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The Number of Snakes in a Box

Christian Barrientos[a*] **and Sarah Minion**[a]

[a]*Valencia College Department of Mathematics 701 N Econlockhatchee Trail, Orlando, FL 32825, USA*
[*]*Corresponding author*

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Abstract

Within the class of polyominoes we work on the enumeration of two subfamilies of the family of snake polyominoes: stairs and snakes of height 2. We consider them from a graph theoretical perspective. In the process of enumeration of these graphs, we use classical ideas, as symmetries, and a new approach that connects these snakes with the partitions of integers.

1. Introduction

A *polyomino* is a planar shape made by connecting a certain number of equal-sized squares, each joined together with at least one other square along an edge. A *snake* of length $n > 1$, is a packing of n congruent geometrical objects, called *cells*, where the first and last cell have only one neighbor and all the other cells have exactly two neighbors. A *snake polyomino* is a snake where all the cells are squares. In Figure 1.1 we show all the polyominoes with six cells, the snakes have been highlighted.

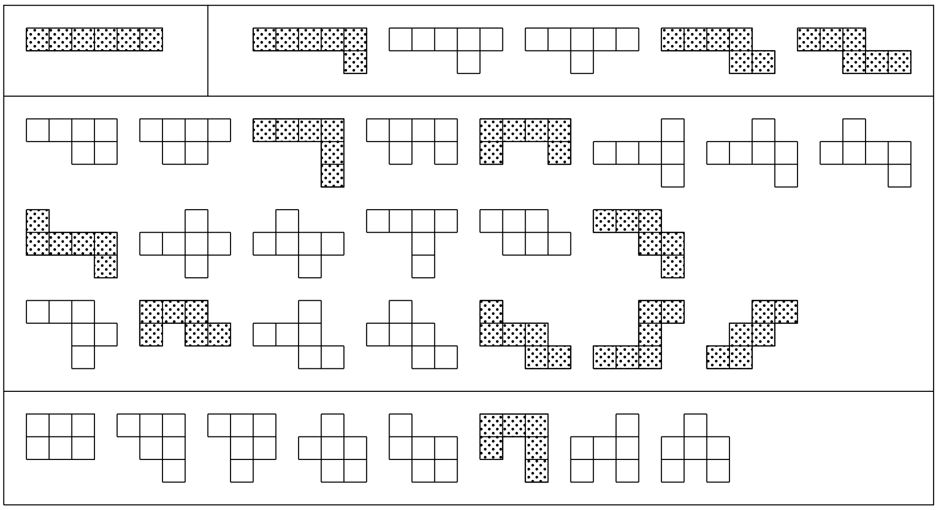

Figure 1.1: All polyominoes with six cells.

In this work, polyominoes are considered graphs, where every cell is a copy of the cycle C_4. Moreover, these graphs are embedded in the integral grid. This last restriction has implications on their number. Thus, snake polyominoes, or simply snakes, form a polyomino class, which can be described by the avoidance of the polyominoes shown in Figure 1.2. This definition is slightly different of the one given in [1]. There, Battaglino et al., only consider, as forbidden substructures, the first two shapes. We included here the third one to be consistent with the definition of snake, where the extreme cells only have one neighbor.

Email addresses and ORCID numbers**: chr_barrientos@yahoo.com, 0000-0003-2838-8687 (C. Barrientos), sarah.m.minion@gmail.com, 0000-0002-8523-3369 (S. Minion)**

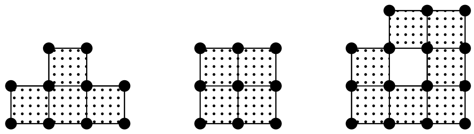

Figure 1.2: Forbidden structures in snake polyominoes.

Golomb [2] introduced the concept of polyomino in 1953, since then, there has been a number of papers centered in the enumeration of subfamilies of them. Several algorithms, that count the number of members of these subfamilies, have been created; however, the general case remains unsolved, that is, for a given value of n, it is unknown the number of polyominoes with n cells.

In Section 2 we present some general results that are used in our counting process in the coming sections. We study two subfamilies of snakes: stairs (Section 3) and snakes of height 2 (Section 4). In both cases, we consider the snakes inscribed into a box, i.e., the boundary of $P_a \times P_b$; partitions of integers are used in the enumeration process of both subfamilies.

2. General results

2.1. Quadrilateral snakes and snake polyominoes

The problem of counting snakes has been considered by several authors. Recently, Goupil et al., [3] studied the problem not only in the plane, they also considered higher dimensions. In that work as well as in [4], the authors accept the third structure in Figure 1.2 as a valid substructure of a snake. Pegg Jr. [4], called these combinatorial structures, 2-sided strip polyominoes with n cells. In Table 1 we show the first values of $p(n)$, i.e., the number of 2-sided strip polyominoes, and $\overline{p}(n)$, the number of snakes that follow our definition. As we may expect, the difference between $\overline{p}(n)$ and $p(n)$ increses with n.

n	1	2	3	4	5	6	7	8	9
$p(n)$	1	1	2	3	7	13	31	65	154
$\overline{p}(n)$	1	1	2	3	7	13	30	64	150

Table 1: Number of quadrilateral snakes and snake polyominoes.

In [5], the first author defined a kC_n*-snake* as a connected graph in which the k cells are isomorphic to the cycle C_n and the block-cutpoint graph is a path. By a *quadrilateral snake* we understand a kC_4-snake. In [6], we established a relationship between quadrilateral snakes and snake polyominoes, showing that for every snake polyomino there exists a quadrilateral snake of the same length. We also show that the converse of this statement is not valid. The reason is that when the number of cells is at least 7, there exist quadrilateral snakes which associated graph is not a snake polyomino because they have a subgraph isomorphic to the third structure in Figure 1.2. In Figure 2.1 we show three, of the 31 quadrilateral snakes of length 7, together with their associated polyomino. We can see that in the third example, the polyomino is not a snake according to our defintion, but it is according to the one used in [3] and [4]. Hence, $p(n)$ actually counts the number of quadrilateral snakes of length n. Therefore, determining a formula for $\overline{p}(n)$, as well as for $p(n)$, is still an open problem.

