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On Semi-Invariant Submanifolds of Trans-Sasakian Finsler Manifolds

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Abstract

We define trans-Sasakian Finsler manifold $\bar{F}^{2n+1} = (\bar{\mathcal{N}}, \bar{\mathcal{N}}', \bar{F})$ and semi-invariant submanifold $F^m = (\mathcal{N}, \mathcal{N}', F)$ of a trans-Sasakian Finsler manifold $\bar{F}^{2n+1}$. Then we study mixed totally geodesic and totally umbilical semi-invariant submanifolds of trans Sasakian Finsler manifold.

1. Introduction

Oubina [1] introduced trans-Sasakian manifolds that reduced to α-Sasakian and $\beta-$Kenmotsu manifolds, in 1985. Then, trans-Sasakian manifolds are studied by many geometers like in [2]. Besides, Kobayashi studied semi-invariant submanifolds for a certain class of almost contact manifolds in [3] in 1986. Afterwards, semi invariant submanifolds of several structures are discussed like nearly trans-Sasakian and nearly Kenmotsu manifolds in [4], in 2004 and in [5], in 2009. Also, Shahid got some fundamental results on almost semi-invariant submanifolds of trans-Sasakian manifolds in [6], in 1993. Besides, Shahid et al. discussed submersion and cohomology class of semi-invariant submanifolds of trans-Sasakian manifolds in [7], in 2013.
B.B. Sinha and R. K. Yadav introduced almost Sasakian Finsler manifold and determined the set of all almost Sasakian Finsler h-connection on almost Sasakian Finsler manifold [8], In 1991. Then Yaliniz and Caliskan studied Sasakian Finsler manifolds in [9] in 2013. In this paper, we discussed mixed totally geodesic and totally umbilical semi- invariant submanifolds of trans-Sasakian Finsler manifolds.

2. Trans-Sasakian Finsler manifolds

Definition 2.1. *Suppose that $\bar{\mathcal{N}}$ be an $(2n+1)$-dimensional Finsler manifold. Then an almost contact metric structure $(\phi^V, \eta^V, \xi^V, G^V)$ on $(\bar{\mathcal{N}}')^v$ is called trans-Sasakian Finsler if the following relation is satisfied:*

$$2(\bar{\nabla}^V_X \phi)Y^V = \alpha \left\{ G^V(X^V, Y^V)\xi^V - \eta^V \left(Y^V\right) X^V \right\} + \beta \left\{ G^V(\phi X^V, Y^V)\xi^V - \eta^V \left(Y^V\right) \phi X^V \right\}$$

where α and β are functions on $(\bar{\mathcal{N}}')^v$, $\bar{\nabla}$ is the Finsler connection with respect to G^V. So, $(\bar{\mathcal{N}}')^v$ is called trans-Sasakian Finsler manifold.

2.1. Semi-invariant submanifolds of trans-Sasakian Finsler manifolds

Definition 2.2. *An m-dimensional Finsler submanifold $(\mathcal{N}')^v$ of a trans-Sasakian Finsler manifold $(\bar{\mathcal{N}}')^v$ is called a semi-invariant submanifold if $\xi^V \in V_{(u,v)}\mathcal{N}'$ and there exist on $(\bar{\mathcal{N}}')^v$ a pair of orthogonal distribution $(D, D^\perp)$ such that*
(i) $V\mathcal{N}' = D \oplus D^\perp \oplus \{\xi^V\}$
(ii) $\phi D_{(u,v)} = D_{(u,v)}$, $\forall (u,v) \in (\mathcal{N}')^v, \forall u \in \mathcal{N}$

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(iii) $\phi\left(D^{\perp}_{(u,v)}\right) \subset \left(V_{(u,v)}\mathcal{N}'\right)^{\perp}$ *for all* $(u,v) \in (\mathcal{N}')^{v}$*, for tangential space* $V_{(u,v)}\mathcal{N}'$ *and normal space* $\left(V_{(u,v)}\mathcal{N}'\right)^{\perp}$ *of* $(\mathcal{N}')^{v}$ *at V with the following decomposition :*

$$V_{(x,y)}\bar{\mathcal{N}}' = (V_{(u,v)}\mathcal{N}') \oplus (V_{(u,v)}\mathcal{N}')^{\perp}$$

