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On Markowitz Geometry

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Abstract

By Markowitz geometry we mean the intersection theory of ellipsoids and affine subspaces in a real finite-dimensional linear space. In the paper we give a meticulous and self-contained treatment of this arch-classical subject, which lays a solid mathematical groundwork of Markowitz mean-variance theory of efficient portfolios in economics.

1. Introduction and notation

1.1. Introduction

In this paper we solve the following extremal problem: Given a positive dimensional affine subspace $C \subset \mathbb{R}^n$, a linear form π which is not constant on C, and a positive definite quadratic form v on $\mathbb{R}^n$, find all points $x_0 \in C$ such that

$$\pi(x_0) = \max_{x\in C, v(x)\leq v(x_0)} \pi(x) \text{ and } v(x_0) = \min_{x\in C,\pi(x)\geq \pi(x_0)} v(x). \tag{1.1}$$

It turns out that the locus of solutions of (1.1) is a ray E in C whose endpoint x_0 is the foot of the perpendicular ρ_0 from the origin O of the coordinate system to the affine space C (perpendicularity is with respect to the scalar product obtained from v via polarization). Let h_r be the hyperplane with equation $\pi(x) = r$, $r \in \mathbb{R}$, and let ρ_r be the perpendicular from O to the affine subspace $C\cap h_r$. If $\pi(x_0) = r_0$, $v(x_0) = a_0$, then $E = \{\rho_r \mid r \geq r_0\}$, $\rho_0 = \rho_{r_0}$, and the levels $r = \pi(x)$ and $a = v(x)$ are quadratically related along E: $a = cr^2$.
Let $x = {}^t(x_1,\dots,x_n)$ be the generic vector in $\mathbb{R}^n$, let M be a proper subset of the set $[n] = \{1,\dots,n\}$ of indices, and let $C = \cap_{j\in M} h^{(j)}$, where $h^{(j)}$ are linearly independent hyperplanes with equations

$$\pi^{(j)}(x) = \tau_j,\ \tau_j \in \mathbb{R}, j \in M. \tag{1.2}$$

In case the hyperplane $\Pi = \{x \mid x_1 + \cdots + x_n = 1\}$ is one of $h^{(j)}$'s, we may interpret $x \in C$ as an n-assets financial portfolio, subject to the linear constraints (1.2). Next, under certain conditions, see 4.2, we may interpret $\pi(x)$ as the expected return on the portfolio x and $v(x)$ as its risk. Finally, we may interpret the elements of E as efficient portfolios from Markowitz mean-variance theory in economics, considered from purely geometrical point of view. The famous pioneering work [1] is written in this fashion and the condition for nonnegativity of the variables (due to lack of short sales) distorts the picture there and forces the use of variants of simplex method in Markowitz's monograph [2]. Thus, instead of the ray E of efficient portfolios, we have to examine a more sophisticated piecewise set E_M of linear segments enclosed in the compact trace Δ of the unit simplex in Π on C. If $x_0 \in E_M \backslash E\cap\Delta$, then

$$\pi(x_0) < \max_{x\in C, v(x)\leq v(x_0)} \pi(x) \text{ or } v(x_0) > \min_{x\in C,\pi(x)\geq \pi(x_0)} v(x),$$

that is, the maximum $\pi(x_0)$ of the expected return decreases or the minimum $v(x_0)$ of the risk increases, which is our point of departure.
In section 1, Theorem 2.3, we show that the trace $Q_a\cap C$ of an ellipsoid Q_a with equation $v(x) = a$ in $\mathbb{R}^n$ on the affine space C is again an ellipsoid in case $a \geq \gamma_M(\tau)$, where $\gamma_M(\tau)$ is a positive definite quadratic form in the variables $\tau = (\tau_j)_{j\in M} \in \mathbb{R}^M$. The center of the ellipsoid $Q_a\cap C$ is the foot of the perpendicular ρ_0 from O to C, and, moreover, we find its equation in terms of appropriate coordinates on C.

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The inequality $a \geq \gamma_M(\tau)$ determines an "elliptic" cone $\hat{\gamma}_M$ in $\mathbb{R} \times \mathbb{R}^M$, which is the base of the bundle ξ described in Theorem 2.11. By dragging the ellipsoids $a = \gamma_M(\tau)$ "upward" (a is increasing) we establish a real algebraic variety Γ_M which is the frontier of $\hat{\gamma}_M$ and branch locus of ξ. The fibres of ξ over the points in the interior of $\hat{\gamma}_M$ are ellipsoids which degenerate into their centers over Γ_M. Using this bundle, we obtain that the image (the shadow) of an ellipsoid in $\mathbb{R}^n$ via projection parallel to some subspace, is again an ellipsoid — see Proposition 2.14.

In section 2 we prove some extremal properties of the tangential points of members of a family of eccentric ellipsoids and parallel hyperplanes in $\mathbb{R}^n$. These two sections stick together in section 3 where we prove that the ray E is the locus of all efficient Markowitz portfolios and give interpretation of the geometrical results in terms of Markowitz mean-variance theory.

1.2. Notation

For any positive integer n we identify the members of the real linear space $\mathbb{R}^n$ with matrices of type $n \times 1$: $x = {}^t(x_1, \ldots, x_n)$, where the sign ^{t} means the transpose of a matrix. We set $O = {}^t(0, \ldots, 0) \in \mathbb{R}^n$ and denote by $(e_i)_{i=1}^n$ the standard basis in $\mathbb{R}^n$. Say that $M = \{j_1, \ldots, j_m\}$, $j_1 < \cdots < j_m$, be a proper subset of the set of indices $[n] = \{1, \ldots, n\}$. Given a vector $x = {}^t(x_1, \ldots, x_n)$, we denote by $x^{(M)}$ the vector ${}^t(x_{j_1}, \ldots, x_{j_m}) \in \mathbb{R}^M$. Moreover, indexed Greek letters $\tau^{(M)}$, etc., mean vectors ${}^t(\tau_{j_1}, \ldots, \tau_{j_m})$, etc., from the linear space $\mathbb{R}^M$. In case K is a proper subset of the set M and we fix all τ_j, $j \in K$, and vary τ_j, $j \in L$, where $L = M \setminus K$, then, with some abuse of notation (the fixed components are supposed to be known), we write $\tau^{(M)} = \tau^{(L,K)}$.

Given a symmetric $n \times n$ matrix Q, by $Q^{(M)}$ we denote the principal $m \times m$ submatrix of Q, obtained by suppressing the rows and columns with indices which are not in M.

For a positive definite quadratic form $v(x) = {}^txQx$ on $\mathbb{R}^n$ with matrix Q we denote $Q_a = \{x \in \mathbb{R}^n \mid v(x) = a\}$, $a \geq 0$. The set Q_a is an ellipsoid with center O in $\mathbb{R}^n$ for all $a > 0$. In case $n = 1$ the "ellipsoid" Q_a consists of two (possibly coinciding) points. We extend this terminology by defining the singleton $\{O\}$ to be an "ellipsoid" when $a = 0$ as well as in the case of zero-dimensional linear space.

For any $a \geq 0$ we denote $Q_{\leq a} = \{x \in \mathbb{R}^n \mid v(x) \leq a\}$ and $Q_{<a} = \{x \in \mathbb{R}^n \mid v(x) < a\}$. Note that $Q_{\leq a}$ and $Q_{<a}$ are strictly convex sets.