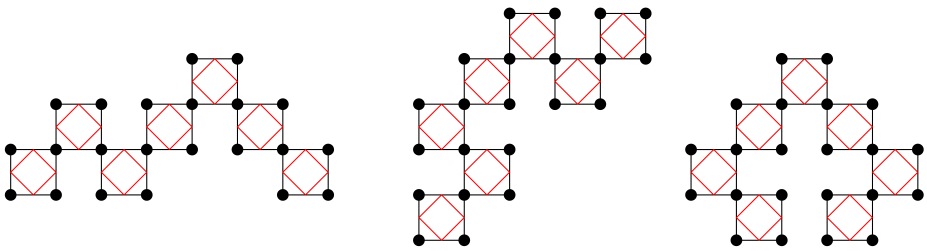

Figure 2.1: Quadrilateral snakes and associated polyominoes.

2.2. Partitioning n into k parts

It is well-known that the number $P(n,k)$ of partitions of n into k parts, where the order is taken under consideration, is given by

$$P(n,k) = C(n-1,k-1),$$

where $C(n-1,k-1)$ is the standard binomial coefficient $\binom{n-1}{k-1}$.

In order to prove this fact, the number n is represented on a line formed by n balls. There are $n-1$ spaces in between the balls where a bar (or separator) can be placed. So, to separate the balls into k groups we need to introduce $k-1$ bars. The number of ways to do this is $C(n-1,k-1)$.

In Figure 2.2 we show an example of this result, exhibiting all the 3-part partitions of 5.

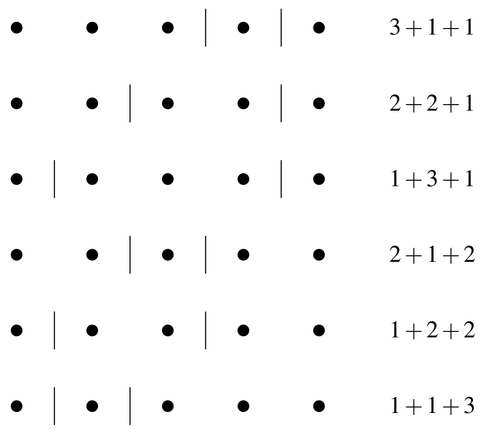

Figure 2.2: 3-part partitions of 5.

In the next sections we use this type of partitions to count the number of snakes considered in each case.

2.3. Snakes in a box

As we mentioned before, our snakes are subgraphs of $P_a \times P_b$. Suppose that a snake S of length n is a subgraph of $P_a \times P_b$ and is not a subgraph of $P_{a-1} \times P_b$ nor $P_a \times P_{b-1}$, then we say that S is inscribed in a box of *base* $b-1$ and *height* $a-1$, or that S has base $b-1$ and height $a-1$. For example, the snakes in Figure 2.1 are inside the boxes $P_4 \times P_8$, $P_6 \times P_6$, and $P_5 \times P_6$, respectively. We use the symmetries of these boxes to count the number of non-isomorphic snakes.

3. The number of stairs

By a *block of cells* of length t we understand the ladder $L_t = P_2 \times P_{t+1}$. Let $L_{p_1}, L_{p_2}, \ldots, L_{p_k}$ be a sequence of these blocks, where $p_1 + p_2 + \cdots + p_k = n$ and each $p_i \neq 0$. The *stair snake polyomino*, or simply *stair*, formed by these blocks, is the graph obtained by placing the first cell of L_{p_i} on top of the last cell of $L_{p_{i-1}}$, for each $2 \leq i \leq k$. In Figure 3.1, we show the stair with base 11 and height 5 with blocks of length 1,4,3,5,2.

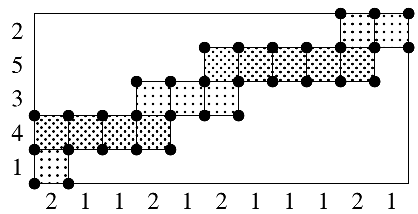

Figure 3.1: A stair of length 15.

First, we must observe that $p_1, p_2, \ldots, p_k$ is a partition of n into k parts. In addition, the construction given above establishes a bijection between the set of partitions of n into k parts, where order matters, and the set of stairs built in this way. So, in order to determine the number of stairs of length n with k steps (i.e., with k blocks of cells), we may count the distinct partitions of n into k parts.

Let S be a stair of length n with k steps built using the partition $p_1, p_2, \ldots, p_k$. Associated with this partition, there are three other partitions that form the same graph. In the case of the example given in Figure 3.1, the numbers on the left of the picture can be read from top to bottom forming a "different" partition of n. The other two partitions are obtained by reading the numbers, on the bottom, from left to right and vice versa. In general, for any given stair S, the other three partitions can be obtained using symmetries; the first one is a 180° rotation of S, while the other two are reflections, of S, around the two diagonals of a square centered at the center of S. In other terms, if the stair is not symmetric, there are two partitions of n into k parts and two partitions of n into $n+1-k$ parts, associated with the same stair. Therefore, we need to analyze the case where the stair is symmetric.

Consider any symmetric stair with n cells. If its first and last cell are deleted, the remaining graph is also a symmetric stair. Thus, all the symmetric stairs with $n+2$ cells can be constructed using the symmetric stairs with n cells, by attaching a new cell, to both, the first and the last cell.

It is easy to see that for $n = 1, 2$ there is only one stair with n cells. In general, every stair with n cells can be inscribed inside a rectangle, that can be a square, in such a way that the extreme cells are located in opposite corners of the rectangle.

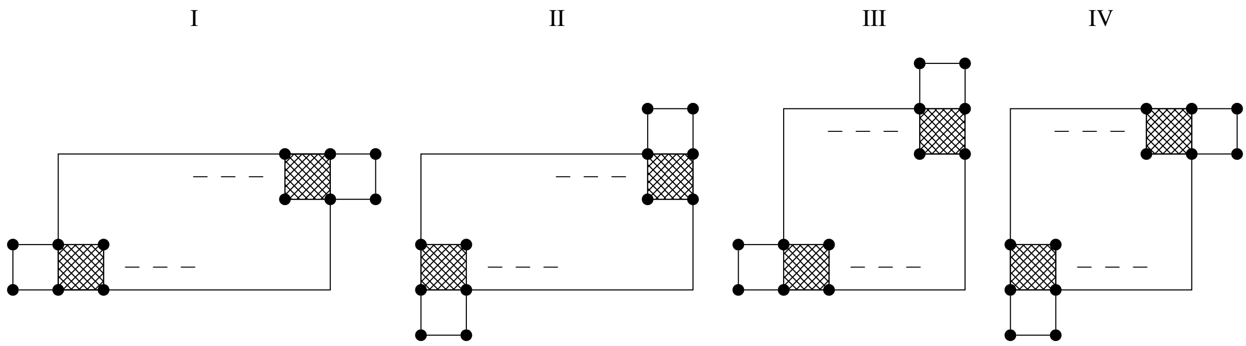

Figure 3.2: Extension schemes for symmetric stairs.

