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Existence and Stability of Solutions of Katugampola-Caputo Type Implicit Fractional Differential Equations with Impulses

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Abstract

This paper investigates the existence and Ulam stability of solutions for impulsive nonlinear fractional implicit differential equations with finite delay via Katugampola fractional derivative in Caputo sense. Our results are based on some standard fixed point theorems. Some examples are presented to illustrate the main results.

1. Introduction

The interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. With this advantage, the fractional-order models become more realistic and practical than the classical integer-order models, in which such effects are not taken into account. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc., see [1]-[3]. For some recent development on the topic, see [4]-[10] and the references therein.

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences. The theory of impulsive differential equations of integer order has found extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to the references [11]-[15]. On the other hand, the implicit differential equations with impulsive and delay have not been addressed so extensively and many aspects of these problems are yet to be explored. For some recent work on impulsive differential equations of fractional order, see [16]-[19] and the references therein. These days generalization of the derivatives of both Riemann-Liouville and Caputo types are introduced and shown the effect of utilizing it in equations of mathematical physics or related to probability. This was done using the definition of generalized fractional derivatives given by Katugampola [20]. The author initiated a new fractional integral, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form. Later, Katugampola [21] introduced a new fractional derivative, which generalizes the two derivatives in question. Motivated by the papers [21]-[23], we apply Katugampola-Caputo derivative for implicit fractional differential equations.

In this paper, we investigate the existence and Ulam stability of solutions for impulsive nonlinear fractional implicit differential equations with delay via Katugampola fractional derivative given by,

$$\begin{cases} {}^{\rho}D^{\omega}_{x_m} u(x) = h(x, u_x, {}^{\rho}D^{\omega}_{x_m} u(x)), & \text{for each } x \in \mathfrak{J} := (x_m, x_{m+1}],\ m = 0, 1, \ldots k,, \\ \Delta u|_{x_m} = I_m(u_{x_m^-}), & m = 1, \ldots, k, \\ u(x) = \psi(x), & x \in [-r, 0],\ r > 0, \end{cases} \tag{1.1}$$

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where ${}^{\rho}D^{\omega}_{x_m}$ is the Katugampola fractional derivative in Caputo sense, $0<\omega\leq 1$, $\rho\in\mathbb{R}^+$, $h:\mathfrak{J}\times\mathfrak{PC}([-r,0],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ is a given function, $I_m:\mathfrak{PC}([-r,0],\mathbb{R})\to\mathbb{R}$, and $\psi\in\mathfrak{PC}([-r,0],\mathbb{R})$, $0=x_0<x_1<\cdots<x_m<x_{m+1}=T$. $\mathfrak{PC}([-r,0],\mathbb{R})$ is a space of piecewice functions defined on $[-r,0]$ to be specified in Section 2.
For each function u defined on $[-r,T]$ and for any $x\in\mathfrak{J}$, we define by u_x the element of $\mathfrak{PC}([-r,0],\mathbb{R})$ defined by:

$$u_x(\theta)=u(x+\theta),\ \theta\in[-r,0],$$

$u_x(\cdot)$ represents the history of the state from time $x-r$ upto time x. Here $\Delta u|_{x_m}=u(x_m^+)-u(x_m^-)$, where $u(x_m^+)=\lim_{l\to 0^+}u(x_m+l)$ and $u(x_m^-)=\lim_{l\to 0^-}u(x_m+l)$ denotes the right and left limits of u_x at $x=x_m$, respectively.

2. Prerequisites

In this section, we introduce notations, definitions, lemmas and theorems that are needed for the proof of the main results.
Let $T>0$, $\mathfrak{J}=[0,T]$ and $\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with the norm

$$\|u\|_\infty=\sup\{|u(x)|:x\in\mathfrak{J}\}.$$

Let $\mathfrak{J}_0=[x_0,x_1]$ and $\mathfrak{J}_m=(x_m,x_{m+1}]$, where $m=1,2,\ldots k$.
Consider the set of functions

$$\mathfrak{PC}([-r,0],\mathbb{R})=\{u:[-r,0]\to\mathbb{R}:u\in\mathfrak{C}\left((t_m,t_{m+1}],\mathbb{R}\right),m=0,1,\ldots,k^{'},\text{ and there exist } u(t_m^-)\text{ and } u(t_m^+),\ m=1,2,\ldots,k\text{ with } u(t_m^-)=u(t_m)\}.$$

$\mathfrak{PC}([-r,0],\mathbb{R})$ is a Banach space with the norm

$$\|u\|_{\mathfrak{PC}}=\sup_{x\in[-r,0]}|u(x)|.$$

$\mathfrak{PC}([-r,T],\mathbb{R})$ is a Banach space with the norm

$$\|u\|_{\mathfrak{PC}_1}=\sup_{x\in[-r,T]}|u(x)|.$$

$\mathfrak{L}^{'}(\mathfrak{J},\mathbb{R})$ is the space of Lebesgue-integrable functions $u:\mathfrak{J}\to\mathbb{R}$ with the norm

$$\|u\|_1=\int_0^T|u(s)|\,\mathrm{d}s.$$

$\mathfrak{AC}^n(\mathfrak{J})=\{h:\mathfrak{J}\to\mathbb{R}:h,h^{'},\ldots h^{(n-1)}\in\mathfrak{C}(\mathfrak{J},\mathbb{R})\text{ and } h^{(n-1)}\text{ is absolutely continuous}\}$.
In what follows $\omega>0$.

Definition 2.1. *[9, 10] The fractional(arbitrary) order integral of the function $h\in\mathfrak{L}^{'}([0,T],\mathbb{R}_+)$ of order $\omega\in\mathbb{R}_+$ is defined by*

$$I^\omega h(x)=\frac{1}{\Gamma(\omega)}\int_0^x(x-s)^{\omega-1}h(s)\mathrm{d}s,$$

where Γ is the Euler gamma function defined by $\Gamma(\omega)=\int_0^\infty x^{\omega-1}e^{-x}\mathrm{d}x$, $\omega>0$.

Definition 2.2. *[9, 10] For a function $h\in\mathfrak{AC}^n(\mathfrak{J})$, the Caputo fractional order derivative of order ω of h is defined by*

$$({}^cD^\omega_{0^+}h)(x)=\frac{1}{\Gamma(n-\omega)}\int_0^x(x-s)^{n-\omega-1}h^n(s)\mathrm{d}s,$$

where $n=[\omega]+1$ and $[\omega]$ denotes the integer part of the real number ω.

Definition 2.3. *[22] The generalized left-sided fractional integral ${}^\rho I^\omega_{0^+}h$ of order $\omega\in\mathbb{C}(Re(\omega)>0)$ is defined by*

$$({}^\rho I^\omega_{0^+}h)(x)=\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}h(s)\mathrm{d}s,$$

for $x>0$, if the integral exists.

Definition 2.4. *[22] The generalized fractional derivative, corresponding to the generalized fractional integral (2.1), is defined by*

$$({}^\rho D^\omega_{0^+}h)(x)=\frac{\rho^{\omega-n+1}}{\Gamma(n-\omega)}\left(x^{1-\rho}\frac{d}{dx}\right)^n\int_0^x(x^\rho-s^\rho)^{n-\omega-1}s^{\rho-1}h(s)\mathrm{d}s, \tag{2.1}$$

if the integral exists.