We let $\pi(x) = p_1x_1 + \cdots + p_nx_n$ be a linear form and let us denote by h_r the hyperplane in $\mathbb{R}^n$, defined by the equation $\pi(x) = r$, $r \in \mathbb{R}$. Let $h_r(\leq)$ denote the half-space $\{x \in \mathbb{R}^n \mid \pi(x) \leq r\}$. The meaning of notation $h_r(\geq)$, $h_r(<)$, and $h_r(>)$ is clear.

The standard scalar product $(x,y) = {}^txy$ in $\mathbb{R}^n$ produces the standard norm $\|x\|$ with $\|x\|^2 = (x,x)$. We set $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ (the unit sphere).

The scalar product $\langle x,y \rangle = {}^txQy$ in $\mathbb{R}^n$ produces the Q-norm $\|x\|_Q$ with $\|x\|_Q^2 = \langle x,x \rangle = v(x)$ and the Q-distance $dist_Q(x,y) = \|x-y\|_Q$. Thus, the ellipsoid Q_a is a Q-sphere with Q-radius $\sqrt{a}$. Two vectors x and y are said to be Q-perpendicular, if $\langle x,y \rangle = 0$.

Throughout the rest of the paper we assume that n is a positive integer and m is a nonnegative integer with $m < n$. Moreover, we suppose that if a proper subset M of the set $[n]$ of indices is given as a list: $M = \{j_1, \ldots, j_m\}$, then $j_1 < \cdots < j_m$.

2. Ellipsoids and affine subspaces

2.1. Intersections of quadric hypersurfaces and affine subspaces

Let $M \subset [n]$ be a set of indices of size m, $M = \{j_1, \ldots, j_m\}$, and let $(h^{(j)})_{j \in M}$ be a family of linearly independent affine hyperplanes in $\mathbb{R}^n$. The system of coordinates can be chosen in such a way that the hyperplane $h^{(j)}$ has equation $x_j = \tau_j$, $\tau_j \in \mathbb{R}$. We denote by $h(\tau^{(M)})$ the intersection $\cap_{j \in M} h^{(j)}$. The family $\{h(\tau^{(M)}) \mid \tau^{(M)} \in \mathbb{R}^M\}$ consists of all $(n-m)$-dimensional affine spaces in $\mathbb{R}^n$, which are orthogonal to the m-dimensional vector subspace generated by the vectors e_j, $j \in M$.

Let $Q = (q_{ij})_{i,j=1}^n$ be a symmetric matrix. For any $j \in M$ we denote by $\rho_{-,j}^{(Q;M^c)}$ the j-th column of the $(n-m) \times n$ matrix obtained from Q by deleting the rows indexed by the elements of M. Thus, $\rho_{-,j}^{(Q;M^c)}$ is a vector in $\mathbb{R}^{n-m}$ with components $\rho_{i,j}^{(Q;M^c)} = q_{ij}$, $i \in M^c$. Given a vector $\tau^{(M)} \in \mathbb{R}^M$, $\tau^{(M)} = {}^t(\tau_{j_1}, \ldots, \tau_{j_m})$, we set $\rho_{-,\tau^{(M)}}^{(Q;M^c)} = \sum_{k=1}^m \tau_{j_k} \rho_{-,j_k}^{(Q;M^c)}$. By

$$\alpha_{(Q;M)}(x) = \sum_{j,k \in M}^n q_{jk}x_jx_k$$

we denote the quadratic form which corresponds to the principal submatrix $Q^{(M)}$ of Q.

Let $M^c = \{i_1, \ldots, i_{n-m}\}$. In case the submatrix $Q^{(M^c)}$ is invertible, let

$$x^{(M^c)} = {}^t(c_{i_1}^{(Q;M^c)}(\tau^{(M)}), \ldots, c_{i_{n-m}}^{(Q;M^c)}(\tau^{(M)}))$$

be the solution of the matrix equation

$$Q^{(M^c)}x^{(M^c)} = -\rho_{-,\tau^{(M)}}^{(Q;M^c)}. \tag{2.1}$$

We set

$$c^{(Q;M^c)}(\tau^{(M)}) = {}^t(c_1^{(Q;M^c)}(\tau^{(M)}), \ldots, c_n^{(Q;M^c)}(\tau^{(M)})),$$

where $c_j^{(Q;M^c)}(\tau^{(M)}) = \tau_j$ for $j \in M$. In particular, $c^{(Q;M^c)}(\tau^{(M)}) \in h(\tau^{(M)})$. In case $L \subset M$, $L = \{\ell_1, \ldots, \ell_\lambda\}$, we set

$$c_L^{(Q;M^c)}(\tau^{(M)}) = {}^t(c_{\ell_1}^{(Q;M^c)}(\tau^{(M)}), \ldots, c_{\ell_\lambda}^{(Q;M^c)}(\tau^{(M)})).$$

Note that if $M = \emptyset$, then $c^{(Q;[n])}(\tau^{(\emptyset)}) = 0$. We write $c^{(Q;M^c)}(\tau^{(M)}) = c^{(M^c)}(\tau)$, and, similarly, $\rho_{-,\tau^{(M)}}^{(Q;M^c)} = \rho_{-,\tau}^{(M^c)}$, etc., when the context allows that.

Since the vector $\rho_{-,\tau}^{(M^c)} \in \mathbb{R}^{n-m}$ depends linearly on $\tau^{(M)}$, the map

$$\psi_M\colon \mathbb{R}^M \to \mathbb{R}^n,\ \tau^{(M)} \mapsto c^{(M^c)}(\tau^{(M)}),$$

is an injective homomorphism of linear spaces. We set $Eff^{(Q;M^c)} = \psi_M\left(\mathbb{R}^M\right)$ and note that $Eff^{(Q;M^c)}$ is an m-dimensional subspace of $\mathbb{R}^n$. Below we use also the short notation $Eff^{(M^c)} = Eff^{(Q;M^c)}$ when the matrix Q is given by default.

Lemma 2.1. *Let K and M be proper subsets of the set of indices $[n]$ with $K \subset M$. Let $Q^{(K^c)}$ and $Q^{(M^c)}$ be invertible submatrices of Q. The following two statements are equivalent:*
(i) *One has $c^{(K^c)}(\tau) \in h(\tau^{(M)})$.*
(ii) *One has $c^{(K^c)}(\tau) = c^{(M^c)}(\tau)$.*

Proof. We have $h(\tau^{(M)}) \subset h(\tau^{(K)})$ and let us assume $K \neq M$. It is enough to prove that (i) implies (ii). Let $c^{(K^c)}(\tau) \in h(\tau^{(M)})$. We remind that the hyperplane $h^{(j)}$ has equation $h^{(j)}\colon x_j = \tau_j$ for any $j \in M$. In particular, for each $j \in K^c \setminus M^c = M \setminus K$ we obtain $c_j^{(K^c)}(\tau) = \tau_j$. Therefore $c^{(K^c)}(\tau)_{M^c}$ is a solution of the equation (2.1). The uniqueness of this solution implies $c^{(K^c)}(\tau) = c^{(M^c)}(\tau)$. □

Corollary 2.2. *One has*

$$Eff^{(K^c)} \cap h\left(\tau^{(M)}\right) \subset Eff\,(M^c)\,.$$

Now, let us fix all components of $\tau^{(M)} \in \mathbb{R}^M$, except $r = \tau_\ell$ for some $\ell \in M$, so $\tau^{(M)} = \tau^{(\{\ell\},M\setminus\{\ell\})}(r)$. When we vary $r \in \mathbb{R}$, then $\tau^{(\{\ell\},M\setminus\{\ell\})}(r)$ describes a straight line in $\mathbb{R}^M$ and hence $c^{(M^c)}(\tau^{(\{\ell\},M\setminus\{\ell\})}(r))$ describes a straight line in $\mathbb{R}^n$ which we denote by $Eff_\ell^{(Q;M^c)}$. Its ray $\{c^{(M^c)}(\tau^{(\{\ell\},M\setminus\{\ell\})}(r)) \mid r \geq b\}$, $b \in \mathbb{R}$, is denoted by $Eff_{\ell b+}^{(Q;M^c)}$.
Let us set

$$\gamma_M^{(Q)}(\tau) = \alpha_{(Q;M)}(\tau) - \alpha_{(Q;M^c)}(c_{i_1}^{(Q;M^c)}(\tau), \ldots, c_{i_{n-m}}^{(Q;M^c)}(\tau)).$$

