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ON TWO TIMES DIFFERENTIABLE PREINVEX AND PREQUASIINVEX FUNCTIONS

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Abstract. The main goal of this paper is to establish a new identity for functions defined on an open invex subset of real numbers. By using this identity, the Hölder integral inequality and power mean integral inequality, we introduce some new type integral inequalities for functions whose powers of second derivatives in absolute values are preinvex and prequasiinvex.

1. PRELIMINARIES

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called convex if the inequality

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v)$$

holds for all $u, v \in I$ and $t \in [0,1]$. If the above inequality reverses, then the function f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a_1, a_2 \in I$ with $a_1 < a_2$. The celebrated inequality

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)\,dx \leq \frac{f(a_1) + f(a_2)}{2}$$

is well-known in the literature as the inequality of Hermite-Hadamard for convex functions [18]. We can estimate the mean value of a continuous convex or concave function by means of the classical Hermite-Hadamard inequality. Hadamard's inequality for convex or concave functions has recently took too much attention and a remarkable variety of refinements and generalizations have been found (see [4, 8, 10, 12, 18]). A usage of Hermite-Hadamard inequality may result in obtaining one of the most useful inequalities in mathematical analysis [5].

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Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex if the inequality

$$f(tu + (1-t)v) \leq \max\{f(u), f(v)\}$$

holds for all $u, v \in I$ and $t \in [0,1]$.

We recall that any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex [6].

Definition 1.2 ([20])**.** Let K be a non-empty subset in $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$. Let $v \in K$, then the set K is said to be invex at v with respect to $\eta(\cdot,\cdot)$, if

$$v + t\eta(u,v) \in K, \ \forall v, u \in K \text{ and } t \in [0,1].$$

K is said to be an invex set with respect to $\eta(\cdot,\cdot)$ if K is invex at each $v \in K$. The invex set K is also called η-connected set.

It is true that every convex set is also an invex set with respect to $\eta(u,v) = u-v$, but the converse is not necessarily true, see [14, 21] and the references therein. For the sake of simplicity, we always assume that $K = [v, v + t\eta(u,v)]$, unless otherwise specified [1].

Definition 1.3 ([20])**.** A function $f : K \to \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}$ is said to be preinvex with respect to $\eta(\cdot,\cdot)$, if

$$f(u + t\eta(v,u)) \leq (1-t)f(u) + tf(v), \ \forall u, v \in K, \ t \in [0,1].$$

It is to be noted that every convex function is preinvex with respect to the map $\eta(u,v) = v - u$ but the converse is not true see for instance.

Definition 1.4 ([2])**.** A function $f : K \to \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}$ is said to be prequasiinvex with respect to $\eta(\cdot,\cdot)$, if

$$f(u + t\eta(v,u)) \leq \max\{f(u), f(v)\}, \ \forall u, v \in K, \ t \in [0,1].$$

We know that every quasi-convex function is a prequasiinvex with respect to the mapping $\eta(v,u) = v - u$ but the converse does not hold, see for example [2].

Definition 1.5 ([14])**.** Let $S \subseteq \mathbb{R}$ be an open invex subset with respect to the mapping $\eta(\cdot,\cdot) : S \times S \to \mathbb{R}$. We say that the function satisfies the Condition C if, for any $x, y \in S$ and any $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y) \tag{1.1}$$

$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y). \tag{1.2}$$

Remark 1.6. Note that, from the Condition C, we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y) \tag{1.3}$$

for any $x, y \in S$ and any $t_1, t_2 \in [0,1]$.

Really: Let $u = y + t_2\eta(x,y)$.

(i) For $t_2 \in [0,1)$,

$$\eta(y+t_2\eta(x,y), y+t_1\eta(x,y)) = \eta(u, u+(t_1-t_2)\eta(x,y)) \tag{1.4}$$

and from (1.2)

$$\begin{aligned}\eta(x,u) &= \eta(x, y+t_2\eta(x,y)) \\ &= (1-t_2)\eta(x,y).\end{aligned}$$

From here we get

$$\eta(x,y) = \frac{\eta(x,u)}{1-t_2}.$$

Using (1.4) and Condition C

$$\begin{aligned}\eta(y+t_2\eta(x,y), y+t_1\eta(x,y)) &= \eta(u, u+(t_1-t_2)\eta(x,y)) \\ &= \eta\left(u, u+\frac{t_1-t_2}{1-t_2}\eta(x,u)\right) \\ &= -\frac{t_1-t_2}{1-t_2}\eta(x,u) \\ &= (t_2-t_1)\eta(x,y).\end{aligned}$$