The distribution D (resp. $D^{\perp}$*) is called the horizontal (resp. vertical) distribution. A semi-invariant Finsler submanifold* $(\mathcal{N}')^{v}$ *is said to be an invariant (resp. anti-invariant) submanifold if we have* $D^{\perp}_{(u,v)} = \{0\}$ *(resp.* $D_{(u,v)} = \{0\}$*) for each* $(u,v) \in (\mathcal{N}')^{v}$*. We also call* $(\mathcal{N}')^{v}$ *proper if neither D nor* $D^{\perp}$ *is null. It is easy to check that each hypersurface of* $(\mathcal{N}')^{v}$ *which is tangent to* ξ^{V} *inherits a structure of semi-invariant Finsler submanifold of* $(\mathcal{N}')^{v}$*.*

We denote by G the metric tensor field of $(\bar{\mathcal{N}}')^{v}$ as well as that ınduced on $(\mathcal{N}')^{v}$. Let $\bar{\nabla}$ be a Finsler connection on $\bar{F}^{2n+1} = (\bar{\mathcal{N}}, \bar{\mathcal{N}}', \bar{F})$. Thus ∇ is a Finsler connection on $F^{m} = (\mathcal{N}, \mathcal{N}', F)$ which we call the induced Finsler connection. Also B is an $\Im(\mathcal{N}')$-bilinear mapping on $\Gamma(V\mathcal{N}') \times \Gamma(V\mathcal{N}')$ and $\Gamma(V\mathcal{N}'^{\perp})$-valued, which we call the second fundamental form of F^{m}.
Using B define the $\Im(\mathcal{N}')$-bilinear mapping:

$$h^{V} : \Gamma(V\mathcal{N}') \times \Gamma(V\mathcal{N}') \to \Gamma(V\mathcal{N}')^{\perp}$$

$$h(X^{V}, Y^{V}) = B(X^{V}, Y^{V})$$

for any $X, Y \in \Gamma(T\mathcal{N}')$. We call h^{V} the v-second fundamental form of $F^{m} = (\mathcal{N}, \mathcal{N}', F)$. From Gauss formula we get;

$$\bar{\nabla}^{V}_{X} Y^{V} = \nabla^{V}_{X} Y^{V} + h^{V}(X^{V}, Y^{V}) \tag{2.1}$$

for any $X, Y \in \Gamma(T\mathcal{N}').(X^{V}, Y^{V} \in \Gamma(V\mathcal{N}'))$.
Now, for any $X \in \Gamma(T\mathcal{N}')$ and $N \in \Gamma(V\mathcal{N}'^{\perp})$, we set

$$\bar{\nabla}_{X} N = -A_{N} X + \nabla^{\perp}_{X} N \tag{2.2}$$

where $A_{N}X \in \Gamma(V\mathcal{N}')$ and $\nabla^{\perp}_{X} N \in \Gamma(V\mathcal{N}') \perp$.
It follows that $\nabla^{\perp}$ is a linear connection on the Finsler normal bundle $(V\mathcal{N}')^{\perp}$ of F^{m}. Therefore $\nabla^{\perp}$ is a vectorial Finsler connection on $V\mathcal{N}'^{\perp}$. We call the normal Finsler connection with respect to $\bar{\nabla}$.