Since $\alpha_{(Q;\emptyset)}(x) = 0$ and $c_1^{(Q;[n])}(\tau) = \cdots = c_n^{(Q;[n])}(\tau) = 0$, we obtain $\gamma_\emptyset^{(Q)}(\tau) = 0$. We write $\gamma_M^{(Q)}(\tau) = \gamma_M(\tau)$ when the matrix Q is known from the context.
It follows from Lemma A.2, (i), that $\gamma_M(\tau)$ is a quadratic form in $\tau^{(M)}$.
Let us move the origin of the coordinate system by the substitution $x = z(\tau^{(M)}) + c^{(M^c)}(\tau^{(M)})$. Then the restrictions of the components of both $x^{(M^c)}$ and $z^{(M^c)}(\tau^{(M)})$ on $h(\tau^{(M)})$ are coordinate functions in this $(n-m)$-dimensional affine space.
Let $v(x) = {}^t xQx$ be the quadratic form produced by the symmetric nonzero $n \times n$-matrix Q. Thus, $Q_a\colon v(x) = a$ is a quadric in $\mathbb{R}^n$ for generic $a \in \mathbb{R}$ and the real variety $q_{a,\tau^{(M)}} = Q_a \cap h(\tau^{(M)})$ is defined in $h(\tau^{(M)})$ by the equation

$$^t x^{(M^c)} Q^{(M^c)} x^{(M^c)} + 2{}^t\rho_{-,\tau}^{(M^c)} x^{(M^c)} + \alpha_M(\tau) - a = 0. \tag{2.2}$$

Let us set

$$v^{(M^c)}(z(\tau^{(M)})) = {}^t z^{(M^c)}(\tau^{(M)}) Q^{(M^c)} z^{(M^c)}(\tau^{(M)}). \tag{2.3}$$

In case the principal submatrix $Q^{(M^c)}$ is invertible, Lemma A.3 implies that $v(x) = v^{(M^c)}(z(\tau^{(M)})) + \gamma_M(\tau^{(M)})$ on $h^{(M)}$, and in terms of z-coordinates the equation (2.2) has the form

$$v^{(M^c)}(z(\tau^{(M)})) = a - \gamma_M(\tau^{(M)}). \tag{2.4}$$

2.2. Intersections of ellipsoids and affine subspaces

Let $v(x) = {}^t xQx$ be a positive definite quadratic form produced by the symmetric (positive definite) $n \times n$-matrix Q. This being so, $Q_a\colon v(x) = a$ is an ellipsoid in $\mathbb{R}^n$ for $a > 0$, $Q_0 = \{0\}$, and $Q_a = \emptyset$ for $a < 0$. In particular, $Q^{(M^c)}$ is a principal, hence positive definite, submatrix of Q. Thus, the quadratic form (2.3) is positive definite.
In accord with (2.2) and (2.4), we establish parts (ii), (iii), and (iv) of the next theorem. Part (i) is proved in Lemma A.2, (ii).

Theorem 2.3. *Let the quadratic form $v(x) = {}^t xQx$ be positive definite.*
(i) *If $M \neq \emptyset$, then the quadratic form $\gamma_M(\tau)$ is positive definite.*
(ii) *If $a > \gamma_M(\tau)$, then $q_{a,\tau^{(M)}}$ is an ellipsoid in the $(n-m)$-dimensional vector space $h(\tau^{(M)})$ with center $c^{(M^c)}(\tau)$ and $Q^{(M^c)}$-radius $\sqrt{a - \gamma_M(\tau)}$.*
(iii) *If $a = \gamma_M(\tau)$, then $q_{a,\tau^{(M)}} = \{c^{(M^c)}(\tau)\}$.*
(iv) *If $a < \gamma_M(\tau)$, then the set $q_{a,\tau^{(M)}}$ is empty.*

Remark 2.4. We remind that ellipsoid in an one-dimensional affine subspace is a set consisting of two points and its center is the midpoint.

Remark 2.5. In accord with Lemma 3.2, the affine subspace $h(\tau^{(M)})$ is tangential to the ellipsoid Q_a, $a = \gamma_M(\tau)$, at the point $x = c^{(M^c)}(\tau)$.

Remark 2.6. In view of the previous remark, Lemma 2.1 has transparent geometrical meaning: If the subspace $h(\tau^{(M)})$ of $h(\tau^{(K)})$ passes through the point $x = c^{(K^c)}(\tau)$, then $h(\tau^{(M)})$ is also tangential to Q_a at x.

We obtain immediately the following corollary:

Corollary 2.7. (i) *For any $x \in h(\tau^{(M)})$ one has $v(x) \geq \gamma_M(\tau)$ and an equality holds if and only if $x = c^{(M^c)}(\tau)$.*
(ii) *The point $c^{(M^c)}(\tau) \in h(\tau^{(M)})$ is the foot of Q-perpendicular from the origin O to the affine subspace $h(\tau^{(M)})$ and one has*

$$dist_Q\left(O,h\left(\tau^{(M)}\right)\right) = c^{(M^c)}(\tau)_Q = \sqrt{\gamma_M(\tau)}.$$

Corollary 2.8. *Let K and L be disjoint subsets of M with $K \cup L = M$. One has*
(i) *If $a = \gamma_M(\tau^{(M)})$, then the trace $q_{a,\tau^{(K)}}$ of the ellipsoid Q_a on the affine space $h(\tau^{(K)})$ is nonempty and the affine subspace $h(\tau^{(M)}) \subset h(\tau^{(K)})$ is tangential to the ellipsoid $q_{a,\tau^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(M)})$.*

$$\text{(ii) } c^{(Q;M^c)}(\tau^{(M)}) = c^{(Q;K^c)}(\tau^{(K)}) + c^{(Q^{(K^c)};M^c)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$$

and

$$\text{(iii) } \gamma_M^{(Q)}(\tau^{(M)}) = \gamma_K^{(Q)}(\tau^{(K)}) + \gamma_L^{(Q^{(K^c)})}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)})).$$

Proof. Both assertions hold when one of the sets M, K, or L, is empty.
(i) The equalities

$$q_{a,\tau^{(M)}} = q_{a,\tau^{(K)}} \cap h(\tau^{(L)}) = q_{a,\tau^{(K)}} \cap h(\tau^{(M)}) = Q_a \cap h(\tau^{(M)})$$

and Theorem 2.3, (ii) – (iv), yield that under the condition $a = \gamma_M(\tau^{(M)})$ we have

$$q_{a,\tau^{(K)}} \cap h(\tau^{(M)}) = \{c^{(Q;M^c)}(\tau^{(M)})\}. \tag{2.5}$$

In particular, $a \geq \gamma_K(\tau^{(K)})$ and in this case $q_{a,\tau^{(K)}}$ is an ellipsoid in the vector space $h(\tau^{(K)})$ endowed with coordinate functions $(z_s^{(K^c)}(\tau^{(K)}))_{s\in K^c}$. The point $\{c^{(Q;K^c)}(\tau^{(K)})\}$ is both the origin of the coordinates and the center of the ellipsoid $q_{a,\tau^{(K)}}$ which has equation

$$^t z^{(K^c)}(\tau^{(K)})Q^{(K^c)}z^{(K^c)}(\tau^{(K)}) = a - \gamma_K(\tau^{(K)}).$$