Lemma 2.5. *Let $\omega\geq 0$ and $n=[\omega]+1$. Then*

$${}^\rho I^\omega_{0^+}\left({}^\rho D^\omega_{0^+}h(x)\right)=h(x)-\sum_{m=0}^{n-1}\frac{h^m(0)}{m!}x^m.$$

Lemma 2.6. *Let $\omega > 0$, then the differential equation ${}^{\rho}D^{\omega}_{0^+}h(x) = 0$ has solutions*

$$h(x) = b_0 + b_1\left(\frac{x^\rho}{\rho}\right) + b_2\left(\frac{x^\rho}{\rho}\right)^2 + \cdots + b_{n-1}\left(\frac{x^\rho}{\rho}\right)^{(n-1)},$$

$b_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\omega]+1$.

Lemma 2.7. *Let $\omega > 0$, then*

$${}^{\rho}I^{\omega}_{0^+}\left({}^{\rho}D^{\omega}_{0^+}h(x)\right) = h(x) + b_0 + b_1\left(\frac{x^\rho}{\rho}\right) + b_2\left(\frac{x^\rho}{\rho}\right)^2 + \cdots + b_{n-1}\left(\frac{x^\rho}{\rho}\right)^{(n-1)},$$

for some $b_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\omega]+1$.

Lemma 2.8. *[24] Let $w : [0,T] \to [0,+\infty)$ be a real function and $\alpha(\cdot)$ is a non-negative, locally integrable function on $[0,T]$ and there are constants $a > 0$ and $0 < \omega \le 1$ such that*

$$w(x) \le \alpha(x) + a\int_0^x (x^\rho - s^\rho)^{-\omega} s^{-\rho} w(s)\mathrm{d}s.$$

Then, there exists a constant $K = K(\omega)$ such that

$$w(x) \le \alpha(x) + Ka\int_0^x (x^\rho - s^\rho)^{-\omega} s^{-\rho}\alpha(s)\mathrm{d}s,$$

for every $x \in [0,T]$.

The following integral inequality of Gronwall type for piecewise continuous functions was introduced by Bainov and Hristova [25] which can be used in the sequel.

Lemma 2.9. *Let for $x \ge x_0 \ge 0$, the following inequality holds,*

$$u(x) \le a(x) + \int_{x_0}^x g(x,s)u(s)\mathrm{d}s + \sum_{x_0<x_m<x}\beta_m(x)u(x_m),$$

where $\beta_m(x)$ $(m \in \mathbb{N})$ are non-decreasing functions for $x \ge x_0$, $a \in \mathfrak{PC}([x_0,\infty),\mathbb{R}_+)$, a is non-decreasing and $g(x,s)$ is a continuous non negative function for $x,s \ge x_0$ and non decreasing with respect to x for any fixed $s \ge x_0$. Then, for $x \ge x_0$, the following inequality is valid:

$$u(x) \le a(x)\prod_{x_0<x_m<x}(1+\beta_m(x))\exp\left(\int_{x_0}^x g(x,s)\mathrm{d}s\right).$$

Now, we consider the concepts of Wang et al. and refer some new concepts about Ulam-Hyers stability and Ulam-Hyers-Rassias stability for considered problem (1.1). See [24, 26, 27, 28, 29].
Let $u \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$, $\varepsilon > 0$, $\phi > 0$ and $\alpha \in \mathfrak{PC}(\mathfrak{J},\mathbb{R}_+)$ is non decreasing. We consider the set of inequalities

$$\begin{cases} |{}^{\rho}D^{\omega}u(x) - h(x,u_x,{}^{\rho}D^{\omega}u(x))| \le \varepsilon, & x \in (x_m,x_{m+1}],\ m = 1,\ldots k, \\ \left|\Delta u|_{x_m} - I_m(u_{x_m^-})\right| \le \varepsilon, & m = 1,\ldots,k; \end{cases} \tag{2.2}$$

the set of inequalities

$$\begin{cases} |{}^{\rho}D^{\omega}u(x) - h(x,u_x,{}^{\rho}D^{\omega}u(x))| \le \alpha(x), & x \in (x_m,x_{m+1}],\ m = 1,\ldots k, \\ \left|\Delta u|_{x_m} - I_m(u_{x_m^-})\right| \le \phi, & m = 1,\ldots,k; \end{cases} \tag{2.3}$$

and the set of inequalities

$$\begin{cases} |{}^{\rho}D^{\omega}u(x) - h(x,u_x,{}^{\rho}D^{\omega}u(x))| \le \varepsilon\alpha(x), & x \in (x_m,x_{m+1}],\ m = 1,\ldots k, \\ \left|\Delta u|_{x_m} - I_m(u_{x_m^-})\right| \le \varepsilon\phi, & m = 1,\ldots,k. \end{cases} \tag{2.4}$$

Definition 2.10. *The problem (1.1) is Ulam-Hyers stable, if there exists a real number $c_{h,k} > 0$ such that for each $\varepsilon > 0$ and for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (2.2), there exists a solution $u_2 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the problem (1.1) with*

$$|u_1(x) - u_2(x)| \le c_{h,k}\varepsilon,\ x \in \mathfrak{J}.$$

Definition 2.11. *The problem (1.1) is generalized Ulam-Hyers stable, if there exists $\theta_{h,k} \in \mathfrak{C}(\mathbb{R}_+,\mathbb{R}_+)$, $\theta_{h,k}(0) = 0$ such that for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (2.2), there exists a solution $u_2 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the problem (1.1) with*

$$|u_1(x) - u_2(x)| \le \theta_{h,k}(\varepsilon),\ x \in \mathfrak{J}.$$

Definition 2.12. *The problem (1.1) is Ulam-Hyers-Rassias stable with respect to (α,ϕ), if there exists $c_{h,k,\alpha} > 0$ such that for each $\varepsilon > 0$ and for each solution $u_1 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (2.4), there exists a solution $u_2 \in \mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the problem (1.1) with*

$$|u_1(x) - u_2(x)| \le c_{h,k,\alpha}\varepsilon(\alpha(x)+\phi),\ x \in \mathfrak{J}.$$

Definition 2.13. *The problem (1.1) is generalized Ulam-Hyers-Rassias stable with respect to (α,ϕ), if there exists $c_{h,k,\alpha}>0$ such that for each solution $u_1\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the inequality (2.3), there exists a solution $u_2\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ of the problem (1.1) with*

$$|u_1(x)-u_2(x)|\leq c_{h,k,\alpha}(\alpha(x)+\phi),\ x\in\mathfrak{J}.$$

Remark 2.14. *From the above definitions, we*

(i) $Definition\ 2.10\Rightarrow Definition\ 2.11$;
(ii) $Definition\ 2.12\Rightarrow Definition\ 2.13$;
(iii) $Definition\ 2.12$ for $\alpha(x)=\phi=1\Rightarrow$ Definition 2.10.