In Figure 3.2 we show the four ways, that exist, to extend a symmetric stair with n cells into a symmetric stair with $n+2$ cells. Schemes I and II show the cases where the original stair is inscribed in a rectangle (that can be a square), and the symmetry is a $180°$ rotation around the center of the rectangle. When the stair is inscribed in a square, schemes III and IV, the symmetry is a $180°$ rotation around the axis formed by the main diagonal of the square.

Independently of the case, the extreme cells have one horizontal and one vertical edge where a new cell can be attached. Thus, scheme I is the connection VV, scheme II is HH, scheme III is VH, and scheme IV is HV. Consequently, if $p_1, p_2, \ldots, p_k$ is the partition of n into k parts associated with a symmetric stair S, with n cells and k steps (or blocks of cells), then the partition of the new stair, for each case is shown in Table 2.

Connection	Partition	Number of Steps
I: VV	$1+p_1, p_2, \ldots, 1+p_k$	k
II: HH	$1, p_1, p_2, \ldots, p_k, 1$	$k+2$
III: VH	$1+p_1, p_2, \ldots, p_k, 1$	$k+1$
IV: HV	$1, p_1, p_2, \ldots, 1+p_k$	$k+1$

Table 2: Types of connections and associated partitions.

One of the consequences of this property is that if $s(n)$ is the number of symmetric stairs with n cells, then $2s(n)$ is the number of symmetric stairs with $n+2$ cells. Since, $s(1) = s(2) = 1$, we may conclude that

$$s(n) = \begin{cases} 2^{\frac{n-2}{2}}, & \text{if } n \text{ is even,} \\ 2^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

The sequence formed by the values of $s(n)$ corresponds to the sequence A016116 in OEIS.

Summarizing, for every stair with n cells and k steps, there is a partition of n into k parts and vice versa. A non-symmetric stair is represented by four different partitions; every symmetric stair is represented by two different partitions.

Since there are 2^{n-1} partitions of n into k parts, the number $e(n)$ of non-isomorphic stairs with $n \geq 3$ cells is:

When n is even:

$$\begin{aligned} e(n) &= \frac{1}{4}\left(2^{n-1} - 2\cdot 2^{\frac{n-2}{2}}\right) + \frac{1}{2}\cdot 2\cdot 2^{\frac{n-2}{2}} \\ &= 2^{n-3} - \frac{1}{2}\cdot 2^{\frac{n-2}{2}} + 2^{\frac{n-2}{2}} \\ &= 2^{n-3} + \frac{1}{2}\cdot 2^{\frac{n-2}{2}} \\ &= 2^{n-3} + 2^{\frac{n-4}{2}}. \end{aligned}$$

When n is odd:

$$\begin{aligned} e(n) &= \frac{1}{4}\left(2^{n-1} - 2\cdot 2^{\frac{n-1}{2}}\right) + \frac{1}{2}\cdot 2\cdot 2^{\frac{n-1}{2}} \\ &= 2^{n-3} - \frac{1}{2}\cdot 2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}} \\ &= 2^{n-3} + \frac{1}{2}\cdot 2^{\frac{n-1}{2}} \\ &= 2^{n-3} + 2^{\frac{n-3}{2}}. \end{aligned}$$

The first values of $e(n)$ are shown in Table 3. For $n \geq 2$, the consecutive values of $e(n)$ form the sequence A005418 in OEIS [7].

We can go even further, using the diagrams in Figure 3.2, we can calculate the number $\sigma(n,k)$ of symmetric stair with n cells and k steps.

n	$e(n)$	n	$e(n)$	n	$e(n)$
1	1	11	272	21	262656
2	1	12	528	22	524800
3	2	13	1056	23	1049600
4	3	14	2080	24	2098176
5	6	15	4160	25	4196352
6	10	16	8256	26	8390656
7	20	17	16512	27	16781312
8	36	18	32896	28	33558528
9	72	19	65792	29	67117056
10	136	20	131328	30	134225920

Table 3: Number of non-isomorphic stairs with n cells.

Recall that $\sigma(1,1)=\sigma(2,1)=\sigma(2,2)=1$. We use the conventions that $\sigma(n,k)=0$ when $k<1$ or $k>n$, and $C(n,k)=0$ if k is not an integer.
Thus, from I and II in Figure 3.2, we know that for all values of $n\geq 3$ and $k\neq \frac{n+1}{2}$,

$$\sigma(n,k)=\sigma(n-2,k)+\sigma(n-2,k-2).$$

When $k=\frac{n+1}{2}$ we get

$$\sigma(n,k)=\sigma(n-2,k)+\sigma(n-2,k-2)+2^{\frac{n-1}{2}}.$$

The number $2^{\frac{n-1}{2}}$ comes from III and IV in Figure 3.2.

Proposition 3.1. *Let n be a positive even number and $k\in\{1,2,\ldots,n\}$. Given that $\sigma(2,1)=\sigma(2,2)=1$, the number $\sigma(n,k)$ of symmetric stairs with n cells and k steps is*

$$\sigma(n,k)=C\left(\frac{n-2}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right).$$

Proof. By induction on n. Recall that $\sigma(2,1)=\sigma(2,2)=1$; for $n=4$:

$$\begin{aligned}\sigma(4,1)=\sigma(2,1)=\sigma(2,-1)=1+0=1\\ \sigma(4,2)=\sigma(2,2)=\sigma(2,0)=1+0=1\\ \sigma(4,3)=\sigma(2,3)=\sigma(2,1)=0+1=1\\ \sigma(4,4)=\sigma(2,4)=\sigma(2,2)=0+1=1\end{aligned}$$

On the other side, $C\left(\frac{4-2}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right)=C\left(1,\left\lfloor\frac{k-1}{2}\right\rfloor\right)$. Thus,

$$\begin{aligned}C\left(1,\left\lfloor\frac{1-1}{2}\right\rfloor\right)=C(1,0)=1\\ C\left(1,\left\lfloor\frac{2-1}{2}\right\rfloor\right)=C(1,0)=1\\ C\left(1,\left\lfloor\frac{3-1}{2}\right\rfloor\right)=C(1,1)=1\\ C\left(1,\left\lfloor\frac{4-1}{2}\right\rfloor\right)=C(1,1)=1.\end{aligned}$$