Therefore we have

$$q_{a,\tau^{(K)}} = Q^{(K^c)}_{a-\gamma_K(\tau^{(K)})}.$$

Because of (2.5), the trace $h(\tau^{(M)})$ of $h(\tau^{(L)})$ on $h(\tau^{(K)})$ is tangential to $q_{a,\tau^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(M)})$ (Note that in case $q_{a,\tau^{(K)}} = \{c^{(Q;M^c)}(\tau^{(M)})\}$ we have $c^{(Q;M^c)}(\tau^{(M)}) = c^{(Q;K^c)}(\tau^{(K)})$ and $h(\tau^{(M)})$ is also tangential to $q_{a,\tau^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(M)})$ — see Remark 3.1).
(ii) The affine subspace $h(\tau^{(M)})$ is defined in $h(\tau^{(K)})$ by the equations $z_s^{(K^c)}(\tau^{(K)}) = \tau_s - c_s^{(Q;K^c)}(\tau^{(K)})$, $s \in L$ (we have $L \subset K^c$). Hence the difference $c^{(Q;M^c)}(\tau^{(M)}) - c^{(Q;K^c)}(\tau^{(K)})$ of points in the affine subspace $h(\tau^{(K)}) \subset \mathbb{R}^n$ coincides with the vector $c^{(Q^{(K^c)};M^c)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$ and we have obtained part (ii). The equalities $a - \gamma_K(\tau^{(K)}) = \gamma_L^{(Q^{(K^c)})}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$ and $a = \gamma_M(\tau^{(M)})$ yield assertion (iii).

□

Remark 2.9. Since the vector $c^{(Q^{(K^c)})}(\tau^{(K)})$ is Q-perpendicular to the affine subspace $h(\tau^{(K)})$ and since the vector $c^{(Q^{(K^c)};M^c)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$ lies in this subspace, part (ii) of the above corollary is Pythagorean theorem.

Remark 2.10. It follows from Theorem of three perpendiculars that the vector $c^{(Q^{(K^c)};M^c)}(\tau^{(L)} - c_L^{(Q;K^c)}(\tau^{(K)}))$ is Q-perpendicular to the affine subspace $h(\tau^{(M)})$.

2.3. A bundle

Let us consider the $(m+1)$-dimensional space $\mathbb{R} \times \mathbb{R}^M$ with generic vector $^t(a, \tau^{(M)})$, endowed with standard topology and let $\hat{\gamma}_M = \{^t(a, \tau^{(M)}) \in \mathbb{R} \times \mathbb{R}^M \mid a \geq \gamma_M(\tau)\}$. The set $\hat{\gamma}_M$ is the closed region in $\mathbb{R} \times \mathbb{R}^M$, which consists of all points above the graph Γ_M of the quadratic function $a = \gamma_M(\tau)$ when $M \neq \emptyset$ and $\hat{\gamma}_\emptyset = [0,\infty) \times \{0\}$. In all cases $pr_a(\hat{\gamma}_M) = [0,\infty)$. The set Γ_M is an algebraic variety (hence a closed set) in $\mathbb{R} \times \mathbb{R}^M$ and the difference $\tilde{\gamma}_M = \hat{\gamma}_M \backslash \Gamma_M$ is an open set, both being nonempty.
Let $\gamma_M(\tau) = {}^t\tau^{(M)} R \tau^{(M)}$, where R is a symmetric $M \times M$-matrix. In accord with Theorem 2.3, (i), in case $M \neq \emptyset$, the matrix R is positive definite. If $M = \emptyset$, then R is the empty matrix. Given $a \geq 0$, we set $R_a = \{\tau^{(M)} \in \mathbb{R}^M \mid \gamma_M(\tau^{(M)}) = a\}$ and note that R_a is an ellipsoid in $\mathbb{R}^M$. Any level set $\Gamma_{a,M} = \{^t(a, \tau^{(M)}) \in \mathbb{R} \times \mathrm{R}^M \mid a = \gamma_M(\tau)\}$, $a > 0$, is isomorphic to the ellipsoid R_a in $\mathbb{R}^M$, and $\Gamma_{0,M} = \{(0,0)\}$. Given $a \geq 0$, let us denote $Eff^{(a;M^c)} = \left\{x \in \mathbb{R}^n \mid x = c^{(M^c)}(\tau),\ ^t\left(a, \tau^{(M)}\right) \in \Gamma_{a,M}\right\}$. We define a morphism of real algebraic varieties by the rule

$$\varphi_M : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^M,\ x \mapsto {}^t\left(v(x), x^{(M)}\right).$$

Theorem 2.3 yields $\varphi_M(\mathbb{R}^n) = \hat{\gamma}_M$, we set $\Phi_M = \varphi_M^{-1}(\hat{\gamma}_M)$, and denote the restriction of φ_M on Φ_M by the same letter. Since $\varphi_M^{-1}(^t(a, \tau^{(M)})) = q_{a,\tau^{(M)}}$, we establish the following:

Theorem 2.11. *Let $\xi = (\Phi_M, \varphi_M, \hat{\gamma}_M)$ be the bundle defined by the map φ_M.*
(i) *The restriction $\xi_{|\tilde{\gamma}_M}$ is a fibration with fibres $\varphi_M^{-1}(^t(a, \tau^{(M)})) = q_{a,\tau^{(M)}}$, $^t(a, \tau^{(M)}) \in \tilde{\gamma}_M$, which are ellipsoids in $\mathbb{R}^{n-m}$ with centers $c^{(M^c)}(\tau)$.*
(ii) *The restriction $\xi_{|\Gamma_M}$ is an isomorphism of real algebraic m-dimensional varieties with inverse isomorphism $\Gamma_M \to Eff^{(M^c)}$, $^t(a, \tau^{(M)}) \mapsto c^{(M^c)}(\tau)$, which maps any level set $\Gamma_{a,M}$ onto $Eff^{(a;M^c)}$.*

Corollary 2.12. *The set $Eff^{(a;M^c)}$ is a real algebraic subvariety of Q_a, which is isomorphic via $\xi \mid \Gamma_M$ to the ellipsoid $\Gamma_{a,M}$.*

Taking into account Remark 2.5, we obtain immediately the following:

Corollary 2.13. *The family $\{h(\tau^{(M)}) \mid \tau^{(M)} \in \Gamma_{a,M}\}$ consists of all $(n-m)$-dimensional affine spaces in $\mathbb{R}^n$, which are both orthogonal to the m-dimensional vector subspace generated by the vectors e_j, $j \in M$, and tangential to the ellipsoid Q_a.*

2.4. A shadow

Let us denote by ζ_M the restriction of the second projection $pr_2\colon \mathbb{R}\times\mathbb{R}^M \to \mathbb{R}^M$ on $\hat{\gamma}_M$. The composition $\phi_M = \zeta_M \circ \varphi_M$ is the restriction on Φ_M of the projection of $\mathbb{R}^n$ parallel to the subspace W defined by $x^{(M)} = 0$: $\phi_M\colon \mathbb{R}^n \to W^\perp$, $\phi_M(x) = x^{(M)}$, and, moreover, $\phi_M^{-1}(\tau^{(M)}) = h(\tau^{(M)})$. Since the set $Eff^{(a;M^c)} \subset Q_a$ is mapped via ϕ_M onto the ellipsoid R_a in $\mathbb{R}^M$ and since the internal points of Q_a are mapped onto the internal points of R_a, we can formulate the result from Corollary 2.13 as solution of a shadow problem:

Proposition 2.14. *All $(n-m)$-dimensional affine spaces in $\mathbb{R}^n$ with common direction vector subspace W, which are also tangential to an ellipsoid Q_a in $\mathbb{R}^n$, intersect the orthogonal complement $W^\perp$ at the points of an ellipsoid R_a in $W^\perp \simeq \mathbb{R}^M$. All affine spaces in $\mathbb{R}^n$ which have nonempty intersection with the interior of Q_a and are parallel to W intersect $W^\perp$ at the internal points of R_a.*

3. Ellipsoids and hyperplanes

3.1. Ellipsoids and their tangent spaces

Let $v(x) = {}^t xQx$ be a positive definite quadratic form. The equation of the tangent space θ_{x_0} of the ellipsoid $Q_a\colon v(x) = a$, $a > 0$, at the point $x_0 \in Q_a$ is

$$\theta_{x_0}(x) = a,$$

where $\theta_{x_0}(x) = {}^t x_0 Qx$. For all $x \in Q_a$ we have $x \neq 0$ and since the matrix Q has rank n, we obtain $Qx_0 \neq 0$. In particular, θ_{x_0} is a hyperplane and Q_a is a smooth hypersurface in $\mathbb{R}^n$.

Remark 3.1. *The tangent space of the "ellipsoid" $Q_0 = \{O\}$ at its only point $x_0 = O$ is $\mathbb{R}^n$. In particular, any linear subspace of $\mathbb{R}^n$ is tangential to Q_0.*

Let $a > 0$ and let us fix a point $x_0 \in Q_a$. For any vector $u \in S^{n-1}$ we denote for short by L_u the line $\{z \in \mathbb{R}^n \mid z = x_0 + tu,\ t \in \mathbb{R}\}$.

Lemma 3.2. *One has*

$$L_u \cap Q_{\leq a} = \{x_0 + tu \mid 0 \leq t \leq -2\frac{\theta_{x_0}(u)}{v(u)}\},\ L_u \cap Q_a = \{x_0, x_0 - 2\frac{\theta_{x_0}(u)}{v(u)}u\}.$$

Proof. The inequality $v(x_0 + tu) \leq a$ is equivalent to $2\theta_{x_0}(u)t + v(u)t^2 \leq 0$ and the equality holds if and only if $t = 0$ or $t = -2\frac{\theta_{x_0}(u)}{v(u)}$. $\square$

Lemma 3.3. *Let $x_0 \in Q_a$.*
(i) *One has $Q_{\leq a} \subset \theta_{x_0}(\leq)$.*
(ii) *One has $Q_{\leq a} \cap \theta_{x_0} = Q_a \cap \theta_{x_0} = \{x_0\}$.*
(iii) *One has $Q_{\leq a} \backslash \{x_0\} \subset \theta_{x_0}(<)$.*

Proof. (i) Let $y \in Q_{\leq a}$, $y \neq x_0$, and let $y \in L_u$. In accord with Lemma 3.2, $y = x_0 + tu$ where $0 \leq t \leq -2\frac{\theta_{x_0}(u)}{v(u)}$. We have $\theta_{x_0}(y) = \theta_{x_0}(x_0) + t\theta_{x_0}(u) = a + t\theta_{x_0}(u) \leq a - 2\frac{(\theta_{x_0}(u))^2}{v(u)} \leq a$.
(ii) Let us suppose that there exists a point y, $y \neq x_0$, with $y \in Q_{\leq a} \cap \theta_{x_0}$ and let $u = \frac{1}{\|y-x_0\|}(y - x_0)$. Then $\theta_{x_0}(u) = 0$, $y \in L_u$, and Lemma 3.2 implies $L_u \cap Q_{\leq a} = \{x_0\}$ — a contradiction with $y \in L_u \cap Q_{\leq a}$. Now, because of the inclusions $\{x_0\} \subset Q_a \cap \theta_{x_0} \subset Q_{\leq a} \cap \theta_{x_0} = \{x_0\}$, part (ii) is proved.
Parts (i) and (ii) yield part (iii).

$\square$

We remind that h_r is a hyperplane in $\mathbb{R}^n$, defined by the equation $\pi(x) = r$, where $\pi(x)$ is a non-zero linear form, and $q_{a,r} = Q_a \cap h_r$.

Lemma 3.4. *Let $x_0 \in q_{a,r}$.*
(i) *If $Q_a \subset h_r(\leq)$, then $h_r = \theta_{x_0}$.*
(ii) *If $Q_{<a_0} \subset h_{r_0}(<)$, then $Q_a \subset h_r(\leq)$.*

Proof. (i) When y varies through $Q_a \backslash \{x_0\}$, then $u = \frac{1}{\|y-x_0\|}(y - x_0)$ varies bijectively through $S^{n-1} \cap \theta_{x_0}(<)$. On the other hand, since $Q_a \subset h_r(\leq)$, then $y \in Q_a \backslash \{x_0\}$ yields $\pi(y) \leq r$, that is, $\pi(x_0 - 2\frac{\theta_{x_0}(u)}{v(u)}u) \leq r$, and hence $\theta_{x_0}(u)\pi(u) \geq 0$ for all $u \in S^{n-1} \cap \theta_{x_0}(<)$. The last inequality also holds for all $u \in S^{n-1} \cap \theta_{x_0}(>)$ because $\theta_{x_0}(-u)\pi(-u) \geq 0$. Thus, we have $\theta_{x_0}(u)\pi(u) \geq 0$ for all $u \in S^{n-1}$, therefore for all vectors $u \in \mathbb{R}^n$. If the linear forms θ_{x_0} and π are not proportional, then after an appropriate change of the coordinates, θ_{x_0} and π can serve as coordinate functions in $\mathbb{R}^n$ — a contradiction.
(ii) Let $y \in Q_a$ and let us set $y_n = (1 - \frac{1}{n})y$ for any positive integer n. Then $y_n \in Q_{<a_0}$ and $\lim_{n\to\infty} y_n = y$. Since $Q_{<a_0} \subset h_{r_0}(<)$, we obtain $h_{r_0}(y_n) < r_0$, hence $h_{r_0}(y) \leq r_0$.

$\square$

3.2. Some extremal properties

Let $h_r\colon \pi(x)=r$ be a hyperplane in $\mathbb{R}^n$, $\pi(x)=p_1x_1+\cdots+p_nx_n$, and let us set $p={}^t(p_1,\ldots,p_n)$. We denote $q_{a,r}=Q_a\cap h_r$.