Remark 2.15. *A function $u\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ is a solution of the inequality (2.4) if and only if there is $\sigma\in\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ and a sequence σ_m, $m=1,2,\dots k$(which depends on u) such that*

(i) $|\sigma(x)|\leq\varepsilon\alpha(x)$, $x\in(x_m,x_{m+1}]$, $m=1,2,\dots k$ and $|\sigma_m|\leq\varepsilon\phi$, $m=1,2,\dots k$;
(ii) ${}^{\rho}D^{\omega}u(x)=h(x,u_x,{}^{\rho}D^{\omega}u(x))+\sigma(x)$, $x\in(x_m,x_{m+1}]$, $m=1,2,\dots k$;
(iii) $\Delta u|_{x_m}=I_m(u_{x_m^-})+\sigma_m$, $m=1,2,\dots k$.

Similarly, we can get remarks for inequalities (2.2) and (2.3).

Theorem 2.16. *[8] (Ascoli-Arzela's Theorem) Let $E\subset\mathfrak{C}(\mathfrak{J},\mathbb{R})$, E is relatively compact(i.e,$\bar{E}$ is compact), if:*

(1) E is uniformly bounded, that is there exists $N>0$ such that $|h(x)|<N$, for every $h\in E$ and $x\in\mathfrak{J}$.
(2) E is equicontinuous, that is for every $\varepsilon>0$, there exists $\delta>0$ such that for each $x_1,x_2\in\mathfrak{J}$, $|x_1-x_2|\leq\delta$ implies $|h(x_1)-h(x_2)|\leq\varepsilon$, for every $h\in E$.

Theorem 2.17. *[30] (Banach's fixed point theorem) Let $\mathscr{C}$ be a non empty closed subset of a Banach space $\mathscr{X}$, then any contraction mapping T of $\mathscr{C}$ into itself has a unique fixed point.*

Theorem 2.18. *[30] (Schaefer's fixed point theorem) Let $\mathscr{X}$ be a Banach space, and $M:\mathscr{X}\to\mathscr{X}$ a completely continuous operator. If the set*

$$S=\{u\in\mathscr{X}:u=\mu Mu,\ \text{for some}\ \mu\in(0,1)\}$$

is bounded, then M has atleast one fixed points.

3. Existence of solutions

Definition 3.1. *A function $u\in\mathfrak{PC}([-r,T],\mathbb{R})$ whose ω-derivative exists on $\mathfrak{J}_m$ is said to be a solution of (1.1), if u satisfies the equation*

$${}^{\rho}D^{\omega}_{x_m}u(x)=h(x,u_x,{}^{\rho}D^{\omega}_{x_m}u(x)),$$

on $\mathfrak{J}_m$, and satisfies the conditions $\Delta u|_{x=x_m}=I_m(u_{x_m^-})$, $m=1,\dots,k$ and $u(x)=\psi(x)$, $x\in[-r,0]$.

The following lemma is required to prove the existence of solutions to (1.1).

Lemma 3.2. *Let $0<\omega\leq1$ and let $\sigma:\mathfrak{J}\to\mathbb{R}$ be continuous. A function u is a solution of the fractional integral equation*

$$u(x)=\begin{cases}\psi(0)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s, & \text{if}\ x\in[0,x_1],\\ \\ \psi(0)+\sum_{i=1}^m I_i(u_{x_i^-}) & \\ +\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^m\int_{x_{i-1}}^{x_i}(x_i^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s & \\ +\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s, & \text{if}\ x\in(x_m,x_{m+1}],\\ \\ \psi(x), & x\in[-r,0],\end{cases}\tag{3.1}$$

where $m=1,2,\dots k$, if and only if u is a solution of the following fractional problem

$$\begin{cases}{}^{\rho}D^{\omega}u(x)=\sigma(x), & x\in\mathfrak{J}_m,\\ \Delta u|_{x=x_m}=I_m(u_{x_m^-}), & m=1,2,\dots k,\\ u(x)=\psi(x), & x\in[-r,0].\end{cases}\tag{3.2}$$

Proof. Assume u satisfies (3.2). If $x\in[0,x_1]$, then ${}^{\rho}D^{\omega}u(x)=\sigma(x)$. From Lemma 2.7, we get

$$u(x)=\psi(0)+{}^{\rho}I^{\omega}\sigma(x)=\psi(0)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s.$$

If $x\in(x_1,x_2]$, then from Lemma 2.7,

$$\begin{aligned}u(x)&=u(x_1^+)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_1}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s\\ &=\Delta u|_{x=x_1}+u(x_1^-)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_1}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s\\ &=\psi(0)+I_1(u_{x_1^-})+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x(x_1^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_1}^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\ \sigma(s)\mathrm{d}s.\end{aligned}$$

If $x \in (x_2, x_3]$, then Lemma 2.7 implies,

$$
\begin{aligned}
u(x) &= u(x_2^+) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s \\
&= \Delta u|_{x=x_2} + u(x_2^-) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s \\
&= I_2(u_{x_2^-}) \\
&+ \left[\psi(0) + I_1(u_{x_1^-}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{0}^{x_1} (x_1^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^{x_2} (x_2^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s\right] \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s. \\
&= \psi(0) + \left[I_1(u_{x_1^-}) + I_2(u_{x_2^-})\right] \\
&+ \left[\frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{0}^{x_1} (x_1^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^{x_2} (x_2^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s\right] \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_2}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s.
\end{aligned}
$$

Continuing this process, we get the solution $u(x)$ for $x \in (x_m, x_{m+1}]$, where $m = 1, 2, \ldots k$. Hence,

$$
u(x) = \psi(0) + \sum_{i=1}^{m} I_i(u_{x_i^-}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} (x_i^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, \sigma(s)\mathrm{d}s.
$$

Conversely, let us assume that u satisfies the equation (3.1). If $x \in [0, x_1]$, then $u(0) = \psi(0)$ and using the concept that ${}^\rho D^\omega$ is the left inverse of ${}^\rho I^\omega$, we get ${}^\rho D^\omega u(x) = \sigma(x)$, for each $x \in [0, x_1]$. If $x \in (x_m, x_{m+1}]$, $m = 1, 2, \ldots k$ and using the fact that ${}^\rho D^\omega L = 0$, where L is a constant, we get

$$
{}^\rho D^\omega u(x) = \sigma(x), \text{ for each } x \in (x_m, x_{m+1}].
$$

Also, we can show that $\Delta u|_{x=x_m} = I_m(u_{x_m^-})$, $m = 1, 2, \ldots k$. □

Now we state and prove the existence results for the problem (1.1), based on Banach's fixed point theorem.

