Suppose that (i) $X$ is regular with respect to $\alpha$ and $\beta$; (ii) $T$ is a cyclic $(\alpha, \beta)$- admissible mapping; (iii) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ or $\beta(x_0) \geq 1$; and (iv) there exist $\mu, \nu \geq 0$, $\theta \in \Theta$ and $\varphi \in \Omega$ such that $T$ is a generalized almost contraction. Then $T$ has a fixed point.*

**Proof.** We know that $\{x\}$ is compact in $X$ for every $x \in X$. We define a multivalued mapping $S : X \longrightarrow C(X)$ as $Sx = \{Tx\}$ for $x \in X$.

Let $x, y \in X$ such that $\alpha(x) \geq 1$ and $\beta(y) \geq 1$. Then by cyclic $(\alpha - \beta)$ - admissibility of $T$, we have

$$\beta(Tx) \geq 1, \text{ that is, } \beta(u) \geq 1 \text{ where } u \in Sx = \{Tx\} \quad \textit{and} \quad \alpha(Ty) \geq 1, \text{ that is, } \alpha(v) \geq 1 \text{ where } v \in Sy = \{Ty\}.$$

Therefore, for $x, y \in X$,

$$\alpha(x) \geq 1 \implies \beta(u) \geq 1 \text{ for all } u \in Sx \text{ and } \beta(y) \geq 1 \implies \alpha(v) \geq 1 \text{ for all } v \in Sy,$$

that is, $S$ is a cyclic $(\alpha - \beta)$- admissible mapping.

Let $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$ or $\alpha(y)\beta(x) \geq 1$. Then

$$\begin{aligned} H(Sx, Sy) &= d(Tx, Ty) \\ &\leq \theta\Big(d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx) + d(x, Ty)}{2}, \frac{d(y, Ty)\,[1 + d(x, Tx)^{\mu}]}{1 + d(x, y)^{\mu}}, \frac{d(y, Tx)\,[1 + d(x, Ty)^{\nu}]}{1 + d(x, y)^{\nu}}\Big) \\ &+ \varphi\Big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big) \\ &= \theta\Big(d(x, y), D(x, Sx), D(y, Sy), \frac{D(y, Sx) + D(x, Sy)}{2}, \frac{D(y, Sy)\,[1 + D(x, Sx)^{\mu}]}{1 + d(x, y)^{\mu}}, \frac{D(y, Sx)\,[1 + D(x, Sy)^{\nu}]}{1 + d(x, y)^{\nu}}\Big) \\ &+ \varphi\Big(D(x, Sx), D(y, Sy), D(x, Sy), D(y, Sx)\Big), \end{aligned}$$

that is, $S$ is a generalized almost contraction. So, all the conditions of Theorem 3.1 are satisfied and hence $S$ has a fixed point $z$ in $X$. Then $z \in Sz = \{Tz\}$, that is, $z = Tz$, that is, $z$ is a fixed point of $T$.

## 4. Stability of fixed point sets

In this section, we investigate the stability of fixed point sets of the setvalued contractions mentioned in Section 3.

**Theorem 4.1.** *Let $(X, d)$ be a complete metric space, $T_l : X \longrightarrow C(X)$, $l = 1, 2$ be two multivalued mappings and $\alpha, \beta : X \longrightarrow [0, \infty)$. Suppose the assumptions (i), (ii) (for each $T_l$), (iii) and (iv) (for each $T_l$), of Theorem 3.1 are satisfied. Then $F(T_l) \neq \emptyset$, for $l = 1, 2$. Also suppose that $\alpha(x) \geq 1$ or $\beta(x) \geq 1$ for any $x \in F(T_l)$, $(l = 1, 2)$. Then $H(F(T_1), F(T_2)) \leq \Phi(M)$, where $M = \sup_{x \in X} H(T_1x, T_2x)$ and $\Phi(M) = \sum_{n=1}^{\infty} \psi^n(M)$.*

**Proof.** By Theorem 3.1, the set of fixed points of $T_l$ $(l = 1, 2)$ are nonempty, that is, $F(T_l) \neq \emptyset$, for $l = 1, 2$. Let $y_0 \in F(T_1)$, that is, $y_0 \in T_1 y_0$. Without loss of generality we assume that $\alpha(y_0) \geq 1$ (the proof is similar if $\beta(y_0) \geq 1$). By Lemma 2.1, there exists $y_1 \in T_2 y_0$ such that

$$d(y_0, y_1) = D(y_0, T_2 y_0). \tag{4.1}$$

By the condition (ii) on $T_2$, $\beta(y_1) \geq 1$. Hence $\alpha(y_0)\beta(y_1) \geq 1$. By Lemma 2.1, there exists $y_2 \in T_2 y_1$ such that $d(y_1, y_2) = D(y_1, T_2 y_1)$. As $\beta(y_1) \geq 1$ and $y_2 \in T_2 y_1$, by the condition (ii) on $T_2$, we have $\alpha(y_2) \geq 1$. Hence $\alpha(y_2)\beta(y_1) \geq 1$. Again by Lemma 2.1, there exists $y_3 \in T_2 y_2$ such that $d(y_2, y_3) = D(y_2, T_2 y_2)$. Then arguing similarly as in the proof of Theorem 3.1, we construct a sequence $\{y_n\}$ such that for all $n \geq 0$,

$$y_{n+1} \in T_2 y_n; \quad \alpha(y_{2n}) \geq 1, \ \beta(y_{2n+1}) \geq 1; \quad d(y_{n+1}, y_{n+2}) \leq \psi(d(y_n, y_{n+1}))$$

and

$$d(y_{n+1}, y_{n+2}) \leq \psi(d(y_n, y_{n+1})) \leq \psi^2(d(y_{n-1}, y_n)) \leq \ldots \leq \psi^{n+1}(d(y_0, y_1)). \tag{4.2}$$