$$A^{V} : \Gamma(V\mathcal{N}'\perp) \times \Gamma(V\mathcal{N}') \to \Gamma(V\mathcal{N}')$$

$$A^{V}(N^{V}, X^{V}) = A_{N^{V}} X^{V}$$

is an $\Im(\mathcal{N}')-$ bilinear mapping for any $N^{V} \in \Gamma(V\mathcal{N}'^{\perp})$. We call A_{N} the shape operator (the Weingarten operator) with respect to N^{V}.
As in the case of the second fundamental form, by means of A we define for any $N^{V} \in \Gamma(V\mathcal{N}') \perp$ the $\Im(\mathcal{N}')-$ linear mappings;

$$A^{V}_{N} : \Gamma(V\mathcal{N}') \to \Gamma(V\mathcal{N}')$$

$$A^{V}_{N} X^{V} = A_{N^{V}} X^{V}$$

and call the v-shape operator. Thus from the Weingarten formula we deduce that

$$\bar{\nabla}_{X^{v}} N^{V} = -A^{V}_{N} X^{V} + \nabla^{\perp}_{X^{v}} N^{V}$$

for any $X \in \Gamma(T\mathcal{N}')$ $X^{V} \in \Gamma(V\mathcal{N}')$ and $N^{V} \in \Gamma(V\mathcal{N}'^{\perp})$.
Moreover we have

$$G(h^{V}(X^{V}, Y^{V}), N^{V}) = G(A^{V}_{N} X^{V}, Y^{V}) \tag{2.3}$$

for a vector field $X^{V} \in V\mathcal{N}'$. We put

$$X^{V} = PX^{V} + QX^{V} + \eta^{V}\left(X^{V}\right)\xi^{V} \tag{2.4}$$

where PX^{V} and QX^{V} belong to the distribution D and $D^{\perp}$ respectively.
For any vector field $N^{V} \in \Gamma(V\mathcal{N}'\perp)$, we put

$$\phi N^{V} = fN^{V} + qN^{V}$$

where fN^{V}(resp. qN^{V}) denotes the tangential (resp. normal) component of ϕN^{V}.

3. Mixed totally geodesic semi-invariant submanifolds of trans-Sasakian Finsler manifolds

Definition 3.1. *A semi-invariant Finsler submanifold is said to be mixed totally geodesic if* $h(X^V, Z^V) = 0$ *for all* $X^V \in D$ *and* $Z^V \in D^\perp$.

Theorem 3.2. *Let* $(\mathscr{N}')v$ *be a semi-invariant submanifold of trans-Sasakian Finsler manifold* $(\mathscr{N}')v$. *Then*

$$P\nabla_{X^V}(fN^V) - PA^V_{qN^V}X^V + \phi PA^V_{N^V}X^V = 0 \tag{3.1}$$

$$Q\nabla_{X^V}(fN^V) - QA^V_{qN^V}X^V - f\nabla^\perp_{X^V}N^V = 0 \tag{3.2}$$

$$h^V(X^V, fN^V) + \nabla^\perp_{X^V}(qN^V) + \phi QA^V_{N^V}X^V - q\nabla^\perp_{X^V}N^V = 0 \tag{3.3}$$

$\forall X^V \in D$ *and* $\forall N^V \in (V_{(u,v)}\mathscr{N}'^\perp)$.

Proof.

$$\bar{\nabla}_{X^V}(\phi N^V) = \bar{\nabla}_{X^V}(fN^V + qN^V) = \bar{\nabla}_{X^V}(fN^V) + \bar{\nabla}_{X^V}(qN^V) \tag{3.4}$$

All $N^V \in (V_{(u,v)}\mathscr{N}'^\perp)$, $\forall X^V \in D$.
For $fN \in V_{(u,v)}\mathscr{N}'$, we have from (2.1)

$$\bar{\nabla}_{X^V}(fN^V) = \nabla_{X^V}(fN^V) + h^V(X^V, fN^V) \tag{3.5}$$

for $qN \in (V_{(u,v)}\mathscr{N}'^\perp)$; we have from (2.2)

$$\bar{\nabla}_{X^V}(qN^V) = -A^V_{qN^V}X^V + \nabla^\perp_{X^V}(qN^V) \tag{3.6}$$