Theorem 3.3. *Assume that*

(A1) $h : \mathfrak{J} \times \mathfrak{PC}([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
(A2) There exist constants $c_1 > 0$ *and* $0 < c_2 < 1$ *such that*

$$
|h(x, z_1, z_2) - h(x, \bar{z_1}, \bar{z_2})| \le c_1 \|z_1 - \bar{z_1}\|_{\mathfrak{PC}} + c_2 |z_2 - \bar{z_2}|,
$$

for any $z_1, \bar{z_1} \in \mathfrak{PC}([-r, 0], \mathbb{R})$*,* $z_2, \bar{z_2} \in \mathbb{R}$ *and* $x \in \mathfrak{J}$*.*
(A3) There exists a constant $c_3 > 0$ *such that*

$$
|I_m(z_1) - I_m(\bar{z_1})| \le c_3 \|z_1 - \bar{z_1}\|_{\mathfrak{PC}},
$$

for each $z_1, \bar{z_1} \in \mathfrak{PC}([-r, 0], \mathbb{R})$ *and* $m = 1, 2, \ldots k$*.*

If

$$
k c_3 + \frac{(k+1) c_1 T^{\rho\omega}}{(1 - c_2) \rho^\omega \Gamma(\omega + 1)} < 1, \tag{3.3}
$$

then there exists a unique solution for the problem (1.1) *on* $\mathfrak{J}$*.*

Proof. Transform the problem (1.1) into a fixed point problem. Consider the operator $M : \mathfrak{PC}([-r, T], \mathbb{R}) \to \mathfrak{PC}([-r, T], \mathbb{R})$ defined by

$$
Mu(x) = \begin{cases} \psi(0) + \sum_{0 < x_m < x} I_m(u_{x_i^-}) \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0 < x_m < x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, g(s)\mathrm{d}s \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}\, g(s)\mathrm{d}s, & x \in [0, T], \\ \\ \psi(x), & x \in [-r, 0], \end{cases} \tag{3.4}
$$

where $g \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ be such that

$$
g(x) = h(x, u_x, g(x)).
$$

Clearly, the fixed points of operator M are solutions of the problem (1.1). Let $y, z \in \mathfrak{PC}([-r, T], \mathbb{R})$. If $x \in [-r, 0]$, then

$$
|M(y)(x) - M(z)(x)| = 0.
$$

For $x \in \mathfrak{J}$, we get

$$\begin{aligned} |M(y)(x)-M(z)(x)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} \, |g_1(s)-g_2(s)| \, \mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \, |g_1(s)-g_2(s)| \, \mathrm{d}s + \sum_{0<x_m<x} \left| I_m(y_{x_m^-}) - I_m(z_{x_m^-}) \right|, \end{aligned}$$

where $g_1, g_2 \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ be such that

$$g_1(x) = h(x, y_x, g_1(x)),$$

and

$$g_2(x) = h(x, z_x, g_2(x)).$$

By $(A2)$, we get

$$|g_1(x)-g_2(x)| = |h(x,y_x,g_1(x)) - h(x,z_x,g_2(x))| \leq c_1 \|y_x - z_x\|_{\mathfrak{PC}} + c_2 |g_1(x)-g_2(x)|.$$

This implies,

$$|g_1(x)-g_2(x)| \leq \frac{c_1}{1-c_2} \|y_x - z_x\|_{\mathfrak{PC}}.$$

Therefore, for each $x \in \mathfrak{J}$,

$$\begin{aligned} |M(y)(x)-M(z)(x)| &\leq \frac{c_1 \rho^{1-\omega}}{(1-c_2)\Gamma(\omega)} \sum_{m=1}^{k} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} \, \|y_s - z_s\|_{\mathfrak{PC}} \mathrm{d}s \\ &+ \frac{c_1 \rho^{1-\omega}}{(1-c_2)\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \, \|y_s - z_s\|_{\mathfrak{PC}} \mathrm{d}s + \sum_{m=1}^{k} c_3 \left\| y_{x_m^-} - z_{x_m^-} \right\|_{\mathfrak{PC}}. \\ &\leq \left[k c_3 + \frac{k c_1 T^{\rho\omega}}{(1-c_2)\rho^\omega \Gamma(\omega+1)} + \frac{c_1 T^{\rho\omega}}{(1-c_2)\rho^\omega \Gamma(\omega+1)} \right] \|y - z\|_{\mathfrak{PC}_1}. \end{aligned}$$

Thus,

$$\|M(y)-M(z)\|_{\mathfrak{PC}_1} \leq \left[k c_3 + \frac{(k+1) c_1 T^{\rho\omega}}{(1-c_2)\rho^\omega \Gamma(\omega+1)} \right] \|y - z\|_{\mathfrak{PC}_1}.$$

By (3.3), the operator M is a contraction. Therefore, by the Banach's contraction principle, M has a unique fixed point which is a unique solution of the problem (1.1). $\square$

Now, Schaefer's fixed point theorem is used to prove the second result.

Theorem 3.4. *Assume that* $(A1), (A2)$ *and*

(A4) There exist $p_1, p_2, p_3 \in \mathfrak{C}(\mathfrak{J}, \mathbb{R}_+)$ *with* $p_3^* = \sup_{x\in\mathfrak{J}} p_3(x) < 1$ *such that*

$$|h(x,y,z)| \leq p_1(x) + p_2(x) \|y\|_{PC} + p_3(x) |z|,$$

where $x \in \mathfrak{J}$, $y \in \mathfrak{PC}([-r,0],\mathbb{R})$ *and* $z \in \mathbb{R}$.

(A5) The functions $I_m : \mathfrak{PC}([-r,0],\mathbb{R}) \to \mathbb{R}$ *are continuous and there exist constants* $M_1^*, M_2^* > 0$ *with* $kM_1^* < 1$ *such that*

$$|I_m(y)| \leq M_1^* \|y\|_{\mathfrak{PC}} + M_2^*,$$

for each $y \in \mathfrak{PC}([-r,0],\mathbb{R})$, $m = 1, 2, \ldots k$. *Then the problem* (1.1) *has at least one solution.*

Proof. Let the operator M defined in (3.4). Now we shall prove that M has atleast one fixed point by using Schaefer's fixed point theorem. The proof contains four steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $\mathfrak{PC}([-r,T],\mathbb{R})$. If $x \in [-r,0]$, then

$$|M(y_n)(x) - M(y)(x)| = 0.$$

For $x \in \mathfrak{J}$, we have

$$\begin{aligned} |M(y_n)(x)-M(y)(x)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} \, |g_n(s)-g(s)| \, \mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \, |g_n(s)-g(s)| \, \mathrm{d}s + \sum_{0<x_m<x} \left| I_m(y_{nx_m^-}) - I_m(y_{x_m^-}) \right|, \end{aligned} \tag{3.5}$$

where $g_n, g \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ such that

$$g_n(x) = h(x, y_{nx}, g_n(x)),$$

$$g(x)=h(x,y_x,g(x)).$$

By $(A2)$, we have

$$\begin{aligned}|g_n(x)-g(x)| &= |h(x,y_{nx},g_n(x))-h(x,y_x,g(x))|\\ &\le c_1\|y_{nx}-y_x\|_{\mathfrak{PC}}+c_2|g_n(x)-g(x)|.\end{aligned}$$

Then,

$$|g_n(x)-g(x)|\le\left(\frac{c_1}{1-c_2}\right)\|y_{nx}-y_x\|_{\mathfrak{PC}}.$$