Arguing similarly as in the proof of Theorem 3.1, we prove $\{y_n\}$ is a Cauchy sequence in $X$ and there exists $u \in X$ such that

$$y_n \longrightarrow u \ \text{ as } \ n \longrightarrow \infty, \tag{4.3}$$

also $u$ is a fixed point of $T_2$, that is, $u \in T_2 u$. From (4.1) and the definition of $M$, we have

$$d(y_0,\ y_1) = D(y_0,\ T_2 y_0) \leq H(T_1 y_0,\ T_2 y_0) \leq M = \sup_{x \in X} H(T_1 x,\ T_2 x). \tag{4.4}$$

Using (4.2), we have

$$d(y_0,\ u) \leq \sum_{i=0}^{n} d(y_i,\ y_{i+1}) + d(y_{n+1},\ u) \leq \sum_{i=0}^{n} \psi^i(d(y_0,\ y_1)) + d(y_{n+1},\ u).$$

Taking limit as $n \longrightarrow \infty$ in the above inequality, using (4.3), (4.4) and the properties of $\theta$, we have

$$d(y_0,\ u) \leq \sum_{i=0}^{\infty} \psi^i(d(y_0,\ y_1)) \leq \sum_{i=0}^{\infty} \psi^i(M) = \Phi(M).$$

Thus given arbitrary $y_0 \in F(T_1)$, we have $u \in F(T_2)$ for which $d(y_0,\ u) \leq \Phi(M)$. Similarly, we can prove that for arbitrary $z_0 \in F(T_2)$, there exists $w \in F(T_1)$ such that $d(z_0,\ w) \leq \Phi(M)$. Hence we conclude that $H(F(T_1),\ F(T_2)) \leq \Phi(M)$.

**Lemma 4.2.** *Let $(X,\ d)$ be a complete metric space, $\{T_n : X \to C(X) : n \in \mathbb{N}\}$ be a sequence of multivalued mappings uniformly convergent to a multivalued mapping $T : X \to C(X)$ and $\alpha,\ \beta : X \to [0,\ \infty)$. Suppose that the assumptions (i), (ii) (for each $T_n$) and (iv) (for each $T_n$), of Theorem 3.1 are satisfied. Then $T$ satisfies the conditions (ii) and (iv) of Theorem 3.1.*

**Proof.** First, we prove that $T$ satisfies the condition (ii) of Theorem 3.1, that is, $T$ is cyclic $(\alpha,\ \beta)$- admissible. Let $\alpha(x) \geq 1$ ( or $\beta(x) \geq 1$), $x \in X$. Suppose $y \in Tx$ is arbitrary. Since $T_n \longrightarrow T$ uniformly, there exists a sequence $\{x_n\}$ in $\{T_n x\}$ such that $x_n \longrightarrow y$ as $n \longrightarrow \infty$. Since $\alpha(x) \geq 1$ ( or $\beta(x) \geq 1$) and each $T_n$ is cyclic $(\alpha,\ \beta)$ - admissible, it follows from Definition 2.4 that $\beta(x_n) \geq 1$, ( or $\alpha(x_n) \geq 1$ ) for every $n \in \mathbb{N}$. Then by regular property of the space with respect to $\beta$( or $\alpha$ ), it follows that $\beta(y) \geq 1$ ( or $\alpha(y) \geq 1$ ). Hence $T$ is cyclic $(\alpha,\ \beta)$-admissible, that is, $T$ satisfies the condition (ii) of Theorem 3.1.

Let $x,\ y \in X$ with $\alpha(x)\beta(y) \geq 1$ or $\alpha(y)\beta(x) \geq 1$. As for every $n \in \mathbb{N}$, $T_n$ satisfies the condition (iv) of Theorem 3.1, we have

$$\begin{aligned} H(T_n x,\ T_n y) \leq \theta\Big(& d(x,\ y),\ D(x,\ T_n x),\ D(y,\ T_n y),\ \frac{D(y,\ T_n x) + D(x,\ T_n y)}{2}, \\ & \frac{D(y,\ T_n y)\ [1 + D(x,\ T_n x)^{\mu}]}{1 + d(x,\ y)^{\mu}},\ \frac{D(y,\ T_n x)\ [1 + D(x,\ T_n y)^{\nu}]}{1 + d(x,\ y)^{\nu}}\Big) + \varphi\Big(D(x,\ T_n x),\ D(y,\ T_n y),\ D(x,\ T_n y),\ D(y,\ T_n x)\Big). \end{aligned}$$