Then, the proposition is correct for $n=2$ and $n=4$.
Suppose that the proposition is correct up to a certain value of n. We want to prove that is also correct for $n+2$; in other terms,

$$\sigma(n+2,k)=C\left(\frac{n}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right).$$

We know that

$$\begin{aligned}\sigma(n+2,k)&=\sigma(n,k)+\sigma(n,k-2)\\ &=C\left(\frac{n-2}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right)+C\left(\frac{n-2}{2},\left\lfloor\frac{k-3}{2}\right\rfloor\right)\\ &=C\left(\frac{n}{2},\left\lfloor\frac{k-1}{2}\right\rfloor\right).\end{aligned}$$

Therefore, the proposition is true for every even value of n. □

Proposition 3.2. *Let n be a positive odd number and $k \in \{1,2,\dots,n\}$. Given that $\sigma(1,1)=1$, the number $\sigma(n,k)$ of symmetric stairs with n cells and k steps is*

$$\sigma(n,k) = C\left(\frac{n-1}{2}, \frac{k-1}{2}\right) + \varepsilon(k),$$

where

$$\varepsilon(k) = \begin{cases} 2^{\frac{n-1}{2}}, & \text{when } k = \frac{n+1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $n \geq 3$ is odd. Note that when k is even, $\frac{k-1}{2}$ is not an integer, then $C\left(\frac{n-1}{2}, \frac{k-1}{2}\right) = 0$. Thus, from this point we are assuming that k is odd.
The term $\varepsilon(k)$ is the number of symmetric stairs with n cells and $k = \frac{n+1}{2}$ steps that are originated by the corresponding stairs with $n-2$ cells and $\frac{n-1}{2}$ steps. Based on the diagrams III and IV in Figure 3.2 and the fact that $\sigma(1,1)=1$, we know that these stairs increase by a factor of 2 in the next generation, so $\varepsilon\left(\frac{n+1}{2}\right) = 2^{\frac{n-1}{2}}$. In addition, we must observe that this $\varepsilon(k)$ is positive only when $k = \frac{n+1}{2}$, otherwise is 0.
For any other value of k, any symmetric stair with $n-2$ cells can be inscribed into a rectangle, that is not a square, implying that this stair produces two stairs with n cells, one with k steps (diagram I) and the other one with $k+2$ steps (diagram II). Since $\sigma(1,1)=1$, we can see that the sequence of values of $\sigma(n,k)$ is exactly the sequence of binomial coefficients, adjusted conveniently. Therefore, $\sigma(n,k) = C\left(\frac{n-1}{2}, \frac{k-1}{2}\right)$ for all odd values of n and k, except when $k = \frac{n+1}{2}$ where we need to add the power $2^{\frac{n-1}{2}}$. □

In Table 4 we show the first values of $\sigma(n,k)$. The triangular arrangement produced by the vales of $\sigma(n,k)$ is quite similar to the one found in the sequence A051159 in OEIS. Both triangles, only differ when n is odd and $k = \frac{n+1}{2}$, that is when we added $\varepsilon(k)$. Thus, $T(n,k) = \sigma(n,k)$ for all n and k except when n is odd and $k = \frac{n+1}{2}$, where $T(n,k)$ are the entries of the triangle in A051159.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1																			
2	1	1																		
3	1	0+2	1																	
4	1	1	1	1																
5	1	0	3+4	0	1															
6	1	1	2	2	1	1														
7	1	0	3	0+8	3	0	1													
8	1	1	3	3	3	3	1	1												
9	1	0	4	0	6+16	0	4	0	1											
10	1	1	4	4	6	6	4	4	1	1										
11	1	0	5	0	10	0+32	10	0	5	0	1									
12	1	1	5	5	10	10	10	10	5	5	1	1								
13	1	0	6	0	15	0	20+64	0	15	0	6	0	1							
14	1	1	6	6	15	15	20	20	15	15	6	6	1	1						
15	1	0	7	0	21	0	35	0+128	35	0	21	0	7	0	1					
16	1	1	7	7	21	21	35	35	35	35	21	21	7	7	1	1				
17	1	0	8	0	28	0	56	0	70+256	0	56	0	28	0	8	0	1			
18	1	1	8	8	28	28	56	56	70	70	56	56	28	28	8	8	1	1		
19	1	0	9	0	36	0	84	0	126	0+512	126	0	84	0	36	0	9	0	1	
20	1	1	9	9	36	36	84	84	126	126	126	126	84	84	36	36	9	9	1	1

Table 4: $\sigma(n,k)$ number of symmetric stairs with n cells and k steps.

There is another alternative to present the problem of counting stairs. We show it for the case where $k = \frac{n+1}{2}$.
Consider the integral grid $\mathbb{N} \times \mathbb{N}$. Determine the number of non-equivalent paths between the points $(0,0)$ and (n,n). Two paths are equivalent if one can be obtained from the other by any of the symmetries of the square where it is inscribed. In Figure 3.2 we show the first instances of these paths, that is, for every $n \in \{1,2,3,4\}$.

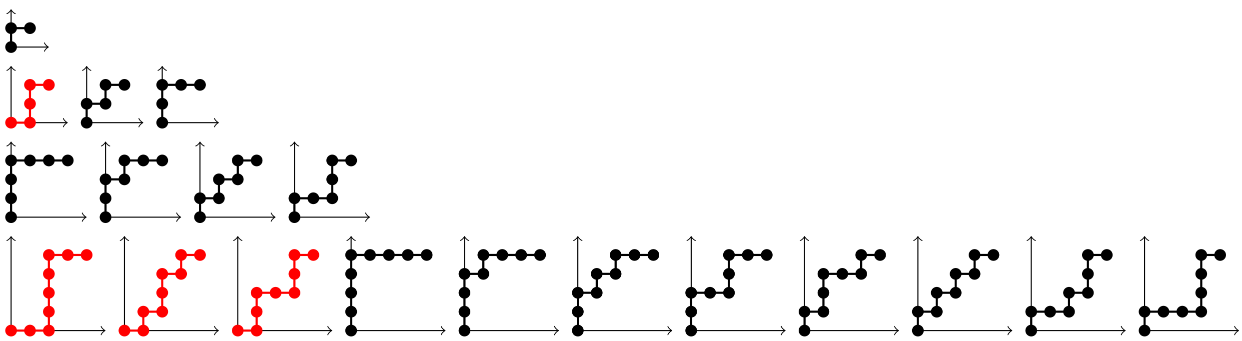

Figure 3.3: Non-equivalent paths from $(0,0)$ to (n,n)