Since $y_n\to y$, then we get $g_n(x)\to g(x)$ as $n\to\infty$ for each $x\in\mathfrak{J}$. And let $\Omega>0$ be such that, for each $x\in\mathfrak{J}$, we have $|g_n(x)|\le\Omega$ and $|g(x)|\le\Omega$. Then, we have

$$\begin{aligned}(x^\rho-s^\rho)^{\omega-1}|g_n(s)-g(s)| &\le (x^\rho-s^\rho)^{\omega-1}[|g_n(s)|+|g(s)|]\\ &\le 2\Omega(x^\rho-s^\rho)^{\omega-1},\end{aligned}$$

and

$$\begin{aligned}(x_m^\rho-s^\rho)^{\omega-1}|g_n(s)-g(s)| &\le (x_m^\rho-s^\rho)^{\omega-1}[|g_n(s)|+|g(s)|]\\ &\le 2\Omega(x_m^\rho-s^\rho)^{\omega-1}.\end{aligned}$$

For each $x\in\mathfrak{J}$, the functions $s\to 2\Omega(x^\rho-s^\rho)^{\omega-1}$ and $s\to 2\Omega(x_m^\rho-s^\rho)^{\omega-1}$ are integrable on $[0,x]$, then by the Lebesgue Dominated Convergence Theorem and (3.5) implies that

$$|M(y_n)(x)-M(y)(x)|\to 0,\ \text{as}\ n\to\infty.$$

and hence,

$$\|M(y_n)-M(y)\|_{\mathfrak{PC}_1}\to 0,\ \text{as}\ n\to\infty,$$

Consequently, M is continuous.

Step 2: *M maps bounded sets into bounded sets in $\mathfrak{PC}([-r,T],\mathbb{R})$.* To prove this, it is enough to show that for any $\Omega^*>0$, there exists a positive constant $\tilde{k}$ such that for each $y\in B_{\Omega^*}=\{y\in\mathfrak{PC}([-r,T],\mathbb{R}):\|y\|_{\mathfrak{PC}_1}\le\Omega^*\}$, we have $\|M(y)\|_{\mathfrak{PC}_1}\le\tilde{k}$. We have for each $x\in\mathfrak{J}$,

$$\begin{aligned}M(y)(x) &= \psi(0)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}\,g(s)\mathrm{d}s\\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\,g(s)\mathrm{d}s+\sum_{0<x_m<x}I_m(y_{x_m^-}),\end{aligned}\tag{3.6}$$

where $g\in\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that

$$g(x)=h(x,y_x,g(x)).$$

By $(A4)$, for each $x\in\mathfrak{J}$, we get

$$\begin{aligned}|g(x)| &= |h(x,y_x,g(x))|\\ &\le p_1(x)+p_2(x)\|y_x\|_{\mathfrak{PC}}+p_3(x)|g(x)|\\ &\le p_1(x)+p_2(x)\|y\|_{\mathfrak{PC}_1}+p_3(x)|g(x)|\\ &\le p_1(x)+p_2(x)\Omega^*+p_3(x)|g(x)|\\ &\le p_1^*+p_2^*\Omega^*+p_3^*|g(x)|,\end{aligned}$$

where $p_1^*=\sup_{x\in\mathfrak{J}}p_1(x)$ and $p_2^*=\sup_{x\in\mathfrak{J}}p_2(x)$. Then,

$$|g(x)|\le\frac{p_1^*+p_2^*\Omega^*}{1-p_3^*}:=N.$$

Thus (3.6) implies

$$\begin{aligned}|M(y)(x)| &\le |\psi(0)|+\frac{kNT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}+\frac{NT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}+k\left(M_1^*\|y_{x_m^-}\|_{\mathfrak{PC}}+M_2^*\right)\\ &\le |\psi(0)|+\frac{(k+1)NT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}+k\left(M_1^*\|y\|_{\mathfrak{PC}_1}+M_2^*\right)\\ &\le |\psi(0)|+\frac{(k+1)NT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}+k(M_1^*\Omega^*+M_2^*):=\tilde{R}.\end{aligned}$$

And if $x\in[-r,0]$, then

$$|M(y)(x)|\le\|\psi\|_{\mathfrak{PC}},$$

thus

$$\|M(y)\|_{\mathfrak{PC}_1} \leq \max\{\tilde{R}, \|\psi\|_{\mathfrak{PC}}\} := \tilde{k}.$$

Step 3: *M maps bounded sets into equicontinuous sets of* $\mathfrak{PC}([-r,T],\mathbb{R})$.
Let $t_1,t_2 \in (0,T]$, $t_1 < t_2$, B_{Ω^*} be a bounded set of $\mathfrak{PC}([-r,T],\mathbb{R})$ as in Step 2, and let $y \in B_{\Omega^*}$. Then

$$\begin{aligned}
|M(y)(t_2)-M(y)(t_1)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^{t_1}\left|(t_2^\rho-s^\rho)^{\omega-1}-(t_1^\rho-s^\rho)^{\omega-1}\right|\left|s^{\rho-1}\right|\,|g(s)|\,\mathrm{d}s\\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{t_1}^{t_2}\left|(t_2^\rho-s^\rho)^{\omega-1}\right|\left|s^{\rho-1}\right||g(s)|\,\mathrm{d}s\\
&+ \sum_{0<x_m<t_2-t_1}\left|I_m(y_{x_m^-})\right|\\
&\leq \frac{N}{\rho^\omega\Gamma(\omega+1)}[2(t_2^\rho-t_1^\rho)^\omega+(t_2^{\rho\omega}-t_1^{\rho\omega})]\\
&+ (t_2^\rho-t_1^\rho)\left(M_1^*\|y_{x_m^-}\|_{\mathfrak{PC}}+M_2^*\right)\\
&\leq \frac{N}{\rho^\omega\Gamma(\omega+1)}[2(t_2^\rho-t_1^\rho)^\omega+(t_2^{\rho\omega}-t_1^{\rho\omega})]\\
&+ (t_2^\rho-t_1^\rho)\left(M_1^*\|y\|_{\mathfrak{PC}_1}+M_2^*\right)\\
&\leq \frac{N}{\rho^\omega\Gamma(\omega+1)}[2(t_2^\rho-t_1^\rho)^\omega+(t_2^{\rho\omega}-t_1^{\rho\omega})]\\
&+ (t_2^\rho-t_1^\rho)(M_1^*\Omega^*+M_2^*).
\end{aligned}$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero. From Step 1 to 3 together with the Ascoli-Arzela theorem, we can conclude that $M : \mathfrak{PC}([-r,T],\mathbb{R}) \to \mathfrak{PC}([-r,T],\mathbb{R})$ is completely continuous.
Step 4: *A priori bounds*. Now, we shall show that the set

$$G = \{y \in \mathfrak{PC}([-r,T],\mathbb{R}) : y = \mu M(y), \text{ for some } 0 < \mu < 1\},$$

is bounded. Let $y \in G$, then $y = \mu M(y)$, for some $0 < \mu < 1$. Thus, for each $x \in \mathfrak{J}$, we get

$$\begin{aligned}
y(x) &= \mu\psi(0)+\frac{\mu\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s\\
&+ \frac{\mu\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}g(s)\mathrm{d}s+\mu\sum_{0<x_m<x}I_m(y_{x_m^-}).
\end{aligned}$$