Since the sequence $\{T_n\}$ is uniformly convergent to $T$ and $\theta$ and $\varphi$ are continuous, taking limit as $n \longrightarrow \infty$ in the above inequality, we get

$$\begin{aligned} H(Tx,\ Ty) \leq \theta\Big(& d(x,\ y),\ D(x,\ Tx),\ D(y,\ Ty),\ \frac{D(y,\ Tx) + D(x,\ Ty)}{2}, \\ & \frac{D(y,\ Ty)\ [1 + D(x,\ Tx)^{\mu}]}{1 + d(x,\ y)^{\mu}},\ \frac{D(y,\ Tx)\ [1 + D(x,\ Ty)^{\nu}]}{1 + d(x,\ y)^{\nu}}\Big) + \varphi\Big(D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\Big), \end{aligned}$$

which shows that $T$ satisfies the condition (iv) of Theorem 3.1.

Now we present our stability result.

**Theorem 4.3.** *Let $(X,\ d)$ be a complete metric space, $\{T_n : X \longrightarrow C(X) : n \in \mathbb{N}\}$ be a sequence of multivalued mappings uniformly convergent to a mapping $T : X \longrightarrow C(X)$ and $\alpha,\ \beta : X \longrightarrow [0,\ \infty)$. Suppose the assumptions (i), (ii) (for each $T_n$), (iii) and (iv) (for each $T_n$), of Theorem 3.1 are satisfied. Then $F(T_n) \neq \emptyset$ for all $n$ and $F(T) \neq \emptyset$. Let $\Phi(t) \longrightarrow 0$ as $t \longrightarrow 0$, where $\Phi(t) = \sum_{n=1}^{\infty} \psi^n(t)$. If $\beta(x) \geq 1$ or $\alpha(x) \geq 1$ for any $x$ belonging to $F(T_n)$, $[n = 1, 2, 3, ..]$ or $F(T)$. Then $\lim_{n \longrightarrow \infty} H(F(T_n),\ F(T)) = 0$, that is, the fixed point sets of $T_n$ are stable.*

**Proof.** By Lemma 4.2 and Theorem 3.1, we have $F(T_n) \neq \emptyset$ for all $n$ and $F(T) \neq \emptyset$. Let $M_n = \sup_{x \in X} H(T_n x,\ Tx)$. Since the sequence $\{T_n\}$ is uniformly convergent to $T$ on $X$,

$$\lim_{n \longrightarrow \infty} M_n = \lim_{n \longrightarrow \infty} \sup_{x \in X} H(T_n x,\ Tx) = 0. \tag{4.5}$$

By Theorem 4.1, we get

$$H(F(T_n),\ F(T)) \leq \Phi(M_n), \ \text{ for every } n \in \mathbb{N}.$$

Since $\Phi$ is continuous and $\Phi(t) \longrightarrow 0$ as $t \longrightarrow 0$, using (4.5), we have

$$\lim_{n\longrightarrow\infty} H(F(T_n),\ F(T)) \leq \lim_{n\longrightarrow\infty} \Phi(M_n) = 0,$$

that is, $\lim_{n\longrightarrow\infty} H(F(T_n),\ F(T)) = 0$, that is, the fixed point sets of $T_n$ are stable.

**Example 4.4.** We take the metric space $(X,\ d)$ and the mappings $\alpha$, $\beta$, $\theta$ and $\varphi$ as taken in Example 3.2. Let $T : X \longrightarrow C(X)$ be defined as $Tx = [0,\ \frac{x}{256}]$, for $x \in X$ and $T_n : X \longrightarrow C(X)$ be defined as $T_n x = [0,\ \frac{x}{256} + \frac{1}{1024\ n}]$, for $x \in X$. Here the sequence $\{T_n\}$ uniformly converges to $T$. Let $\mu,\ \nu \geq 0$ be any real numbers. Now for every $n$, $T_n x = [0,\ \frac{x}{256} + \frac{1}{1024\ n}] \subseteq [0,\ 1]$ for every $x \in [0,\ 1]$ and $H(T_n x,\ T_n y) = \dfrac{|\,x-y\,|}{256}$ for $x,\ y \in [0,\ 1]$. Then as explained in Example 3.2, we can show that the assumptions (i), (ii) (for each $T_n$), (iii) and (iv) (for each $T_n$), of Theorem 3.1 are satisfied. Here $F(T_n) = [0,\ \frac{1}{1020n}]$, for each $n$ and $F(T) = \{0\}$. Here $\Phi(t) \to 0$ as $t \to 0$, where $\Phi(t) = \sum_{n=1}^{\infty} \psi^n(t)$, and also $\beta(x) \geq 1$ and $\alpha(x) \geq 1$ for any $x$ belonging to $F(T_n)$, $[n = 1, 2, 3, ..]$ or $F(T)$. So we see all the conditions of Theorem 4.3 are satisfied. Here $\lim_{n\longrightarrow\infty} H(F(T_n),\ F(T)) = 0$, that is, the fixed point sets of $T_n$ are stable.

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