Before closing this section, we want to note a connection between stairs and caterpillars. Suppose that the rows, of the box containing the stair, are labeled $0,1,\dots,\lambda$ from the top to the bottom, and the columns are labeled, from left to right, $\lambda+1, \lambda+2, \dots, n$. Thus, this representation of the stair corresponds to the α-labeling of a caterpillar. This labeling scheme was given by Rosa [8]; he proved that all caterpillars admit an α-labeling. Barrientos and Minion [9] used an extension of the adjacency matrix of α-labeled graphs and realized that all the adjacencies lie in a rectange. In the case of Rosa's labeling of caterpillars, the distribution of the adjacencies follows the stair pattern of these polyominoes. This fact is ratified in [7], where one of the interpretations of the sequence A005418 is that it represents the number of caterpillars of order n. For more information about labelings and, in particular, α-labelings, the interested reader is referred to [10].

4. Snakes in $P_3 \times P_{t+1}$

In this section we determine the number of snakes that can be inscribed in a box of base t and height 2. Consider the six snakes shown in Figure 4.1. Each of them is formed by blocks of cells of the form $L_p = P_2 \times P_{p+1}$, where $p \geq 1$. For example, the snake in part E is formed by the sequence of blocks L_4, L_3, L_5, L_2. In general, when $L_{p_1}, L_{p_2}, \ldots, L_{p_k}$ is the sequence of blocks of cells associated to a snake polyomino of height 2, the last cell (from left to right) of L_{p_i} is adjacent to the first cell of $L_{p_{i+1}}$. We use the convention that the odd numbered blocks are placed on the top row, as shown in Figure 4.1; in this figure we show all the different posibilites for the end blocks L_{p_1} and L_{p_k}. Note that for every $2 \leq i \leq k-1$, each L_{p_i} must have at least three cells, otherwise, the associated polyomino would not be a snake because it would have a subgraph isomorphic to the second graph in Figure 1.2.

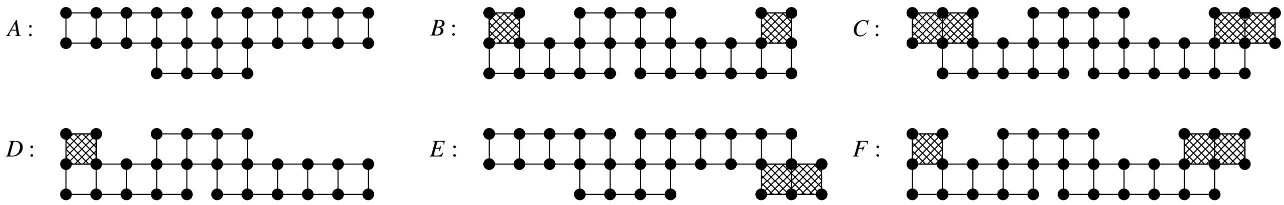

Figure 4.1: All general configurations for snakes of height 2.

Hence, if a snake of height 2 with n cells is represented by the sequence of blocks $L_{p_1}, L_{p_2}, \ldots, L_{p_k}$, the associated sequence $p_1, p_2, \ldots, p_k$ is a partition of n into k parts where $p_2, p_3, \ldots, p_{k-1}$ are at least 3. Thus, instead of counting snakes we may count partitions that satisfy these conditions.

In [11], Deutsch showed that in the OEIS sequence A102547, the term $T(n,k)$ is the number of compositions of $n+3$ with $k+1$ parts, all at least 3. He calculated this number to be

$$T(n,k) = C(n-2k, k)$$

where $n \geq 0$ and $0 \leq k \leq \frac{n}{3}$. Adjusting this expression to our terminology we can say that for every $n \geq 3$ and $1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$, the number of partitions of n into k parts, where every part is at least 3 is given by

$$\pi_3(n,k) = C(n-2k-1, k-1).$$

Therefore the number of partitions of n where every part is at least three is:

$$\sum_{k=1}^{\left\lfloor \frac{n}{3} \right\rfloor} \pi_3(n,k) = \sum_{k=1}^{\left\lfloor \frac{n}{3} \right\rfloor} C(n-2k-1, k-1).$$

Table 5 shows the values of $\pi_3(n,k)$ from $n = 3$ up to $n = 29$.

Before completing the counting process, we need to calculate the number $s_3(n,k)$ of symmetric partitions of n into k parts where every part is at least three.

Let $p_1, p_2, \ldots, p_k$ be a partition of n into k parts. We say that this partition is *symmetric* (or *reversible*) if for every $1 \leq i \leq k$, $p_i = p_{k_1 - i}$.

Proposition 4.1. *If n is odd and k is even, then $s_3(n,k) = 0$.*

Proof. By contradiction. Suppose that $s_3(n,k) \neq 0$, that is, there exists a symmetric partition of n into k parts, where each part is at least 3. Since the partition is symmetric and k is even

$$\sum_{i=1}^{\frac{k}{2}} p_i = \sum_{i=\frac{k}{2}+1}^{k} p_i$$

and

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{\frac{k}{2}} p_i + \sum_{i=\frac{k}{2}+1}^{k} p_i = 2\sum_{i=1}^{\frac{k}{2}} p_i$$

which is even. But this is a contradiction because $n = \sum_{i=1}^{k} p_i$ is odd. Therefore $s_3(n,k) = 0$. □

Proposition 4.2. *If both n and k are odd, then $s_3(n,k) = s_3(n+1,k)$.*

$n\backslash k$	1	2	3	4	5	6	7	8	9	Total
3	1									1
4	1									1
5	1									1
6	1	1								2
7	1	2								3
8	1	3								4
9	1	4	1							6
10	1	5	3							9
11	1	6	6							13
12	1	7	10	1						19
13	1	8	15	4						28
14	1	9	21	10						41
15	1	10	28	20	1					60
16	1	11	36	35	5					88
17	1	12	45	56	15					129
18	1	13	55	84	35	1				189
19	1	14	66	120	70	6				277
20	1	15	78	165	126	21				406
21	1	16	91	220	210	56	1			595
22	1	17	105	286	330	126	7			872
23	1	18	120	364	495	252	28			1278
24	1	19	136	455	715	462	84	1		1873
25	1	20	153	560	1001	792	210	8		2745
26	1	21	171	680	1365	1287	462	36		4023
27	1	22	190	816	1820	2002	924	120	1	5896
28	1	23	210	969	2380	3003	1716	330	9	8641
29	1	24	231	1140	3060	4368	3003	792	45	12664

Table 5: Partitions of n into k parts p_i, where $p_i \geq 3$.