And by $(A4)$, for each $x \in \mathfrak{J}$, we get,

$$\begin{aligned}
|g(x)| &= |h(x,y_x,g(x))|\\
&\leq p_1(x)+p_2(x)\|y_x\|_{\mathfrak{PC}}+p_3(x)|g(x)|\\
&\leq p_1^*+p_2^*\|y_x\|_{\mathfrak{PC}}+p_3^*|g(x)|.
\end{aligned}$$

Thus,

$$|g(x)| \leq \frac{1}{1-p_3^*}\left(p_1^*+p_2^*\|y_x\|_{\mathfrak{PC}}\right).$$

This implies, by (3.7) and $(A5)$, that for each $x \in \mathfrak{J}$, we have

$$\begin{aligned}
|y(x)| &\leq |\psi(0)|+\frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}(p_1^*+p_2^*\|y_s\|_{\mathfrak{PC}})\mathrm{d}s\\
&+ \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}(p_1^*+p_2^*\|y_s\|_{\mathfrak{PC}})\mathrm{d}s\\
&+ k\left(M_1^*\|y_{x_m^-}\|_{\mathfrak{PC}}+M_2^*\right).
\end{aligned}$$

Now, we consider the function q defined by

$$q(x) = \sup\{|q(s)| : -r \leq s \leq x\},\ 0 \leq x \leq T,$$

then there exists $x^* \in [-r,T]$ such that $q(x) = |y(x^*)|$. If $x^* \in [0,T]$, then by the previous inequality, we have for $x \in \mathfrak{J}$,

$$\begin{aligned}
q(x) &\leq |\psi(0)|+\frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}(p_1^*+p_2^*q(s))\mathrm{d}s\\
&+ \frac{\rho^{1-\omega}}{(1-p_3^*)\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}(p_1^*+p_2^*q(s))\mathrm{d}s\\
&+ k(M_1^*q(x)+M_2^*).
\end{aligned}$$

Thus,

$$\begin{aligned} q(x) &\leq \frac{|\psi(0)|+kM_2^*}{1-kM_1^*} + \frac{\rho^{1-\omega}}{(1-kM_1^*)(1-p_3^*)\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1}(p_1^* + p_2^* q(s))\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{(1-kM_1^*)(1-p_3^*)\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1}(p_1^* + p_2^* q(s))\mathrm{d}s \\ &\leq \frac{|\psi(0)|+kM_2^*}{1-kM_1^*} + \frac{(k+1)p_1^* T^{\rho\omega}}{(1-kM_1^*)(1-p_3^*)\rho^\omega \Gamma(\omega+1)} \\ &+ \frac{(k+1)p_2^*}{(1-kM_1^*)(1-p_3^*)\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} q(s)\mathrm{d}s. \end{aligned}$$

Applying Lemma 2.8, we get

$$q(x) \leq \left[\frac{|\psi(0)|+kM_2^*}{1-kM_1^*} + \frac{(k+1)p_1^* T^{\rho\omega}}{(1-kM_1^*)(1-p_3^*)\rho^\omega \Gamma(\omega+1)}\right] \times \left[1 + \frac{\lambda(k+1)p_2^* T^{\rho\omega}}{(1-kM_1^*)(1-p_3^*)\rho^\omega \Gamma(\omega+1)}\right] := \tilde{A},$$

where $\lambda = \lambda(\omega)$ a constant. If $x^* \in [-r,0]$, then $q(x) = \|\psi\|_{\mathfrak{PC}}$, thus for any $x \in [-r,T]$, $\|y\|_{\mathfrak{PC}_1} \leq q(x)$, we get

$$\|y\|_{\mathfrak{PC}_1} \leq \max\{\|\psi\|_{\mathfrak{PC}}, \tilde{A}\},$$

which implies the set G is bounded. From Schaefer's fixed point theorem, we conclude that M has atleast one fixed point which is a solution of the problem (1.1). □

4. Ulam-Hyers-Rassias stability

Now, we present the following Ulam-Hyers-Rassias stable result.

Theorem 4.1. *Assume that* $(A1)$-$(A3)$, (3.3) *and*

(A6) There exists a nondecreasing function $\alpha \in \mathfrak{PC}(\mathfrak{J}, \mathbb{R}_+)$ *and there exists* $\mu_\alpha > 0$ *such that for any* $x \in \mathfrak{J}$*:*

$${}^\rho I^\omega \alpha(x) \leq \mu_\alpha \alpha(x),$$

are satisfied, then the problem (1.1) *is Ulam-Hyers-Rassias stable with respect to* (α, ϕ).

Proof. Let $v \in \mathfrak{PC}([-r,T], \mathbb{R})$ be a solution of the inequality (2.4). Denote by u the unique solution of the problem

$$\begin{cases} {}^\rho D_{x_m}^\omega u(x) = h(x, u_x, {}^\rho D_{x_m}^\omega u(x)), & \text{for each } x \in (x_m, x_{m+1}],\ m = 1, \ldots k; \\ \Delta u|_{x=x_m} = I_m(u_{x_m^-}), & m = 1, \ldots, k; \\ u(x) = v(x) = \psi(x), & x \in [-r, 0], \end{cases}$$

using Lemma 3.2, we obtain for each $x \in (x_m, x_{m+1}]$,

$$\begin{aligned} u(x) &= \psi(0) + \sum_{i=1}^m I_i(u_{x_i^-}) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{i=1}^m \int_{x_{i-1}}^{x_i} (x_i^\rho - s^\rho)^{\omega-1} s^{\rho-1} g(s)\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} g(s)\mathrm{d}s,\ x \in (x_m, x_{m+1}], \end{aligned}$$

where $g \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ be such that

$$g(x) = h(x, u_x, g(x)).$$

Since v is a solution of the inequality (2.4) and by Remark 2.15, we get

$$\begin{cases} {}^\rho D_{x_m}^\omega v(x) = h(x, v_x, {}^\rho D_{x_m}^\omega v(x)) + \sigma(x), & x \in (x_m, x_{m+1}],\ m = 1, \ldots k; \\ \Delta v|_{x=x_m} = I_m(v_{x_m^-}) + \sigma_m, & m = 1, \ldots, k. \end{cases} \tag{4.1}$$

Clearly the solution of (4.1) is given by,

$$\begin{aligned} v(x) &= \psi(0) + \sum_{i=1}^m I_i(v_{x_i^-}) + \sum_{i=1}^m \sigma_i \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{i=1}^m \int_{x_{i-1}}^{x_i} (x_i^\rho - s^\rho)^{\omega-1} s^{\rho-1} f(s)\mathrm{d}s + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{i=1}^m \int_{x_{i-1}}^{x_i} (x_i^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s)\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} f(s)\mathrm{d}s + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sigma(s)\mathrm{d}s,\ x \in (x_m, x_{m+1}], \end{aligned}$$

where $f \in \mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that $f(x) = h(x, v_x, f(x))$. Hence for each $x \in (x_m, x_{m+1}]$, we get,

$$\begin{aligned}
|v(x)-u(x)| &\leq \sum_{i=1}^{m}|\sigma_i| + \sum_{i=1}^{m}\left|I_i(v_{x_i^-}) - I_i(u_{x_i^-})\right| \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho - s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho - s^\rho)^{\omega-1}s^{\rho-1}|\sigma(s)|\,\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}|\sigma(s)|\,\mathrm{d}s.
\end{aligned}$$