(ii) For $t_2=1$, let $a=y+\eta(x,y)$. From (1.1)

$$\begin{aligned}\eta(y+\eta(x,y), y+t_1\eta(x,y)) &= \eta(a, a-(1-t_1)\eta(x,y)) \\ &= \eta(a, a+(1-t_1)\eta(y,a)) \\ &= -(1-t_1)\eta(y,a) \\ &= (1-t_1)\eta(x,y).\end{aligned}$$

Consequently, the equality (1.3) is true.

In recent years, many mathematicians have been studying about preinvexity and types of preinvexity. See for more information [7, 16, 17, 19, 20, 22].

Theorem 1.7 ([15]). *Let $f:[a_1, a_1+t\eta(a_2,a_1)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and $a_1, a_2 \in K^\circ$ with $\eta(a_2,a_1)>0$. Then the following inequalities hold:*

$$f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right) \le \frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)} f(x)dx \le \frac{f(a_1)+f(a_2)}{2}. \tag{1.5}$$

For several recent results on inequalities for preinvex and prequasiinvex functions which are connected to (1.3), we refer the reader to [3, 9, 11, 13] and the references therein.

Let $0<a_1<a_2$, throughout this paper we will use

$$\begin{aligned}A &= A(a_1,a_2) = \frac{a_1+a_2}{2} \\ L_p(a_1,a_2) &= \left(\frac{a_2^{p+1}-a_1^{p+1}}{(p+1)(a_2-a_1)}\right)^{\frac{1}{p}}, \quad a_1 \neq a_2,\ p \in \mathbb{R},\quad p \neq -1, 0\end{aligned}$$

for the arithmetic and generalized logarithmic mean, respectively. Moreover, for shortness, the notations

$$\alpha=\alpha\left(a_1,a_2,\eta\right)=a_1+\frac{\eta(a_2,a_1)}{2},\ \alpha_t=\alpha_t\left(a_1,a_2,\eta\right)=a_1+t\frac{\eta(a_2,a_1)}{2}$$

and

$$\begin{aligned}I_f(a_1,a_2,\eta)&:=\eta(a_2,a_1)\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)f'\left(a_1+\eta(a_2,a_1)\right)\\&-f\left(a_1+\eta(a_2,a_1)\right)\left(a_1+\eta(a_2,a_1)\right)+f(a_1)a_1+\int_{a_1}^{a_1+\eta(a_2,a_1)}f(x)dx\end{aligned}$$

will be used.

2. MAIN RESULTS FOR OUR LEMMA

We will use the next Lemma to obtain our main results related with the preinvexity and prequasiinvexity.

Lemma 2.1. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta\left(\cdot,\cdot\right):K\times K\to\mathbb{R}^n$ and $a_1,a_2\in K$ with $\eta(a_2,a_1)>0$. Suppose that $f:K\to\mathbb{R}$ is a twice differentiable function on K such that $f''\in L\left[a_1,a_1+\eta(a_2,a_1)\right]$. Then the following identity holds:*

$$\begin{aligned}&\eta(a_2,a_1)\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)f'\left(a_1+\eta(a_2,a_1)\right)\\&-f\left(a_1+\eta(a_2,a_1)\right)\left(a_1+\eta(a_2,a_1)\right)+f(a_1)a_1+\int_{a_1}^{a_1+\eta(a_2,a_1)}f(x)dx\\&\qquad=\eta^2(a_2,a_1)\int_0^1 t\left(a_1+t\frac{\eta(a_2,a_1)}{2}\right)f''(a_1+t\eta(a_2,a_1))dt.\end{aligned}$$