Lemma 3.5. *Let $x_0\in R^n\backslash\{0\}$, $a>0$, and $r>0$. The following four statements are equivalent:*
(i) *One has $x_0\in q_{a,r}$ and $Qx_0\in\mathbb{R}p$.*
(ii) *One has $rQx_0=ap$ and $a=r^2({}^tpQ^{-1}p)^{-1}$.*
(iii) *One has $x_0\in q_{a,r}$ and $\theta_{x_0}=h_r$.*
(iv) *One has $q_{a,r}=\{x_0\}$.*

Proof. (i) $\Longrightarrow$ (ii) Let $Qx_0=bp$, $b\in\mathbb{R}$. We have

$$a=v(x_0)={}^tx_0Qx_0={}^tx_0(bp)=b{}^tpx_0=b\pi(x_0)=br,$$

therefore $rQx_0=ap$. On the other hand, we obtain

$$a={}^tx_0Qx_0=\frac{a}{r}{}^tpQ^{-1}\frac{a}{r}p=\frac{a^2}{r^2}{}^tpQ^{-1}p,$$

hence $a=r^2({}^tpQ^{-1}p)^{-1}$.
(ii) $\Longrightarrow$ (i) We have $Qx_0\in\mathbb{R}p$, and, moreover, ${}^tx_0=\frac{a}{r}{}^tpQ^{-1}$. $\pi(x_0)={}^tpx_0={}^tx_0p=\frac{a}{r}{}^tpQ^{-1}p=\frac{a}{r}\frac{r^2}{a}=r$, hence $x_0\in h_r$. Finally, $v(x_0)={}^tx_0Qx_0=\frac{a}{r}{}^tpQ^{-1}Qx_0=\frac{a}{r}{}^tpx=\frac{a}{r}\pi(x_0)=a$, therefore $x_0\in Q_a$.
The equivalence of parts (i) and (iii) is straightforward. Part (iii) and Lemma 3.3, (ii), imply part (iv).
(iv) $\Longrightarrow$ (iii) Let $L=\{x_0+tz\mid t\in\mathbb{R}\}$, $z\neq 0$, be a line in h_r, that is, $\pi(z)=0$. The roots of the quadratic equation $v(x_0+tz)=a$ correspond to the intersection points of the line L and the ellipsoid Q_a. Taking into account that $v(x_0+tz)=v(x_0)+2\theta_{x_0}(z)t+v(z)t^2$, we obtain the equivalent equation $2\theta_{x_0}(z)t+v(z)t^2=0$. Since $q_{a,r}=\{x_0\}$, this quadratic equation has a double root $t=0$, that is, $\theta_{x_0}(z)=0$. Thus, we obtain $L\subset\theta_{x_0}$ and therefore $\theta_{x_0}=h_r$. □

Corollary 3.6. *Under conditions* (i) – (iv) *one has $\theta_{x_0}(x)=\frac{a}{r}\pi(x)$.*

Remark 3.7. If $x_0=0$, then parts (i), (ii), and (iv) of Lemma 3.5 hold for $a=r=0$.

Let us set $c_p=({}^tpQ^{-1}p)^{-1}$, $E_p^{(Q)}=\{(a,r)\mid a=c_pr^2,r\geq 0\}$, $x(a,r)=\frac{a}{r}Q^{-1}p$ for any $(a,r)\in E_p^{(Q)}$ with $r>0$, $x(0,0)=0$, and

$$Ef_p^{(Q)}=\{x\in\mathbb{R}^n\mid x=x(a,r),\ (a,r)\in E_p^{(Q)}\}.$$

Thus, the set $ef_p^{(Q)}$ consists of all vectors $x\in\mathbb{R}^n$ which satisfy the four equivalent conditions from Lemma 3.5. Note that $0\in ef_p^{(Q)}$ and if $x(a,r)\in ef_p^{(Q)}$, then $\{x(a,r)\}=q_{a,r}$. In other words, Lemma 3.5 implies

Corollary 3.8. *One has*

$$ef_p^{(Q)}=\cup_{r\geq 0,a=c_pr^2}q_{a,r}.$$

In case M is a singleton, Theorem 2.3 yields the following two corollaries:

Corollary 3.9. *Let $x,x_0\in ef_p^{(Q)}$, $x=x(a,r)$, $x_0=x(a_0,r_0)$.*
(i) *If $a=a_0$, then $q_{a,r_0}=\{x_0\}$.*
(ii) *If $a>a_0$, then q_{a,r_0} is an ellipsoid in the hyperplane h_{r_0}.*
(iii) *If $a<a_0$, then $q_{a,r_0}=\emptyset$.*

Corollary 3.10. *Let $x,x_0\in ef_p^{(Q)}$, $x=x(a,r)$, $x_0=x(a_0,r_0)$.*
(i) *If $r=r_0$, then $q_{a_0,r}=\{x_0\}$.*
(ii) *If $r<r_0$, then $q_{a_0,r}$ is an ellipsoid in the hyperplane h_{r_0}.*
(iii) *If $r>r_0$, then $q_{a_0,r}=\emptyset$.*

Corollaries 3.9 and (3.10) imply the following two equivalent propositions:

Proposition 3.11. *Let $x,x_0\in ef_p^{(Q)}$, $x=x(a,r)$, $x_0=x(a_0,r_0)$. One has*

$$r_0=\max_{q_{a_0,r}\neq\emptyset}r \text{ and } a_0=\min_{q_{a,r_0}\neq\emptyset}a.$$

Proposition 3.12. *Given $x_0\in ef_p^{(Q)}$, one has*

$$\pi(x_0)=\max_{x\in ef_p^{(Q)},v(x)\leq v(x_0)}\pi(x) \text{ and } v(x_0)=\min_{x\in ef_p^{(Q)},\pi(x)\geq\pi(x_0)}v(x).$$

It turns out that we can trow out the constraint condition $x\in ef_p^{(Q)}$ from Proposition 3.12. We have the following theorem (compare, for example, with [3, Section 2]).

Theorem 3.13. *Let $x_0 \in q_{a_0,r_0}$ and $r_0 \geq 0$. The following six statements are equivalent:*
(i) *One has $x_0 \in ef_p^{(Q)}$.*
(ii) *One has*

$$\pi(x_0) = \max_{v(x) \leq a_0} \pi(x) \text{ and } v(x_0) = \min_{\pi(x) \geq r_0} v(x).$$

(iii) *One has*

$$\pi(x_0) = \max_{v(x) \leq a_0} \pi(x). \tag{3.1}$$

(iv) *One has*

$$\pi(x_0) = \max_{v(x) = a_0} \pi(x).$$

(v) *One has*

$$v(x_0) = \min_{\pi(x) \geq r_0} v(x).$$

(vi) *One has*

$$v(x_0) = \min_{\pi(x) = r_0} v(x).$$

Proof. Below we prove only these implications which are not straightforward.
If $r_0 = 0$ and $x_0 = x(a_0, 0) \in ef_p^{(Q)}$, then $a_0 = 0$, $x_0 = 0$, and the equivalences hold. Now, let $r_0 > 0$. In particular, we have $x_0 \neq 0$.
(i) $\Longrightarrow$ (ii) According to Lemma 3.5, (iii), and Corollary 3.6 we have $x_0 \in q_{a_0,r_0}$ and $\theta_{x_0}(x) = \frac{a_0}{r_0}\pi(x)$. Let us suppose $v(x) \leq a_0$ for $x \in \mathbb{R}^n$. Then Lemma 3.3, (i), imply $\pi(x) \leq r_0$. Now, let $\pi(x) \geq r_0$, that is, $\theta_{x_0}(x) \geq a_0$ for some $x \in \mathbb{R}^n$. In this case Lemma 3.3, (iii), yields $v(x) \geq a_0$.
(iii) $\Longrightarrow$ (i) Let x_0 satisfies condition (3.1). Lemma 3.4, (i), imply $\theta_{x_0} = h_{r_0}$. Now Lemma 3.5, (iii), finishes the proof.
(v) $\Longrightarrow$ (i). Since $Q_{<a_0} \subset h_{r_0}(<)$, Lemma 3.4 yields $\theta_{x_0} = h_{r_0}$. In accord with Lemma 3.5, (iii), part (i) holds. □

4. Markowitz geometry

In this section we unite the results from the previous two sections and give complete characterization of the tangent points of a family of concentric ellipsoids and a family of parallel hyperplanes in an affine subspace of $\mathbb{R}^n$.