Thus,

$$\begin{aligned}
|v(x)-u(x)| &\leq k\varepsilon\phi + (k+1)\varepsilon\mu_\alpha\alpha(x) + \sum_{i=1}^{m}c_3\left\|v_{x_i^-} - u_{x_i^-}\right\|_{\mathfrak{PC}} \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho - s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,\mathrm{d}s \\
&+ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}|f(s)-g(s)|\,\mathrm{d}s.
\end{aligned}$$

By $(A2)$, we get

$$\begin{aligned}
|f(x)-g(x)| &= |h(x,v_x,f(x)) - h(x,u_x,g(x))| \\
&\leq c_1\|v_x - u_x\|_{\mathfrak{PC}} + c_2|f(x)-g(x)|.
\end{aligned}$$

Then,

$$|f(x)-g(x)| \leq \frac{c_1}{1-c_2}\|v_x - u_x\|_{\mathfrak{PC}}.$$

Therefore, for each $x \in \mathfrak{J}$,

$$\begin{aligned}
|v(x)-u(x)| &\leq k\varepsilon\phi + (k+1)\varepsilon\mu_\alpha\alpha(x) + \sum_{i=1}^{m}c_3\left\|v_{x_i^-} - u_{x_i^-}\right\|_{\mathfrak{PC}} \\
&+ \frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\sum_{i=1}^{m}\int_{x_{i-1}}^{x_i}(x_i^\rho - s^\rho)^{\omega-1}s^{\rho-1}\|v_s - u_s\|_{\mathfrak{PC}}\,\mathrm{d}s \\
&+ \frac{c_1\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_{x_m}^{x}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}\|v_s - u_s\|_{\mathfrak{PC}}\,\mathrm{d}s.
\end{aligned}$$

Thus,

$$\begin{aligned}
|v(x)-u(x)| &\leq \varepsilon(\phi+\alpha(x))(k+(k+1)\mu_\alpha) + \sum_{0<x_i<x}c_3\left\|v_{x_i^-} - u_{x_i^-}\right\|_{\mathfrak{PC}} \\
&+ \frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_0^x(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}\|v_s - u_s\|_{\mathfrak{PC}}\,\mathrm{d}s. \qquad (4.2)
\end{aligned}$$

Now, we consider the function q_1 defined by

$$q_1(x) = \sup\{|v(s)-u(s)| : -r \leq s \leq x\},\ 0 \leq x \leq T,$$

then, there exists $x^* \in [-r,T]$ such that $q_1(x) = |v(x^*) - u(x^*)|$. If $x^* \in [-r,0]$, then $q_1(x) = 0$. If $x^* \in [0,T]$, then by the equation (4.2), we get

$$\begin{aligned}
q_1(x) &\leq \varepsilon(\phi+\alpha(x))(k+(k+1)\mu_\alpha) + \sum_{0<x_i<x}c_3q_1(x_i^-) \\
&+ \frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}\int_0^x(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}q_1(s)\mathrm{d}s.
\end{aligned}$$

Applying Lemma 2.9, we have,

$$\begin{aligned}
q_1(x) &\leq \varepsilon(\phi+\alpha(x))(k+(k+1)\mu_\alpha) \times \left[\prod_{0<x_i<x}(1+c_3)\exp\left(\int_0^x\frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)\Gamma(\omega)}(x^\rho - s^\rho)^{\omega-1}s^{\rho-1}\mathrm{d}s\right)\right] \\
&\leq l_\alpha\varepsilon(\phi+\alpha(x)),
\end{aligned}$$

where

$$\begin{aligned} l_\alpha &= (k+(k+1)\mu_\alpha)\times\left[\prod_{i=1}^{k}(1+c_3)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right)\right] \\ &= (k+(k+1)\mu_\alpha)\left[(1+c_3)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right)\right]^k. \end{aligned}$$

Thus, the problem (1.1) is Ulam-Hyers-Rassias stable with respect to (α,ϕ). Hence the proof is complete. □

Now, we present the following Ulam-Hyers stable result.

Theorem 4.2. *Assume that* $(A1)$-$(A3)$ *and* (3.3) *are satisfied, then the problem* (1.1) *is Ulam-Hyers stable.*

Proof. Let $v\in\mathfrak{PC}([-r,T],\mathbb{R})$ be a solution of (2.2). Denote by u the unique solution of the problem.

$$\begin{cases} {}^\rho D^\omega_{x_m}u(x)=h(x,u_x,{}^\rho D^\omega_{x_m}u(x)), & x\in(x_m,x_{m+1}],\, m=1,\dots k;\\ \Delta u|_{x=x_m}=I_m(u_{x_m^-}), & m=1,\dots,k;\\ u(x)=v(x)=\psi(x), & x\in[-r,0]. \end{cases}$$

From the proof of the Theorem 4.1, we get

$$q_1(x) \le \sum_{0<x_i<x} c_3q_1(x_i^-)+k\varepsilon+\frac{\varepsilon(k+1)T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}+\frac{c_1(k+1)\rho^{1-\omega}}{(1-c_2)\Gamma(\omega+1)}\int_0^x(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}q_1(s)\mathrm{d}s.$$

Applying Lemma 2.9, we have

$$\begin{aligned} q_1(x) &\le \varepsilon\left(\frac{k\rho^\omega\Gamma(\omega+1)+(k+1)T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}\right)\times\left[\prod_{0<x_i<x}(1+c_3)\exp\left(\int_0^x\frac{c_1(k+1)\rho^{1-\omega}(x^\rho-s^\rho)^{\omega-1}}{(1-c_2)\Gamma(\omega)}s^{\rho-1}\mathrm{d}s\right)\right] \\ &\le l_\alpha\varepsilon, \end{aligned}$$

where,

$$\begin{aligned} l_\alpha &= \left(\frac{k\rho^\omega\Gamma(\omega+1)+(k+1)T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}\right)\left[\prod_{i=1}^{k}(1+c_3)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right)\right] \\ &= \left(\frac{k\rho^\omega\Gamma(\omega+1)+(k+1)T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}\right)\left[(1+c_3)\exp\left(\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\right)\right]^k, \end{aligned}$$

which completes the proof of the theorem. □

Moreover, if we set $\gamma(\varepsilon)=l_\alpha\varepsilon$; $\gamma(0)=0$, then the problem (1.1) is generalized Ulam-Hyers stable.