Proof. Integrating by parts and changing the variable and we have

$$\begin{aligned}&\eta^2(a_2,a_1)\int_0^1 t\left(a_1+t\frac{\eta(a_2,a_1)}{2}\right)f''(a_1+t\eta(a_2,a_1))dt\\&=\eta(a_2,a_1)\left(a_1t+t^2\frac{\eta(a_2,a_1)}{2}\right)f'\left(a_1+t\eta(a_2,a_1)\right)\Big|_0^1\\&\quad-\left(a_1+t\eta(a_2,a_1)\right)f\left(a_1+t\eta(a_2,a_1)\right)\Big|_0^1+\eta(a_2,a_1)\int_0^1 f\left(a_1+t\eta(a_2,a_1)\right)dt\\&=\quad\eta(a_2,a_1)\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)f'\left(a_1+\eta(a_2,a_1)\right)\\&\quad-\left(a_1+\eta(a_2,a_1)\right)f\left(a_1+\eta(a_2,a_1)\right)+f(a_1)a_1+\int_{a_1}^{a_1+\eta(a_2,a_1)}f(x)dx.\end{aligned}$$

□

Theorem 2.2. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\rightarrow\mathbb{R}^n$ and $a_1,a_2\in K$ with $\eta(a_2,a_1)>0$. Suppose that $f:K\rightarrow\mathbb{R}$ is a twice differentiable function on K such that $f''\in L\left[a_1,a_1+\eta(a_2,a_1)\right]$. If $|f''|^q$ is preinvex on K for $q>1$, then the following inequality holds:*

$$\begin{aligned}|I_f(a_1,a_2,\eta)| &\leq 2^{\frac{1}{q}}\left(\frac{\eta^2(a_2,a_1)}{p+1}\right)^{\frac{1}{p}}\\ &\quad\times\left[|f''(a_2)|^q C_{1,\eta}(a_1,a_2)+|f''(a_1)|^q C_{2,\eta}(a_1,a_2)\right]^{\frac{1}{q}}. \quad (2.1)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$C_{1,\eta}(a_1,a_2):=\begin{cases}\eta(a_2,a_1)\left[L_{q+1}^{q+1}(\alpha,a_1)-a_1L_q^q(\alpha,a_1)\right], & a_1>0,\ \alpha>0,\\ 2(a_1+\alpha)L_{q+1}^{q+1}(\alpha,-a_1)-\frac{4a_1}{q+1}A\left(\alpha^{q+1},(-a_1)^{q+1}\right), & a_1<0,\ \alpha>0,\\ -\eta(a_2,a_1)\left[L_{q+1}^{q+1}(-a_1,-\alpha)+aL_q^q(-a_1,-\alpha)\right], & a_1<0,\ \alpha<0.\end{cases}$$

$$C_{2,\eta}(a_1,a_2):=\begin{cases}-\eta(a_2,a_1)\left[L_{q+1}^{q+1}(\alpha,a_1)-\alpha L_q^q(\alpha,a_1)\right], & a_1>0,\ \alpha>0,\\ -2(a_1+\alpha)L_{q+1}^{q+1}(\alpha,-a_1)+\frac{4\alpha}{q+1}A\left(\alpha^{q+1},(-a_1)^{q+1}\right), & a_1<0,\ \alpha>0,\\ \eta(a_2,a_1)\left[L_{q+1}^{q+1}(-a_1,-\alpha)+\alpha L_q^q(-a_1,-\alpha)\right], & a_1<0,\ \alpha<0.\end{cases}$$

Proof. If $|f''|^q$ for $q>1$ is preinvex on $[a_1,a_1+\eta(a_2,a_1)]$, using Lemma 2.1, the Hölder integral inequality and