By using (3.5) and (3.6) in (3.4), we get

$$\bar{\nabla}_{X^V}(\phi N^V) = \nabla_{X^V}(fN^V) + h^V(X^V, fN^V) - A^V_{qN^V}X^V + \nabla^\perp_{X^V}(qN^V) \tag{3.7}$$

where $\nabla_{X^V}(fN^V) \in \left(V_{(u,v)}\mathscr{N}'\right)$ and $A^V_{qN^V}X^V \in \left(V_{(u,v)}\mathscr{N}'\right)$, We have from (2.3)

$$\nabla_{X^V}(fN^V) = P\nabla_{X^V}(fN^V) + Q\nabla_{X^V}(fN^V) + \eta^V(\nabla_{X^V}(fN^V))\xi^V \tag{3.8}$$

and

$$A^V_{qN^V}X^V = PA^V_{qN^V}X^V + QA^V_{qN^V}X^V + \eta^V(A^V_{qN^V}X^V)\xi^V \tag{3.9}$$

by using (3.8) and (3.9) in (3.7) we obtain

$$\begin{aligned} \bar{\nabla}_{X^V}(\phi N^V) &= (\bar{\nabla}_{X^V}\phi)N^V + \phi(\bar{\nabla}_{X^V}N^V) \\ &= P\nabla_{X^V}(fN^V) + Q\nabla_{X^V}(fN^V) + \eta^V(\nabla_{X^V}(fN^V))\xi^V \\ &\quad + h^V(X^V, fN^V) - PA^V_{qN^V}X^V - QA^V_{qN^V}X^V \\ &\quad - \eta(A^V_{qN^V}X^V)\xi^V + \nabla^\perp_{X^V}(qN^V) \end{aligned}$$

where

$$\begin{aligned} (\bar{\nabla}_{X^V}\phi)N^V &= \frac{\alpha}{2}\left\{G(X^V, N^V)\xi^V - \eta^V(N^V)X^V\right\} \\ &\quad + \frac{\beta}{2}\left\{G(\phi X^V, N^V)\xi^V - \eta^V(N^V)\phi X^V\right\} \end{aligned}$$

Since $G(X^V, N^V) = 0 = G(N^V, \xi^V) = G(\phi X^V, N^V)$, we get $\left(\bar{\nabla}_X\phi\right)N = 0$. Thus, we using (2.3) and (2.4) from (3.10) then we obtain

$$\begin{aligned} \bar{\nabla}_{X^V}(\phi N^V) &= \phi(\bar{\nabla}_{X^V}N^V) = \phi(-A^V_{N^V}X^V + \bar{\nabla}^{\perp N^V}_{X^V}) \\ &= -\phi A^V_{N^V}X^V + \phi\nabla^\perp_{X^V}N^V \\ &= -\phi PA^V_{N^V}X^V - \phi Q\phi A^V_{N^V}X^V + f\nabla^\perp_{X^V}N^V + q\nabla^\perp_{X^V}N^V \end{aligned} \tag{3.10}$$

where $A_N \in V_{(u,v)}\mathscr{N}' and \nabla^\perp_{X^V}N^V \in (V_{(u,v)}\mathscr{N}'^\perp)$. By seperating the components of $D\ D^\perp$ and $(V_{(u,v)}\mathscr{N}'^\perp)$ from (3.10) and (3.10) we get (3.1),(3.2) and (3.3). □

Theorem 3.3. *Let* $(\mathscr{N}')^v$ *be a semi-invariant submanifold of trans-Sasakian Finsler manifold* $(\bar{\mathscr{N}}')^v$. *Then the following propositions are equivalent:*
(a) $(\mathscr{N}')^v$ *is a totally geodesic.*
(b) $\nabla^\perp_{X^V}N^V \in \phi\ D^\perp$ *and D is invariant with respect to* A^V_N *(all* $N^V \in \phi\ D^\perp$*), that is* $\nabla^\perp_D(\phi\ D^\perp) \subset \phi\ D^\perp$ *and* $A^V_{\phi D^\perp}D \subset D$.