5. Examples

Example 5.1. *Consider the following Katugampola-type impulsive problem,*

$$\begin{cases} {}^\rho D^{\frac{1}{2}}_{x_m}u(x)=\frac{e^{-x}}{(22+e^x)}\left[\frac{u_x}{1+u_x}-\frac{{}^\rho D^{\frac{1}{2}}_{x_m}u(x)}{1+{}^\rho D^{\frac{1}{2}}_{x_m}u(x)}\right], & \textit{for each } x\in\mathfrak{J}_0\cup\mathfrak{J}_1,\\ \Delta u|_{x=\frac{1}{2}}=\frac{u(\frac{1}{2}^-)}{20+u(\frac{1}{2}^-)}, & \\ u(x)=\psi(x), & x\in[-r,0],\ r>0, \end{cases} \tag{5.1}$$

where $\psi\in\mathfrak{PC}([-r,0],\mathbb{R})$, $\mathfrak{J}_0=[0,\frac{1}{2}]$, $\mathfrak{J}_1=(\frac{1}{2},1]$, $x_0=0$, *and* $x_1=\frac{1}{2}$.

Let

$$h(x,u_1,u_2)=\frac{e^{-x}}{(22+e^x)}\left[\frac{u_1}{1+u_1}-\frac{u_2}{1+u_2}\right],$$

$x\in[0,1]$, $u_1\in\mathfrak{PC}([-r,0],\mathbb{R})$ and $u_2\in\mathbb{R}$. Clearly, the function h is jointly continuous.
Let $u_1,\bar{u_1}\in\mathfrak{PC}([-r,0],\mathbb{R})$, $u_2,\bar{u_2}\in\mathbb{R}$ and $x\in[0,1]$:

$$\begin{aligned} |h(x,u_1,u_2)-h(x,\bar{u_1},\bar{u_2})| &\le \frac{e^{-x}}{22+e^x}\left(\|u_1-\bar{u_1}\|_{\mathfrak{PC}}+|u_2-\bar{u_2}|\right) \\ &\le \frac{1}{23}\left(\|u_1-\bar{u_1}\|_{\mathfrak{PC}}+|u_2-\bar{u_2}|\right). \end{aligned}$$

Hence the condition $(A2)$ is satisfied with $c_1=c_2=\frac{1}{23}$. And let,

$$I_1(u_1)=\frac{u_1}{20+u_1},\ u_1\in\mathfrak{PC}([-r,0],\mathbb{R})$$

Let $u_1,u_2 \in \mathfrak{PC}([-r,0],\mathbb{R})$, then we have,

$$\begin{aligned}|I_1(u_1)-I_2(u_2)| &= \left|\frac{u_1}{20+u_1}-\frac{u_2}{20+u_2}\right| \\ &\leq \frac{1}{20}\|u_1-u_2\|_{\mathfrak{PC}}.\end{aligned}$$

Let us assume $k=1, T=1, \rho=1, \omega=\frac{1}{2}, c_1=c_2=\frac{1}{23}, c_3=\frac{1}{20}$, Substitute these values in the inequality (3.3), we get

$$kc_3+\frac{(k+1)c_1T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)}=0.2551<1,$$

It follows from Theorem 3.3, we get that the problem (5.1) has a unique solution on $\mathfrak{J}$. Now, we consider for any $x\in[0,1]$, $\alpha(x)=x$, $\phi=1, \rho=1$. Since

$$\begin{aligned}{}^{\rho}I^{\omega}\alpha(x) &= \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x (x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}\mathrm{d}s \\ &= \frac{2}{\pi}\int_0^x (x^{\rho}-s^{\rho})^{\frac{-1}{2}}\mathrm{d}s \\ &\leq \frac{2x}{\sqrt{\pi}},\end{aligned}$$

then the condition $(A6)$ is satisfied with $\mu_{\alpha}=\frac{2}{\sqrt{\pi}}$. Therefore, we get that the problem (5.1) is Ulam-Hyers-Rassias stable with respect to (α,ϕ).

Example 5.2. *Consider the following Katugampola-type impulsive problem,*

$$\begin{cases}{}^{\rho}D^{\frac{1}{2}}_{x_m}u(x)=\dfrac{2+|u_x|+\left|{}^{\rho}D^{\frac{1}{2}}_{x_m}u(x)\right|}{110e^{x+3}\left(1+|u_x|+\left|{}^{\rho}D^{\frac{1}{2}}_{x_m}u(x)\right|\right)}, & \text{for each } x\in\mathfrak{J}_0\cup\mathfrak{J}_1,\\ \Delta u|_{x=\frac{1}{3}}=\dfrac{\left|u(\frac{1}{3}^-)\right|}{8+\left|u(\frac{1}{3}^-)\right|}, & \\ u(x)=\psi(x), & x\in[-r,0],\ r>0,\end{cases} \tag{5.2}$$

where $\psi\in\mathfrak{PC}([-r,0],\mathbb{R})$, $\mathfrak{J}_0=[0,\frac{1}{3}]$, $\mathfrak{J}_1=(\frac{1}{3},1]$, $x_0=0$, *and* $x_1=\frac{1}{3}$.

Let

$$h(x,u_1,u_2)=\frac{2+|u_1|+|u_2|}{110e^{x+3}(1+|u_1|+|u_2|)},\ x\in[0,1],\ u_1\in\mathfrak{PC}([-r,0],\mathbb{R}) \text{ and } u_2\in\mathbb{R}.$$

Clearly, the function h is jointly continuous. For any $u_1,\bar{u_1}\in\mathfrak{PC}([-r,0],\mathbb{R})$, $u_2,\bar{u_2}\in\mathbb{R}$ and $x\in[0,1]$:

$$|h(x,u_1,u_2)-h(x,\bar{u_1},\bar{u_2})|\leq\frac{1}{110e^3}\left(\|u_1-\bar{u_1}\|_{\mathfrak{PC}}+|u_2-\bar{u_2}|\right).$$

Hence the condition $(A2)$ is satisfied with $c_1=c_2=\frac{1}{110e^3}$. We have, for each $x\in[0,1]$,

$$|h(x,u_1,u_2)|\leq\frac{1}{110e^{x+3}}\left(2+\|u_1\|_{\mathfrak{PC}}+|u_2|\right).$$

Thus, the condition $(A4)$ is satisfied with $p_1(x)=\frac{1}{55e^{x+3}}$ and $p_2(x)=p_3(x)=\frac{1}{110e^{x+3}}$. Let

$$I_1(u_1)=\frac{|u_1|}{8+|u_1|},\ u_1\in\mathfrak{PC}([-r,0],\mathbb{R}).$$

We have, for each $u_1\in\mathfrak{PC}([-r,0],\mathbb{R})$,

$$|I_1(u_1)|\leq\frac{1}{8}\|u_1\|_{\mathfrak{PC}}+1.$$

Thus, the condition $(A5)$ is satisfied with $M_1^*=\frac{1}{8}$ and $M_2^*=1$. It follows from Theorem 3.4 that the problem (5.2) has at least one solution on $\mathfrak{J}$.

6. Conclusion

In this article, with the help of standard fixed point theorem of Schaefer's and Banach contraction type, we successfully developed existence of solutions of Katugampola-Caputo type implicit fractional differential equations with impulses. The obtained conditions ensure that the existence of at least one solution to the proposed problem. Further different kinds of Ulam-Hyers and Ulam-Hyers-Rassias stability have been investigated.

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