$$|f''(a_1+t\eta(a_2,a_1))|^q\leq t\,|f''(a_2)|^q+(1-t)\,|f''(a_1)|^q,$$

we get

$$\begin{aligned}|I_f(a_1,a_2,\eta)| &\leq \eta^2(a_2,a_1)\int_0^1 t\,|\alpha_t|\,|f''(a_1+t\eta(a_2,a_1))|\,dt\\ &\leq \eta^2(a_2,a_1)\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1|\alpha_t|^q\,|f''(a_1+t\eta(a_2,a_1))|^q\,dt\right)^{\frac{1}{q}}\\ &\leq \frac{\eta^2(a_2,a_1)}{(p+1)^{\frac{1}{p}}}\left(|f''(a_2)|^q\int_0^1 t\,|\alpha_t|^q\,dt+|f''(a_1)|^q\int_0^1(1-t)\,|\alpha_t|^q\,dt\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\eta^2(a_2,a_1)}{(p+1)^{\frac{1}{p}}}\left(\frac{2\left|f''(a_2)\right|^q}{\eta^2(a_2,a_1)}\int_{a_1}^{\alpha} 2\left(x-a_1\right)|x|^q\,dx\right.\\
&\quad \left.+\frac{2\left|f''(a_1)\right|^q}{\eta^2(b,a_1)}\int_{a_1}^{\alpha}\left(\eta(a_2,a_1)-2(x-a_1)\right)|x|^q\,dx\right)^{\frac{1}{q}}\\
&= 2^{\frac{1}{q}}\left(\frac{\eta^2(a_2,a_1)}{p+1}\right)^{\frac{1}{p}}\left(\left|f''(a_2)\right|^q\int_{a_1}^{\alpha} 2\left(x-a_1\right)|x|^q\,dx\right.\\
&\quad \left.+\left|f''(a_1)\right|^q\int_{a_1}^{\alpha}\left(\eta(a_2,a_1)-2(x-a_1)\right)|x|^q\,dx\right)^{\frac{1}{q}}\\
&= 2^{\frac{1}{q}}\left(\frac{\eta^2(a_2,a_1)}{p+1}\right)^{\frac{1}{p}}\left[\left|f''(a_2)\right|^q C_{1,\eta}\left(a_1,a_2\right)+\left|f''(a_1)\right|^q C_{2,\eta}\left(a_1,a_2\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

The proof is completed. □

Corollary 1. *Suppose that all the assumptions of Theorem 2.2 are satisfied. If we choose* $\eta(a_2,a_1)=a_2-a_1$ *then when* $|f''|^q$ *is convex on* K *for* $q>1$ *we obtain*

$$
\begin{aligned}
&\left|\frac{a_1+a_2}{2}f'(a_2)-\frac{f(a_2)a_2-f(a_1)a_1}{a_2-a_1}-\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)dx\right|\\
&\leq 2^{\frac{1}{q}}\left(\frac{(a_2-a_1)^2}{p+1}\right)^{\frac{1}{p}}\left[\left|f''(a_2)\right|^q C_1\left(a_1,a_2\right)+\left|f''(a_1)\right|^q C_2\left(a_1,a_2\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

where

$$
C_1\left(a_1,a_2\right)=\begin{cases}
(a_2-a_1)\left[L_{q+1}^{q+1}\left(A,a_1\right)-aL_q^q\left(A,a_1\right)\right], & a_1>0,\ A>0,\\
(3a_1+a_2)L_{q+1}^{q+1}\left(A,-a_1\right)-\frac{4a_1}{q+1}A\left(A^{q+1},(-a_1)^{q+1}\right), & a_1<0,\ A>0,\\
-(a_2-a_1)\left[L_{q+1}^{q+1}\left(-a_1,-A\right)+a_1L_q^q\left(-a_1,-A\right)\right], & a_1<0,\ A<0.
\end{cases}
$$

$$
C_2\left(a_1,a_2\right)=\begin{cases}
-(a_2-a_1)\left[L_{q+1}^{q+1}\left(A,a_1\right)-\alpha L_q^q\left(A,a_1\right)\right], & a_1>0,\ A>0,\\
-(3a_1+a_2)L_{q+1}^{q+1}\left(A,-a_1\right)+\frac{4\alpha}{q+1}A\left(A^{q+1},(-a_1)^{q+1}\right), & a_1<0,\ A>0,\\
(a_2-a_1)\left[L_{q+1}^{q+1}\left(-a_1,-A\right)+\alpha L_q^q\left(-a_1,-A\right)\right], & a_1<0,\ A<0.
\end{cases}
$$

Remark 2.3. If the mapping η satisfies condition C then by use of the preinvexity of $|f''|^q$ we have

$$
\begin{aligned}
\left|f''\left(a_1+t\eta(a_2,a_1)\right)\right|^q &= \left|f''\left(a_1+\eta(a_2,a_1)+(1-t)\eta(a_1,a_1+\eta(a_2,a_1))\right)\right|^q\\
&\leq t\left|f''\left(a_1+\eta(a_2,a_1)\right)\right|^q+(1-t)\left|f''(a_1)\right|^q. \qquad (2.2)
\end{aligned}
$$

for every $t \in [0,1]$. If we use the inequality (2.2) in the proof of Theorem 2.2, then the inequality (2.1) becomes the following inequality:

$$\begin{aligned} |I_f(a_1,a_2,\eta)| &\leq 2^{\frac{1}{q}}\left(\frac{\eta^2(a_2,a_1)}{p+1}\right)^{\frac{1}{p}}\left[\left|f''(a_1+\eta(a_2,a_1))\right|^q C_{1,\eta}(a_1,a_2)\right. \\ &\quad \left. +\left|f''(a_1)\right|^q C_{2,\eta}(a_1,a_2)\right]^{\frac{1}{q}}. \end{aligned} \tag{2.3}$$