4.1. The equality

Let $M \neq \emptyset$, $\ell \in M$, and let us set $L = \{\ell\}$, $K = M \setminus L$. Let us fix all components of $\tau^{(K)} \in \mathbb{R}^K$: $\tau^{(K)} = \mu^{(K)}$, and set $h^{(K)} = h(\mu^{(K)})$, $\rho^{(K)} = c^{(Q;K^c)}(\mu^{(K)})$, $\gamma^{(K)} = \gamma_K(\mu^{(K)})$. We denote $r = \tau_\ell$, $\rho = \rho_\ell^{(K)}$, $r' = r - \rho$, so $\tau^{(M)} = \tau^{(L,K)}(r)$. Finally, we set $a = \gamma_M(\tau^{(L,K)}(r))$.
We remind that after the translation $z = x - \rho^{(K)}$ of the coordinate system, $(z_s)_{s \in K^c}$, where $z_s = z_s^{(K^c)}$, is a system of coordinate functions on the affine subspace $h^{(K)}$ with origin $\rho^{(K)}$. In this case $h(\tau^{(M)}) = h(\tau^{(L,K)}(r))$ is a hyperplane in $h^{(K)}$ with equation $z_\ell = r'$. In particular, the corresponding ℓ-th coordinate vector $p \in \mathbb{R}^{K^c}$ (the ℓ-th component of p is 1 and all other components are zeroes) is a normal vector of $h(\tau^{(L,K)}(r))$ in $h^{(K)}$. We set $\pi(x) = x_\ell$, $\pi^{(K^c)}(z) = z_\ell$, and note that the linear form $\pi^{(K^c)}(z)$ is the restriction on $h^{(K)}$ of the linear form $\pi(x)$, written in terms of z. It follows from Corollary 2.8, (i), that the trace $q_{a,\mu^{(K)}}$ of the ellipsoid Q_a on affine space $h^{(K)}$ is nonempty and the hyperplane $h(\tau^{(L,K)}(r))$ is tangential to the ellipsoid $q_{a,\mu^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(L,K)}(r))$.
In order to stick together notation from sections 2 and 3 in this case, we set $a' = a - \gamma^{(K)}$, $h(\tau^{(L,K)}(r)) = h_{r'}$, $q_{a',r'} = q_{a,\mu^{(K)}} \cap h_{r'} = Q_{a'}^{(K^c)} \cap h_{r'}$.

Theorem 4.1. (i) *If $r' \geq 0$, then*

$$x(a', r') = c^{(Q;M^c)}(\tau^{(L,K)}(r)) \tag{4.1}$$

and $x(0,0) = \rho^{(K)}$.
(ii) *One has*

$$eff_{\ell\rho+}^{(Q;M^c)} = ef_p^{(Q^{(K^c)})}.$$

Proof. (i) The affine space $h(\tau^{(L,K)}(r))$ is a hyperplane in $h^{(K)}$, which is tangential to the ellipsoid $q_{a,\mu^{(K)}}$ at the point $c^{(Q;M^c)}(\tau^{(L,K)}(r))$. In particular, $Q_{a'}^{(K^c)} \cap h_{r'} = \{c^{(Q;M^c)}(\tau^{(L,K)}(r))\}$ and Lemma 3.5, (ii), yields $a' = c_p r'^2$ for $c_p = ({}^t p Q^{-1} p)^{-1}$. Therefore, when $r' \geq 0$, we have $(a', r') \in E_p^{(Q)}$ and the equality (4.1) holds. In addition, if $r' = 0$, then $a' = 0$, $\gamma_M(\tau^{(L,K)}(r)) = \gamma^{(K)}$, and Corollary 2.8, (ii), (iii), implies $\gamma_L^{(Q^{(K^c)})}(\tau^{(L)} - c_L^{(Q;K^c)}(\mu^{(K)})) = 0$, hence

$$c^{(Q^{(K^c)};M^c)}((\tau^{(L)} - c_L^{(Q;K^c)}(\mu^{(K)}))) = 0.$$

In other words,

$$c^{(Q;M^c)}(\tau^{(L,K)}(\rho)) = c^{(Q;K^c)}(\mu^{(K)}).$$

This shows that $x(0,0) = c^{(Q;K^c)}(\mu^{(K)}) = \rho^{(K)}$ and the equality (4.1) proves part (i) which, in turn, yields part (ii). □

Theorem 4.2. *Let $x_0 = c^{(Q;M^c)}(\tau^{(L,K)}(r_0)) \in eff^{(Q;M^c)}_{\ell\rho+}$. One has $r_0 = \pi(x_0)$ and if $a_0 = v(x_0)$, then*

$$\pi(x_0) = \max_{x\in h^{(K)}, v(x)\le a_0} \pi(x) \text{ and } v(x_0) = \min_{x\in h^{(K)}, \pi(x)\ge r_0} v(x). \tag{4.2}$$

Proof. According to Theorem 4.1, we have $r'_0 = r_0 - \rho \ge 0$, hence $x_0 = x(a_0, r_0) \in ef^{(Q^{(K^c)})}_{p}$. Let $x_0 = z_0 + \rho^{(K)}$. Theorem 3.13, (i), (ii), implies

$$\pi^{(K^c)}(z_0) = \max_{z\in h^{(K)}, v^{(K^c)}(z)\le a'_0} \pi^{(K^c)}(z)$$

and

$$v^{(K^c)}(z_0) = \min_{z\in h^{(K)}, \pi^{(K^c)}(z)\ge r'_0} v^{(K^c)}(z).$$

Since $\pi^{(K^c)}(x) = \pi(z) + \rho$, $v(x) = v^{(K^c)}(z) + \gamma^{(K)}$ on $h^{(K)}$, and since $r'_0 = r_0 - \rho$, $a'_0 = a_0 - \gamma^{(K)}$, we establish the extremal property (4.2). □

4.2. The interpretation

Let k, m, and n be integers with $n \ge 2$, $0 \le k < n-1$, $m = k+1$, and let $M = \{n-k, n-k+1, \dots, n\}$, $K = \{n-k+1, \dots, n\}$, $L = \{n-k\}$. Let $h^{(j)}\colon \pi^{(j)}(y) = \tau_j,\ j \in M$, be linearly independent affine hyperplanes in $\mathbb{R}^n$. We fix $h^{(n)}\colon y_1 + \dots + y_n = 1$, so $\tau_n = 1$, and denote this hyperplane by Π. Since $\pi^{(j)}(y)$ are linearly independent linear forms, we can change the coordinates in $\mathbb{R}^n$: $y = Ax$, in such a way that the hyperplane $h^{(j)}$ has equation $x_j = \tau_j$, $j \in M$, and, moreover, $x_i = y_i$, $i \in [n]\setminus M$.
We fix $\tau^{(K)}$: $\tau^{(K)} = \mu^{(K)}$ ($\mu_n = 1$), and interpret $h^{(n)} = \Pi$ as the hyperplane consisting of all *financial portfolios* with n assets (here y_s is the relative amount of money invested in the s-th asset, $s = 1, \dots, n$). The affine subspace $h^{(n-k+1)} \cap \dots \cap h^{(n-1)}$ (which is equal to $\mathbb{R}^n$ if $m = 2$) represents several additional *linear constrain conditions* and its trace on Π is the affine space $C = h^{(K)} = h^{(n-k+1)} \cap h^{(n-1)} \cap \dots \cap \Pi$ of *linear constrain conditions on* Π.
We denote $\ell = n-k$, $\pi^{(\ell)}(y) = \pi(y)$ and let $r = \tau_\ell$ be variable. When the coefficient in front of y_s in the linear form $\pi(y)$ is the expected return on s-th asset, $s = 1, \dots, n$, the trace of the hyperplane $h = h^{(\ell)}$, $h\colon \pi(y) = r$, on Π may be interpreted as the set of all financial portfolios with *expected return* r. Moreover, the trace of the hyperplane h on C may be interpreted as the set of all financial portfolios with *expected return* r, that obey the above linear constrain conditions on Π.
On the other hand, if $v(x) = {}^t xQx$, where ${}^t A^{-1} Q A^{-1}$ is the $n \times n$ covariance matrix produced by the expected returns of the individual assets, we may interpret $v(x)$ as the risk of the portfolio x. Theorem 4.2 yields that the ray $E = eff^{(Q;M^c)}_{ell\rho+}$ with endpoint $\rho^{(K)}$ is the locus of all Markowitz efficient portfolios which satisfy the linear constraint conditions C. It turns out that the value $v(\rho^{(K)})$ is the absolute minimum of the risk and in terms of x-coordinates the ℓ-th component of $\rho^{(K)}$ is the absolute minimum of the corresponding expected return r under the given constrains.
In order to relate this approach to the classical one, we have to study the intersection $E \cap \Delta$, where Δ is the trace of the unit simplex in Π on C, because the members of $E \cap \Delta$ are the efficient portfolios that have no short sales. Moreover, the properties of this intersection characterize the financial market.