Proof. From (2.4) we know that,

$$\phi N^V = fN^V = Y^V, \text{ all } Y^V \in D^\perp, N \in \phi D^\perp \subset (V_{(u,v)}\mathcal{N}'^\perp)$$

by using (3.2) from (3.3) we have

$$h^V(X^V,Y^V) + \nabla^\perp_{X^V}(qN^V) - \phi QA^V_{N^V}X^V - q\nabla^\perp_{X^V}N^V = 0 \tag{3.11}$$

where, since $Y^V \in D^\perp, N \in \phi D^\perp$, we can write $qN^V = 0$. Thus from (3.11) we have

$$h^V(X^V,Y^V) = q\nabla^\perp_{X^V}N^V - \phi QA^V_{N^V}X^V \tag{3.12}$$

Now, suppose that $(\mathcal{N}')^v$ a total geodesic. Because of $h^V(X^V,Y^V) = 0$, $\forall X^V \in D$ and $Y^V \in D^\perp$, from (3.12) we get

$$0 = q\nabla^\perp_{X^V}N^V - \phi QA^V_{N^V}X^V$$

where $A^V_{N^V}X^V \in (V_{(u,v)}\mathcal{N}'^\perp)$, $QA^V_{N^V}X^V \in D^\perp$ and $\phi QA^V_{N^V}X^V \in \phi D^\perp \subset (V_{(u,v)}\mathcal{N}'^\perp)$. If $q\nabla^\perp_{X^V}N^V \in \phi D^\perp$, it must be $\phi N^V = fN^V = Y^V \in D^\perp$, $\forall N^V \in \phi D^\perp$. Thus we have $\phi(q\nabla^\perp_{X^V}N^V) \in D^\perp$. Also from (2.4) we can write

$$\phi\nabla^\perp_{X^V}N^V = f\nabla^\perp_{X^V}N^V + q\nabla^\perp_{X^V}N^V, \nabla^\perp_X N \in (V_{(u,v)}\mathcal{N}'^\perp)$$

if we apply ϕ on both sides of the equation we get

$$-\nabla^\perp_{X^V}N^V = \phi(f\nabla^\perp_{X^V}N^V) + \phi(q\nabla^\perp_{X^V}N^V) \tag{3.13}$$

where if $f\nabla^\perp_{X^V}N^V \in D^\perp \subset V_{(u,v)}\mathcal{N}'$, then it means $\phi f\nabla^\perp_{X^V}N^V \in \phi D^\perp \subset (V_{(u,v)}\mathcal{N}'^\perp)$. In equation (3.13), since $\nabla^\perp_{X^V}N^V \in (V_{(u,v)}\mathcal{N}'^\perp)$ and $\phi\left(f\nabla^\perp_{X^V}N^V\right) \in \phi D^\perp$, it means that $\phi\left(q\nabla^\perp_{X^V}N^V\right) \notin D^\perp.(\phi(q\nabla^\perp_X N) \in (V_{(u,v)}\mathcal{N}'^\perp))$. If $f\nabla^\perp_{X^V}N^V \notin D^\perp$, then $\phi(f\nabla^\perp_X N) \in V\mathcal{N}'_v$, while $\nabla^\perp_X N \in (V_{(u,v)v}\mathcal{N}'^\perp)$ and $\phi(f\nabla^\perp_X N) \in V_{(u,v)}\mathcal{Y}'$ either $\phi(q\nabla^\perp_X N) \in \left(V_{(u,v)}\mathcal{N}'^\perp\right)$ or $\phi(q\nabla^\perp_X N) \in V_{(u,v)}\mathcal{N}'$. If $\phi(q\nabla^\perp_X N) \in V_{(u,v)}\mathcal{N}'$, we get the following contradiction