We note that by use of the preinvexity of $|f''|^q$ we get

$$\left|f''(a_1+\eta(a_2,a_1))\right|^q \leq \left|f''(a_2)\right|^q.$$

Therefore, the inequality (2.3) is better than the inequality (2.1).

Theorem 2.4. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot) : K \times K \to \mathbb{R}^n$ and $a_1, a_2 \in K$ with $\eta(a_2,a_1) > 0$. Suppose that $f : K \to \mathbb{R}$ is a twice differentiable function on K such that $f'' \in L[a_1, a_1+\eta(a_2,a_1)]$. If $|f''|^q$ is preinvex on K for $q > 1$, then the following inequality holds:*

$$\begin{aligned} |I_f(a_1,a_2,\eta)| &\leq 2^{\frac{1}{p}}\eta^{1+\frac{1}{q}}(a_2,a_1)C_{3,\eta}^{\frac{1}{p}}(a_1,a_2) \\ &\quad \times\left[\frac{(q+1)\left|f''(a_2)\right|^q+\left|f''(a_1)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}, \end{aligned} \tag{2.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$C_{3,\eta}(a_1,a_2) := \begin{cases} \frac{\eta(a_2,a_1)}{2}L_p^p(\alpha,a_1), & a_1>0,\ \alpha>0, \\ \frac{2}{p+1}A\left(\alpha^{p+1},(-a_1)^{p+1}\right), & a_1<0,\ \alpha>0, \\ \frac{\eta(a_2,a_1)}{2}L_p^p(-a_1,-\alpha), & a_1<0,\ \alpha<0. \end{cases}$$

Proof. If $|f''|^q$ for $q > 1$ is preinvex on $[a_1, a_1+\eta(b,a_1)]$, using Lemma 2.1, the Hölder integral inequality and

$$\left|f''(a_1+t\eta(a_2,a_1))\right|^q \leq t\left|f''(a_2)\right|^q+(1-t)\left|f''(a_1)\right|^q,$$

we obtain

$$\begin{aligned} &|I_f(a_1,a_2,\eta)| \leq \eta^2(a_2,a_1)\int_0^1 t\left|\alpha_t\right|\left|f''(a_1+t\eta(a_2,a_1))\right|dt \\ &\leq \eta^2(a_2,a_1)\left(\int_0^1\left|\alpha_t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t^q\left|f''(a_1+t\eta(a_2,a_1))\right|^q dt\right)^{\frac{1}{q}} \\ &\leq \eta^2(a_2,a_1)\left(\int_0^1\left|\alpha_t\right|^p dt\right)^{\frac{1}{p}}\left(\left|f''(a_2)\right|^q\int_0^1 t^{q+1}dt+\left|f''(a_1)\right|^q\int_0^1\left(t^q-t^{q+1}\right)dt\right)^{\frac{1}{q}} \end{aligned}$$

$$= 2^{\frac{1}{p}}\eta^{1+\frac{1}{q}}(a_2,a_1)\left(\int_{a_1}^{\alpha}|x|^p\,dx\right)^{\frac{1}{p}}\left[\frac{|f''(a_2)|^q}{q+2}+|f''(a_1)|^q\left(\frac{1}{q+1}-\frac{1}{q+2}\right)\right]^{\frac{1}{q}}$$

$$= 2^{\frac{1}{p}}\eta^{1+\frac{1}{q}}(a_2,a_1)C_{3,\eta}^{\frac{1}{p}}(a_1,a_2)\left[\frac{(q+1)\,|f''(a_2)|^q+|f''(a_1)|^q}{(q+1)\,(q+2)}\right]^{\frac{1}{q}}.$$