Proof. Suppose that both n and k are odd numbers. Let $p_1, p_2, \ldots, p_k$ be a symmetric partition of n into k parts where every part is at least 3. This partition can be transformed into a symmetric partition of $n+1$ by adding one unit to the part $p_{\frac{k+1}{2}}$. Thus, every symmetric partition of n corresponds to a symmetric partition of $n+1$, where each $p_i \geq 3$.
Let $p'_1, p'_2, \ldots, p'_k$ be a symmetric partition of $n+1$ where each $p'_i \geq 3$. Then, for every $1 \leq i \leq \frac{k-1}{2}$, $p'_i = p'_{k+1-i}$. Hence, $p'_{\frac{k+1}{2}} \geq 3$ must be an even number. So, by making $p_{\frac{k+1}{2}} = p'_{\frac{k+1}{2}} - 1$ and $p_i = p'_i$ for every $1 \leq i \leq \frac{k-1}{2}$, we obtain a symmetric partition of n. That is, every symmetric partition of $n+1$ corresponds to a symmetric partition of n.
Therefore, $s_3(n,k) = s_3(n+1,k)$ when n and k are odd. □

Proposition 4.3. *If both n and k are odd, then* $s_3(n,k) = \pi_3\left(\frac{n+3}{2}, \frac{k+1}{2}\right)$.

Proof. Let us assume that $k=3$ and p_1, p_2, p_3 is a symmetric partition of n where each part is at least 3. Because of the symmetry, we know that p_2 is odd; so $p_2 \in \{3, 5, \ldots, n-6\}$. This implies that $p_1 = p_3$ and it belongs to $\{3, 4, \ldots, \frac{n-3}{2}\}$. Then, there are $\frac{n-3}{2} - 3 + 1 = \frac{n-7}{2}$ partitions of n; that is, $s_3(n,3) = \frac{n-7}{2}$.
On the other side,

$$\pi_3\left(\frac{n+3}{2}, 2\right) = C\left(\frac{n+3}{2} - 4 - 1, 1\right) = \frac{n+3}{2} - 5 = \frac{n-7}{2}.$$

So, $s_3 = (n,3) = \pi_3\left(\frac{n+3}{2}, 2\right)$ as we claimed.
Suppose now that $k > 3$. If $p_1, p_2, \ldots, p_k$ is a symmetric partition of n into k parts where every part is at least 3. Then $p_{\frac{k+1}{2}}$ is odd and belongs to $\{3, 5, \ldots, n-6\}$. Moreover, $p_1, p_2, \ldots, p_{\frac{k-1}{2}}$ is a partition of $\frac{1}{2}\left(n - p_{\frac{k+1}{2}}\right)$ into $\frac{k-1}{2}$ parts. Thus,

$$\begin{aligned}
\sum_{p_{\frac{k+1}{2}} \in \{3,5,\ldots,n-5\}} \pi_3\left(\frac{1}{2}\left(n - p_{\frac{k+1}{2}}\right), \frac{k-1}{2}\right) &= \pi_3\left(\frac{n-3}{2}, \frac{k-1}{2}\right) + \pi_3\left(\frac{n-5}{2}, \frac{k-1}{2}\right) + \cdots + \pi_3\left(\frac{n-n+6}{2}, \frac{k-1}{2}\right) \\
&= \pi_3\left(3, \frac{k-1}{2}\right) + \pi_3\left(5, \frac{k-1}{2}\right) + \cdots + \pi_3\left(\frac{n-3}{2}, \frac{k-1}{2}\right) \\
&= \sum_{i=3}^{\frac{n-3}{2}} \pi_3\left(i, \frac{k-1}{2}\right) = \sum_{i=3}^{\frac{n+3}{2}-3} \pi_3\left(i, \frac{k+1}{2} - 1\right) \\
&= \sum_{i=3\cdot\frac{k+1}{2}-3}^{\frac{n+3}{2}-3} \pi_3\left(i, \frac{k+1}{2} - 1\right) = \pi_3\left(\frac{n+3}{2}, \frac{k+1}{2}\right)
\end{aligned}$$

because $\pi_3\left(i, \frac{k+1}{2}-1\right) = 0$ for all the values of i such that $3 \le i < 3\left(\frac{k+1}{2}-1\right)$. □

Proposition 4.4. *For every $n < 3k$, $\pi_3(n,k) = 0$.*

Proof. By contradiction. Suppose that $n < 3k$ and $\pi_3(n,k) > 0$. Then, there exists a partition of n into k parts where every part is at least 3. Let $p_1, p_2, \ldots, p_k$ be this partition. Thus, $p_1 + p_2 + \cdots + p_k = n$. Since $p_i \ge 3$ for every $i \in \{1,2,\ldots,k\}$, we have that $p_1 + p_2 + \cdots + p_k \ge 3k$. Hence $3k = n$, which is a contradiction.
Therefore, for every $n < 3k$, $\pi_3(n,k) = 0$. □

Proposition 4.5. *If n and k are even, then $s_3(n,k) = \pi_3\left(\frac{n}{2}, \frac{k}{2}\right)$.*

Proof. Suppose that both, n and k, are even. Let $p_1, p_2, \ldots, p_k$ be a symmetric partition of n into k parts, where each part is at least 3. Then, for every $1 \le i \le \frac{k}{2}$, $p_i = p_{k+1-i}$ and $p_1, p_2, \ldots, p_{\frac{k}{2}}$ is a partition of $\frac{n}{2}$ where every part is at least 3. There are $\pi_3\left(\frac{n}{2}, \frac{k}{2}\right)$ of these partitions. Therefore, when n and k are even, $s_3(n,k) = \pi_3\left(\frac{n}{2}, \frac{k}{2}\right)$. □

We summarize these results in the next theorem.