$$\phi(q\nabla^\perp_X N) = \phi(f\nabla^\perp_X N) \tag{3.14}$$

$$q\nabla^\perp_X N = f\nabla^\perp_X N$$

In that case $\phi(q\nabla^\perp_{X^V}N^V) \notin V_{(u,v)}\mathcal{N}'\ (\notin D^\perp)$. Thus we get $q\nabla^\perp_X N \in \left(V_{(u,v)}\mathcal{N}'^\perp - \phi D^\perp\right)$ in (3.14). Since $q\nabla^\perp_{X^V}N^V \in \left\{(VM'^\perp_v) - \phi D^\perp\right\}$ and $\phi QA^V_{N^V}X^V \in \phi D^\perp$, it must be $q\nabla^\perp_{X^V}N^V = 0$ and $\phi QA^V_{N^V}X^V = 0$. Since $q\nabla^\perp_{X^V}N^V = 0$, it means that $\nabla^\perp_X N \in \phi D^\perp$ and since $QA^V_{N^V}X^V = 0$, then $A^V_{N^V}X^V \in D$. Thus we get $\nabla^\perp_D \phi D^\perp$ and $A_{\phi D^\perp}D \subset D$. □

Theorem 3.4. *Let $(\mathcal{N}')^v$ be a semi-invariant submanifolds of trans-Sasakian Finsler manifold $(\bar{\mathcal{N}}')^v$. If $\beta \neq 0$, then each $M'^\perp_v$ leaf of $D^\perp$ is not totally geodesic at $(\mathcal{N}')^v$.*

Proof. Suppose that $((\mathcal{N}')^v)^\perp$ is totally geodesic in $(\mathcal{N}')^v$. Then $\nabla_{X^V}Y^V \in D^\perp$, for each $X^V, Y^V \in D^\perp$ or equivalent to $G(\nabla_{X^V}Y^V, Z^V) = 0$, for each $Z^V \in D \oplus \left\{\xi^V\right\}$. Using the

$$\nabla^V_Y \xi^V = \frac{\beta}{2}Y^V \text{ and } h^V(Y^V,\xi^V) = -\frac{\alpha}{2}\phi Y^V$$

we get

$$G(\nabla_{X^V}Y^V,\xi^V) = -G(Y^V,\nabla_{X^V}\xi^V) = -G(Y^V,\frac{\beta}{2}X^V) = -\frac{\beta}{2}G(Y^V,X^V)$$

Thus, we find the following contradiction

$$0 = G(\nabla_{X^V}X^V,\xi^V) = -\frac{\beta}{2}G(X^V,X^V)$$

That is, $((\mathcal{N}')^v)^\perp$ is not total geodesic at $(\mathcal{N}')^v$. □

4. Totally umbilical semi-invariant submanifolds of trans- Sasakian Finsler manifolds

Definition 4.1. *$\forall X^V, Y^V \in V\mathcal{N}'$ and $N^V \in V\mathcal{N}'^{\perp}$*
(1) If $A^V_{N^V} = aI$ (for $a \in \Im(\mathcal{N}')^v$), N^V is called umbilical section of $(\mathcal{N}')^v$. (2) If N^V is umbilical section of $(\mathcal{N}')^v$ then $(\mathcal{N}')^v$ is umbilical with respect to N^V. (3) If $(\mathcal{N}')^v$ is umbilical for each $N^V \in V_{(u,v)}\mathcal{N}'^{\perp}$ then $(\mathcal{N}')^v$ is called totally umbilical submanifold of $(\mathcal{N}')^v$. (4) Suppose that $\{E^V_1, ..., E^V_m\}$ orthonormal base of $V_{(u,v)}\mathcal{N}'$. Then

$$H = \frac{1}{m} iz(h_{(u,v)}) = \frac{1}{m}\sum_{i=1}^{m} h^V(E^V_i, E^V_i)$$

is called mean curvature vector of $(\mathcal{N}')^v$ at $u \in (\mathcal{N}')^v$.