The proof is completed. □

Corollary 2. *Suppose that all the assumptions of Theorem 2.4 are satisfied. If we choose $\eta(a_2,a_1)=a_2-a_1$ then when $|f''|^q$ is convex on K for $q>1$ we obtain*

$$\left|\frac{a_1+a_2}{2}f'(a_2)-\frac{f(a_2)a_2-f(a_1)a_1}{a_2-a_1}-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)dx\right|$$

$$\leq 2^{\frac{1}{p}}(a_2-a_1)^{1+\frac{1}{q}}C_3^{\frac{1}{p}}(a_1,a_2)\left[\frac{(q+1)\,|f''(a_2)|^q+|f''(a_1)|^q}{(q+1)\,(q+2)}\right]^{\frac{1}{q}},$$

where

$$C_3(a_1,a_2)=\begin{cases}\frac{a_2-a_1}{2}L_p^p(A,a_1)\,, & a_1>0, A>0,\\ \frac{2}{p+1}A\left(A^{p+1},(-a_1)^{p+1}\right), & a_1<0, A>0,\\ \frac{a_2-a_1}{2}L_p^p(-a_1,-A)\,, & a_1<0,\ A<0.\end{cases}$$

Remark 2.5. If the mapping η satisfies condition C then using the inequality (2.2) in the proof of Theorem 2.4, then the inequality (2.4) becomes the following inequality:

$$|I_f(a_1,a_2,\eta)| \leq 2^{\frac{1}{p}}\eta^{1+\frac{1}{q}}(a_2,a_1)C_{3,\eta}^{\frac{1}{p}}(a_1,a_2)\left[\frac{(q+1)\,|f''(a_1+\eta(a_2,a_1))|^q+|f''(a_1)|^q}{(q+1)\,(q+2)}\right]^{\frac{1}{q}}. \quad (2.5)$$

We note that by use of the preinvexity of $|f''|^q$ we get

$$|f''(a_1+\eta(a_2,a_1))|^q\leq|f''(a_2)|^q.$$

Therefore, the inequality (2.5) is better than the inequality (2.4).

Theorem 2.6. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\to\mathbb{R}^n$ and $a_1,a_2\in K$ with $\eta(a_2,a_1)>0$. Suppose that $f:K\to\mathbb{R}$ is a twice differentiable function on K such that $f''\in L[a_1,a_1+\eta(a_2,a_1)]$. If $|f''|^q$ is preinvex on K for $q\geq1$, then the following inequality holds:*

$$|I_f(a_1,a_2,\eta)| \leq 2^{1+\frac{1}{q}}\eta^{1-\frac{2}{q}}(a_2,a_1)D_{1,\eta}^{1-\frac{1}{q}}(a_1,a_2)\left[|f''(a_2)|^q D_{2,\eta}(a_1,a_2)+|f''(a_1)|^q D_{3,\eta}(a_1,a_2)\right]^{\frac{1}{q}} \quad (2.6)$$

where

$$D_{1,\eta}(a_1,a_2):=\begin{cases}\frac{\eta(a_2,a_1)}{2}\left[L_2^2(\alpha,a_1)-a_1L(\alpha,a_1)\right], & a_1>0,\alpha>0\\ \frac{2}{3}A\left(\alpha^3,a_1^3\right)-a_1A\left(\alpha^2,a_1^2\right), & a_1<0,\ \alpha>0\\ -\frac{\eta(a_2,a_1)}{2}\left[L_2^2(\alpha,a_1)-a_1L(\alpha,a_1)\right], & a_1<0,\alpha<0\end{cases},$$

$$D_{2,\eta}(a_1,a_2):=\begin{cases}\frac{\theta\eta^3(a_2,a_1)}{4}, & a_1>0,\alpha>0\\ \frac{\theta\eta^3(a_2,a_1)}{4}+\frac{a_1^4}{3}, & a_1<0,\alpha>0\\ -\frac{\theta\eta^3(a_2,a_1)}{4}, & a_1<0,\alpha<0\end{cases},$$

$$D_{3,\eta}(a_1,a_2):=\begin{cases}\frac{\eta^2(a_2,a_1)}{2}\left[L_2^2(\alpha,a_1)-a_1L(\alpha,a_1)\right]-\frac{\theta\eta^3(b,a_1)}{4}, & a_1>0,\ \alpha>0\\ \eta(a_2,a_1)\left[\frac{2}{3}A\left(\alpha^3,a_1^3\right)-aA\left(\alpha^2,a^2\right)\right]-\frac{\theta\eta^3(b,a)}{4}-\frac{a^4}{3} & a_1<0,\ \alpha>0\\ -\frac{\eta^2(a_2,a_1)}{2}\left[L_2^2(\alpha,a_1)-aL(\alpha,a_1)\right]+\frac{\theta\eta^3(b,a)}{4}, & a_1<0,\ \alpha<0.\end{cases}$$

and $\theta=\frac{\eta(a_2,a_1)}{8}+\frac{a_1}{3}$.