Theorem 4.6. *Let $n \ge 3$ be an integer and $1 \le k \le \left\lfloor \frac{n}{3} \right\rfloor$. The number of symmetric partitions of n into k parts where each part is at least 3 is given by*

$$s_3(n,k) = \begin{cases} 0 & \text{if } n \text{ is odd and } k \text{ is even,} \\ \pi_3\left(\frac{n+3}{2}, \frac{k+1}{2}\right) = C\left(\frac{n-1}{2} - k, \frac{k-1}{2}\right) & \text{if } n \text{ is odd and } k \text{ is odd,} \\ \pi_3\left(\frac{n+2}{2}, \frac{k+1}{2}\right) = C\left(\frac{n-2}{2} - k, \frac{k-1}{2}\right) & \text{if } n \text{ is even and } k \text{ is odd,} \\ \pi_3\left(\frac{n}{2}, \frac{k}{2}\right) = C\left(\frac{n-2}{2} - k, \frac{k-2}{2}\right) & \text{if } n \text{ is even and } k \text{ is even.} \end{cases}$$

Table 6 contains the first values of $s_3(n,k)$. This sequence of numbers can be found in the OEIS, sequence A317489. The column of totals, obtained by adding the $s_3(n,k)$ for all possible values of k, can be also found in OEIS, sequence A226916, see [12].

$n\backslash k$	1	2	3	4	5	6	7	8	9	Total
3	1									1
4	1									1
5	1									1
6	1	1								2
7	1	0								1
8	1	1								2
9	1	0	1							2
10	1	1	1							3
11	1	0	2							3
12	1	1	2	1						5
13	1	0	3	0						4
14	1	1	3	2						7
15	1	0	4	0	1					6
16	1	1	4	3	1					10
17	1	0	5	0	3					9
18	1	1	5	4	3	1				15
19	1	0	6	0	6	0				13
20	1	1	6	5	6	3				22
21	1	0	7	0	10	0	1			19
22	1	1	7	6	10	6	1			32
23	1	0	8	0	15	0	4			28
24	1	1	8	7	15	10	4	1		47
25	1	0	9	0	21	0	10	0		41
26	1	1	9	8	21	15	10	4		69
27	1	0	10	0	28	0	20	0	1	60
28	1	1	10	9	28	21	20	10	1	101
29	1	0	11	0	36	0	35	0	5	88

Table 6: Number of symmetric partitions of n into k parts, where $p_i \ge 3$.

Similarly to what we did in the previous section, we use the values of $\pi_3(n,k)$ and $s_3(n,k)$ to find the number of non-isomorphic snake polyominoes with n cells and height 2. Note that any of these snakes must fit in exactly one of the six cases shown in Figure 4.1; so we analyze six cases:
Case I: The snake has the shape A, i.e., every block of cells has length at least 3. Thus, the number $\beta_2(n)$ of snake polyominoes of length n and height 2 is the same that the number of different partitions of n where every part is at least 3. In order to determine this number, we must

remember that the graphs produced by the partition $p_1, p_2, \ldots, p_k$ and its reverse, $p_k, p_{k-1}, \ldots, p_1$, are isomorphic; furthermore, some of these partitions are symmetric, thus for a fixed value of k

$$\frac{1}{2}\left(\pi_3(n,k) - s_3(n,k)\right)$$

is the number of different non-symmetric partitions of n into k parts where each part is at least 3. So, adding $s_3(n,k)$ to this expression we get

$$\frac{1}{2}\left(\pi_3(n,k) + s_3(n,k)\right).$$

Adding these numbers over all the possible values of k we obtain

$$\beta_2^1(n) = \frac{1}{2}\sum_{k=2}^{\lfloor \frac{n}{3} \rfloor}\left(\pi_3(n,k) + s_3(n,k)\right).$$

Note that the case $k = 1$ cannot be used here because the resulting snake has height 1.

Case II: The snake has the shape B, i.e., every block of cells has length at least 3 except the first and the last one that have length 1. Thus, the number of non-isomorphic snakes is the same that the number of different partitions of $n-2$ where every part is at least 3. Following the same steps than the previous case, we get

$$\beta_2^2(n) = \frac{1}{2}\sum_{k=1}^{\lfloor \frac{n-2}{3} \rfloor}\left(\pi_3(n-2,k) + s_3(n-2,k)\right).$$

Case III: The snake has the shape C, i.e., every block of cells has length 3 except the first and the last one that have length 2. Thus, the number of non-isomorphic snakes is the same that the number of different partitions of $n-4$ where every part is at least 3. This number is given by

$$\beta_2^3(n) = \frac{1}{2}\sum_{k=1}^{\lfloor \frac{n-4}{3} \rfloor}\left(\pi_3(n-4,k) + s_3(n-4,k)\right).$$

Case IV: The snake has the shape D, i.e., every block of cells has length 3 except the first one that has length 1. Thus, the number of non-isomorphic snakes is the same that the number of different partitions of $n-1$ where every part is at least 3. This number is given by

$$\beta_2^4(n) = \sum_{k=1}^{\lfloor \frac{n-1}{3} \rfloor}\pi_3(n-1,k).$$

Case V: The polyomino has the shape E, i.e., every block of cells has length 3 except the last one that has length 2. Thus, the number of non-isomorphic snakes is the same that the number of partitions of $n-2$ whre every part is at least 3. This number is given by

$$\beta_2^5(n) = \sum_{k=1}^{\lfloor \frac{n-2}{3} \rfloor}\pi_3(n-2,k).$$

Case VI: The polyomino has the shape F, i.e., every block of cells has length 3 except the first one that has length 1 and the last one that has length 2. Thus, the number of non-isomorphic snakes is the same that the number of different partitions of $n-3$ where every part is at least 3. This number is given by

$$\beta_2^6(n) = \sum_{k=1}^{\lfloor \frac{n-3}{3} \rfloor}\pi_3(n-3,k).$$

Adding all these quantities we obtain the total number of non-isomorphic snake polyominoes of length n and width 2. In this way we have proven the following theorem.

Theorem 4.7. *The number $\beta_2(n)$ of non-isomorphic snake polyominoes of length n and height 2 is*

$$\beta_2(n) = \sum_{i=1}^{6}\beta_2^i(n).$$

In Figure 4.2 we show a complete example for the case $n = 12$. In this case we have: $\beta_2^1(12) = 11, \beta_2^2(12) = 6, \beta_2^3(12) = 3, \beta_2^4(12) = 13, \beta_2^5(12) = 9, \beta_2^6(12) = 6$, and $\beta_2(12) = 48$.