If $\{E^V_{m+1}, ..., E^V_{2n+1}\}$ is orthonormal base of $V_{(u,v)}\mathcal{N}'^{\perp}$, then we can write

$$H = \frac{1}{m}\sum_{a=m+1}^{2n+1} iz(A^V_a)E^V_a,\ A^V_a = A^V_{E^V_a} \tag{4.1}$$

Let $(\mathcal{N}')^v$ be a semi-invariant submanifold of trans-Sasakian Finsler manifold $(\bar{\mathcal{N}}')^v$. Since

$$h^V(X^V, Y^V) = \sum_{a=m+1}^{2n+1} G(h^V(X^V, Y^V), E^V_a)E^V_a$$

and

$$G(h^V(X^V, Y^V), E^V_a) = G(A^V_{E^V_a} X^V, Y^V)$$

we have

$$h^V(X^V, Y^V) = \sum_{a=m+1}^{2n+1} G(A^V_{E^V_a} X^V, Y^V)E^V_a$$

Since $(\mathcal{N}')^v$ is totally umbilical submanifold of $(\bar{\mathcal{N}}')^v$, we have

$$A^V_{E^V_a} X^V = C_a X^V,\ C_a \in \Im(\mathcal{N}')^v \tag{4.2}$$

Thus we get

$$\begin{aligned} h^V(X^V, Y^V) &= \sum_{a=m+1}^{2n+1} G(C_a X^V, Y^V)E^V_a \\ &= \sum_{a=m+1}^{2n+1} C_a G(X^V, Y^V)E^V_a \\ &= G(X^V, Y^V)\left(\sum_{a=m+1}^{2n+1} C_a E^V_a\right) \end{aligned} \tag{4.3}$$

by using (4.1) and (4.2), we get

$$\begin{aligned} H &= \frac{1}{m}\sum_{a=m+1}^{2n+1} iz(A^V_{E^V_a})E^V_a = \frac{1}{m}\sum_{a=m+1}^{2n+1} iz(C_a I)E^V_a \\ &= \frac{1}{m}\sum_{a=m+1}^{2n+1} (mC_a)E^V_a = \sum_{a=m+1}^{2n+1} C_a E^V_a \end{aligned} \tag{4.4}$$

from (4.3) and (4.4) we obtain

$$h^V(X^V, Y^V) = G(X^V, Y^V)H \tag{4.5}$$

Theorem 4.2. *Let $(\mathcal{N}')^v$ be a semi-invariant submanifolds of trans-Sasakian Finsler manifold $(\bar{\mathcal{N}}')^v$. Then*
(a) $(\mathcal{N}')^v$ is a totally geodesic.
(b) If $\alpha \neq 0$ for every point of $(\mathcal{N}')^v$, then $(\mathcal{N}')^v$ is an invariant submanifold, that is $D^{\perp} = 0$.

Proof. For $X^V = \xi^V$, from (3.1) we get $\bar{\nabla}_{\xi^V}\xi^V = 0$. Later, we take ξ^V instead of X^V and Y^V from (2.1), we obtain

$$\bar{\nabla}_{\xi^V}\xi^V = \nabla_{\xi^V}\xi^V + h^V(\xi^V, \xi^V)$$

since $\bar{\nabla}_{\xi^V}\xi^V = 0$, we have

$$0 = \nabla_{\xi^V}\xi^V + h^V(\xi^V, \xi^V)$$

that is $\nabla_{\xi^V}\xi^V = 0$ and $h^V(\xi^V, \xi^V) = 0$. Since $(\mathscr{N}')^v$ is totally umbilical submanifold, we have from (4.5)

$$0 = h^V(\xi^V, \xi^V) = G(\xi^V, \xi^V)H$$

since $G(\xi^V, \xi^V) \neq 0$, it must be $H = 0$. Thus we have

$$h^V(X^V, Y^V) = G^V(X^V, Y^V)0 = 0$$

This means that $(\mathscr{N}')^v$ is totally geodesic. We know that $\nabla^V_Y \xi^V = \frac{\beta}{2}Y^V$ and $h^V(Y^V, \xi^V) = -\frac{\alpha}{2}\phi Y^V$ for all $Y^V \in D^\perp$. Since $(\mathscr{N}')^v$ is totally geodesic and totally umbilical, we get

$$-\frac{\alpha}{2}\phi Y^V = G^V(Y^V, \xi^V)0 = 0$$

Since $\alpha \neq 0$, this means that

$$\phi Y^V = 0 \to Y^V = 0 \to D^\perp = 0$$

□

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