Proof. Using Lemma 2.1 and Power-mean integral inequality, we obtain

$$\begin{aligned}|I_f(a_1,a_2,\eta)| &\leq \eta^2(a_2,a_1)\int_0^1 t\,|\alpha_t|\,|f''(a_1+t\eta(a_2,a_1))|\,dt\\ &\leq \eta^2(a_2,a_1)\left(\int_0^1 t\,|\alpha_t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|\alpha_t|\,|f''(a_1+t\eta(a_2,a_1))|^q\,dt\right)^{\frac{1}{q}}\end{aligned}$$

$$\begin{aligned}&\leq \eta^2(a_2,a_1)\left(\int_0^1 t\,|\alpha_t|\,dt\right)^{1-\frac{1}{q}}\\ &\quad\times\left(|f''(a_2)|^q\int_0^1 t^2\,|\alpha_t|\,dt+|f''(a_1)|^q\int_0^1 t(1-t)\,|\alpha_t|\,dt\right)^{\frac{1}{q}}\\ &= 2^{1+\frac{1}{q}}\eta^{1-\frac{2}{q}}(a_2,a_1)\left(\int_a^\alpha (x-a_1)\,|x|\,dx\right)^{1-\frac{1}{q}}\left(|f''(a_2)|^q\int_a^\alpha 2(x-a_1)^2\,|x|\,dx\right.\\ &\quad\left.+|f''(a_1)|^q\int_{a_1}^\alpha (x-a_1)\left[\eta(a_2,a_1)-2(x-a_1)\right]|x|\,dx\right)^{\frac{1}{q}}\\ &= 2^{1+\frac{1}{q}}\eta^{1-\frac{2}{q}}(a_2,a_1)D_{1,\eta}^{1-\frac{1}{q}}(a_1,a_2)\\ &\quad\times\left[|f''(a_2)|^q D_{2,\eta}(a_1,a_2)+|f''(a_1)|^q D_{3,\eta}(a_1,a_2)\right]^{\frac{1}{q}}\end{aligned}$$

The proof is completed. □

Corollary 3. *Suppose that all the assumptions of Theorem 3 are satisfied. If we choose* $\eta(a_2,a_1)=a_2-a_1$ *then when* $|f''|^q$ *is convex on* K *for* $q\geq 1$ *we get*

$$\begin{aligned}&\left|\frac{a_1+a_2}{2}f'(b)-\frac{f(a_2)a_2-f(a_1)a_1}{a_2-a_1}-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)dx\right|\\ &\leq 2^{1+\frac{1}{q}}(a_2-a_1)^{1-\frac{2}{q}}D_1^{1-\frac{1}{q}}(a_1,a_2)\left[|f''(a_2)|^q D_2(a_1,a_2)+|f''(a_1)|^q D_3(a_1,a_2)\right]^{\frac{1}{q}},\end{aligned}$$

where

$$D_1(a_1,a_2)=\begin{cases}\frac{a_2-a_1}{2}\left[L_2^2\left(A,a_1\right)-a_1L\left(A,a_1\right)\right], & a_1>0,\ A>0\\ \frac{2}{3}A\left(A^3,a_1^3\right)-a_1A\left(A^2,a_1^2\right), & a_1<0,\ A>0\\ -\frac{a_2-a_1}{2}\left[L_2^2\left(A,a_1\right)-a_1L\left(A,a_1\right)\right], & a_1<0,\ A<0\end{cases},$$

$$D_2(a_1,a_2)=\begin{cases}\frac{(a_2-a_1)^3}{4}\left(\frac{3a_2+5a_1}{24}\right), & a_1>0,\ A>0\\ \frac{(a_2-a_1)^3}{4}\left(\frac{3a_2+5a_1}{24}\right)+\frac{a^4}{3}, & a_1<0,\ A>0\\ -\frac{(a_2-a_1)^3}{4}\left(\frac{3a_2+5a_1}{24}\right), & a_1<0,\ A<0\end{cases},$$