A. Appendix

In this appendix we use freely notation introduced in the main body of the paper.

A.1. Three lemmas

The partition $M^c \cup M = [n]$ of the set of indices $[n]$ produces the following partitioned matrices: Any vector $x = {}^t(x_1, \dots, x_n) \in \mathbb{R}^n$ can be visualized as $x = {}^t(x^{(M^c)}, x^{(M)})$ and any $n \times n$-matrix Q can be visualized as

$$\begin{pmatrix} Q^{(M^c)} & Q^{(M^c\times M)} \\ Q^{(M\times M^c)} & Q^{(M)} \end{pmatrix}.$$

Lemma A.1. *Let Q be a symmetric $n \times n$-matrix and let $v(x) = {}^t xQx$ be the corresponding quadratic form. One has*

$$v(x) = {}^t x^{(M^c)} Q^{(M^c)} x^{(M^c)} + 2\,{}^t x^{(M^c)} Q^{(M^c\times M)} x^{(M)} + {}^t x^{(M)} Q^{(M)} x^{(M)}.$$

Proof. We have

$$v(x) = {}^t xQx = ({}^t x^{(M^c)}, {}^t x^{(M)}) \begin{pmatrix} Q^{(M^c)} & Q^{(M^c\times M)} \\ {}^t Q^{(M^c\times M)} & Q^{(M)} \end{pmatrix} {}^t(x^{(M^c)}, x^{(M)}) =$$

$${}^t x^{(M^c)} Q^{(M^c)} x^{(M^c)} + 2\,{}^t x^{(M^c)} Q^{(M^c\times M)} x^{(M)} + {}^t x^{(M)} Q^{(M)} x^{(M)}.$$

□

Below we assume that $Q^{(M^c)}$ is an invertible matrix.

Lemma A.2. *Let*

$$c_{M^c}^{(M^c)}(x^{(M)}) = -(Q^{(M^c)})^{-1}Q^{(M^c\times M)}x^{(M)},\ c^{(M^c)}(x^{(M)}) = {}^t(c_{M^c}^{(M^c)}(x^{(M)}), x^{(M)}),$$

$$\text{and let } \gamma_M(x^{(M)}) = -{}^tc_{M^c}^{(M^c)}(x^{(M)})Q^{(M^c)}c_{M^c}^{(M^c)}(x^{(M)}) + {}^tx^{(M)}Q^{(M)}x^{(M)}.$$

(i) *$\gamma_M(x^{(M)})$ is a quadratic form in $x^{(M)}$,*

$$\gamma_M(x^{(M)}) = {}^tx^{(M)}[Q^{(M)} - {}^tQ^{(M^c\times M)}(Q^{(M^c)})^{-1}Q^{(M^c\times M)}]x^{(M)},$$

and one has $\gamma_M(x^{(M)}) = v(c^{(M^c)}(x^{(M)}))$.
(ii) *If $v(x)$ is a positive definite quadratic form in x, then $\gamma_M(x^{(M)})$ is a positive definite quadratic form in $x^{(M)}$.*

Proof. (i) We begin by noting that since

$${}^tc_{M^c}^{(M^c)}(x^{(M)})Q^{(M^c)}c_{M^c}^{(M^c)}(x^{(M)}) = {}^tx^{(M)t}Q^{(M^c\times M)}(Q^{(M^c)})^{-1}Q^{M^c\times M}x^{(M)},$$

we obtain the above expression for $\gamma_M(x^{(M)})$. On the other hand, Lemma A.1 implies

$$v(c^{(M^c)}(x^{(M)})) =$$

$${}^tc_{M^c}^{(M^c)}(x^{(M)})Q^{(M^c)}c_{M^c}^{(M^c)}(x^{(M)}) + 2{}^tc_{M^c}^{(M^c)}(x^{(M)})Q^{(M^c\times M)}x^{(M)} + {}^tx^{(M)}Q^{(M)}x^{(M)}.$$

Taking into account that $Q^{(M^c\times M)}x^{(M)} = -Q^{(M^c)}c_{M^c}^{(M^c)}(x^{(M)})$, we establish the identity.
(ii) In is enough to note that $c^{(M^c)}(x^{(M)}) = 0$ if and only if $x^{(M)} = 0$.

□

Now, let us translate the system of coordinates by the rule

$$z(\tau^{(M)}) = x - c^{(M^c)}(\tau^{(M)}).$$

Lemma A.3. *If $x^{(M)} = \tau^{(M)}$, then*

$$v(x) = {}^tz^{(M^c)}Q^{(M^c)}z^{(M^c)} + \gamma_M(\tau^{(M)}).$$

Proof. In accord with Lemma A.1, we have

$$v(x) = {}^tx^{(M^c)}Q^{(M^c)}x^{(M^c)} + 2{}^tx^{(M^c)}Q^{(M^c\times M)}\tau^{(M)} + {}^t\tau^{(M)}Q^{(M)}\tau^{(M)} =$$

$${}^tx^{(M^c)}Q^{(M^c)}x^{(M^c)} - 2{}^tx^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)}(\tau^{(M)}) + {}^t\tau^{(M)}Q^{(M)}\tau^{(M)} =$$

$${}^t(z^{(M^c)} + c_{M^c}^{(M^c)}(\tau^{(M)}))Q^{(M^c)}(z^{(M^c)} + c_{M^c}^{(M^c)}(\tau^{(M)}))$$

$$-2{}^t(z^{(M^c)} + c_{M^c}^{(M^c)}(\tau^{(M)}))Q^{(M^c)}c_{M^c}^{(M^c)}(\tau^{(M)}) + {}^t\tau^{(M)}Q^{(M)}\tau^{(M)} =$$

$${}^tz^{(M^c)}Q^{(M^c)}z^{(M^c)} + {}^tc_{M^c}^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)} + 2{}^tz^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)} -$$

$$2{}^tz^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)} - 2{}^tc_{M^c}^{(M^c)}Q^{(M^c)}c_{M^c}^{(M^c)} + {}^t\tau^{(M)}Q^{(M)}\tau^{(M)} =$$

$${}^tz^{(M^c)}Q^{(M^c)}z^{(M^c)} + \gamma_M(\tau^{(M)}).$$

□

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