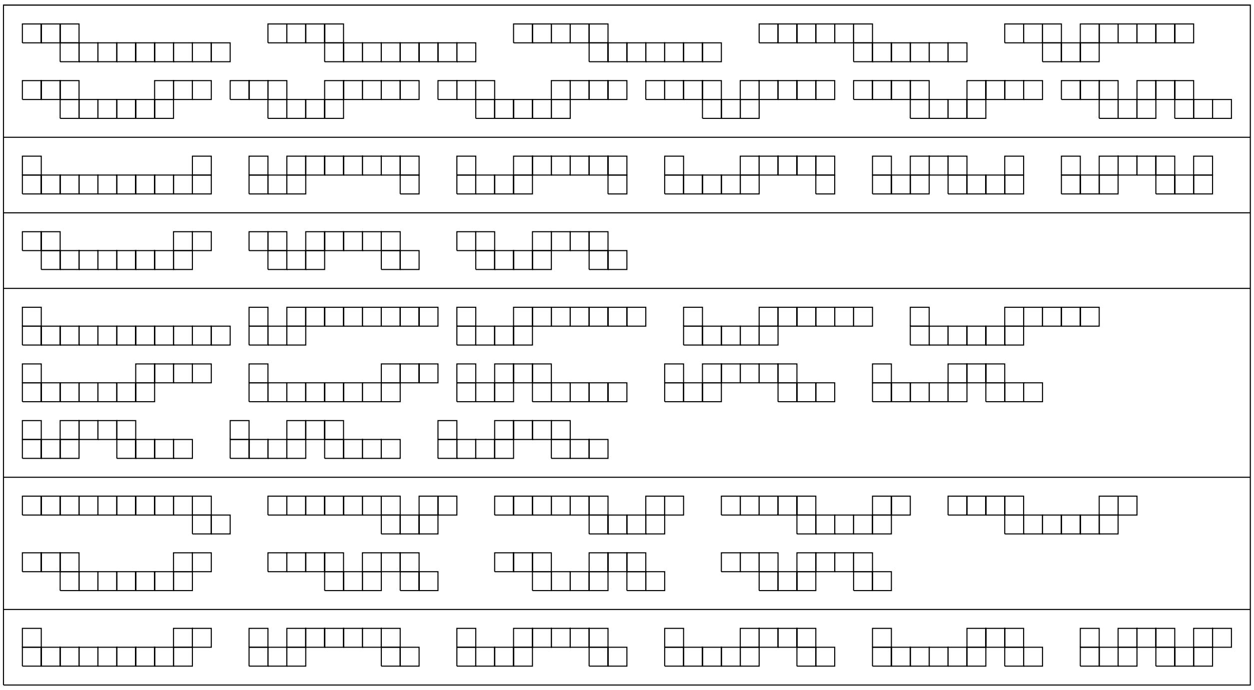

Figure 4.2: Non-isomorphic snake polyominoes of length 12 and height 2.

In Table 7 we show the inital values of these numbers. The last column, that corresponds to $\beta_2(n)$, can be obtained from A102543 in OEIS [13]. In fact, for $n \geq 5$, $\beta_2(n) = a(n+1) - 1$, where the values of $a(n)$ form the sequence A102543. We must also observe that the values of $\beta_2^4(n), \beta_2^5(n)$, and $\beta_2^6(n)$ can be found, with some shiftings, in A078012 [14].

n	$\beta_2^1(n)$	$\beta_2^2(n)$	$\beta_2^3(n)$	$\beta_2^4(n)$	$\beta_2^5(n)$	$\beta_2^6(n)$	$\beta_2(n)$
3	0	0	0	0	0	0	0
4	0	0	0	1	0	0	1
5	0	1	0	1	1	0	3
6	1	1	0	1	1	1	5
7	1	1	1	2	1	1	7
8	2	2	1	3	2	1	11
9	3	2	1	4	3	2	15
10	5	3	2	6	4	3	23
11	7	4	2	9	6	4	32
12	11	6	3	13	9	6	48
13	15	8	4	19	13	9	68
14	23	12	6	28	19	13	101
15	32	16	8	41	28	19	144
16	48	24	12	60	41	28	213
17	68	33	16	88	60	41	306
18	101	49	24	129	88	60	451
19	144	69	33	189	129	88	652
20	213	102	49	277	189	129	959
21	306	145	69	406	277	189	1392
22	451	214	102	595	406	277	2045
23	652	307	145	872	595	406	2977
24	959	452	214	1278	872	595	4370
25	1392	653	307	1873	1278	872	6375
26	2045	960	542	2745	1873	1278	9353
27	2977	1393	653	4023	2745	1873	13664
28	4370	2046	960	5896	4023	2745	20040
29	6375	2978	1393	8641	5896	4023	29306

Table 7: $\beta_2(n)$ is the number of non-isomorphic snake polyominoes of length n and height 2.

References

[1] D. Battaglino, A. Frosini, V. Guerrini, *Binary pictures with excluded patters*, Discrete Geometry for Computer Imagery, 18^{th} IAPR International Conference, (2014), 25–38.
[2] S. W. Golomb, (2^{nd} ed.), *Polyominoes: Ouzzles, Patterns, Problems, and Packings*, Princeton University Press, Princeton, MA, 1996.
[3] A. Goupil, M.-E. Pellerin, J. de Wouters d'Oplinter, *Partially directed snake polyominoes*, Discrete Appl. Math., **236** (2018), 223–234.
[4] E. Pegg Jr., https://oeis.org/A002013, 2009.
[5] C. Barrientos, *Graceful labelings of cyclic snakes*, Ars Combin., **60** (2001), 85–96.
[6] C. Barrientos, S. Minion, *Snakes: from graceful to harmonious*, Bull. Inst. Combin. Appl., **79** (2017), 95–107.
[7] N. J. A. Sloane, R. K. Guy, https://oeis.org/A005418, 2012.
[8] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris, (1967), 349–355.
[9] C. Barrientos, S. Minion, *On the number of α-labeled graphs*, Discuss. Math. Graph Theory, **38** (2018), 177–188.
[10] J. A. Gallian, *A dynamic survey of graph labelings*, Electron. J. Combin., (2017), #DS6.
[11] E. Deutsch, https://oeis.org/A102547, 2010.
[12] N. J. A. Sloane, https://oeis.org/A226916, 2013.
[13] G. McGarvey,, https://oeis.org/A102543, 2005.
[14] N. J .A. Sloane, https://oeis.org/A078012, 2002.