$$D_3(a_1,a_2)=\begin{cases}\frac{(a_2-a_1)^2}{2}\left[L_2^2\left(A,a_1\right)-a_1L\left(A,a_1\right)\right]-\frac{(a_2-a_1)^3(3a_2+5a_1)}{96}, \\ \qquad a_1>0,\ A>0\\ (a_2-a_1)\left[\frac{2}{3}A\left(A^3,a_1^3\right)-a_1A\left(A^2,a_1^2\right)\right]-\frac{(a_2-a_1)^3(3a_2+5a_1)}{96}-\frac{a^4}{3}, \\ \qquad a_1<0,\ A>0\\ -\frac{(a_2-a_1)^2}{2}\left[L_2^2\left(A,a_1\right)-a_1L\left(A,a_1\right)\right]+\frac{(a_2-a_1)^3(3a_2+5a_1)}{96}, \\ \qquad a_1<0,\ A<0.\end{cases}$$

Remark 2.7. If the mapping η satisfies condition C then using the inequality (2.2) in the proof of Theorem 2.6, then the inequality (2.6) becomes the following inequality:

$$\begin{aligned}|I_f(a,b,\eta)| \quad &\leq \quad 2^{1+\frac{1}{q}}\eta^{1-\frac{2}{q}}(b,a)D_{1,\eta}^{1-\frac{1}{q}}(a,b)\\ &\times\left[|f''(a+\eta(b,a)))|^q D_{2,\eta}(a,b)+|f''(a)|^q D_{3,\eta}(a,b)\right]^{\frac{1}{q}}. \quad (2.7)\end{aligned}$$

We note that by use of the preinvexity of $|f''|^q$ we get

$$|f''(a+\eta(b,a))|^q\leq|f''(b)|^q.$$

Therefore, the inequality (2.7) is better than the inequality (2.6).

Corollary 4. *If we take $q=1$ in Theorem 2.6, then we have the following inequality:*

$$|I_f(a_1,a_2,\eta)|\leq\frac{4}{\eta(a_2,a_1)}\left[|f''(a_2)|D_{2,\eta}(a_1,a_2)+|f''(a_1)|D_{3,\eta}(a_1,a_2)\right]$$

Theorem 2.8. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot)$: $K\times K\to\mathbb{R}^n$ and $a_1,a_2\in K$ with $\eta(a_2,a_1)>0$. Suppose that $f:K\to\mathbb{R}$ is a twice differentiable function on K such that $f''\in L[a_1,a_1+\eta(a_2,a_1)]$. If $|f''|^q$ is prequasiinvex on K for $q>1$, then the following inequality holds:*

$$\begin{aligned}&|I_f(a_1,a_2,\eta)|\\ &\leq 2^{\frac{1}{q}}\eta^{1+\frac{1}{p}}(a_2,a_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{|f''(a_1)|^q,|f''(a_2)|^q\right\}\right)^{\frac{1}{q}}C_\eta^{\frac{1}{q}}(q,a_1,a_2) \quad (2.8)\end{aligned}$$

where

$$C_\eta(q,a_1,a_2):=\begin{cases}\frac{\eta(a_2,a_1)}{2}L_q^q(\alpha,a_1), & a_1>0,\alpha>0,\\ \frac{2}{q+1}A\left[\alpha^{q+1},(-a_1)^{q+1}\right], & a_1<0,\alpha>0,\\ \frac{\eta(a_2,a_1)}{2}L_q^q(-a_1,-\alpha), & a_1<0,\alpha<0.\end{cases}$$

Proof. If $|f''|^q$ for $q>1$ is prequasiinvex on $[a_1, a_1+\eta(a_2,a_1)]$, using Lemma 2.1, the Hölder integral inequality and

$$|f''(a_1+t\eta(a_2,a_1))|^q \leq \max\left\{|f''(a_1)|^q, |f''(a_2)|^q\right\}$$

we obtain

$$\begin{aligned}
|I_f(a_1,a_2,\eta)| &\leq \eta^2(a_2,a_1)\int_0^1 t\,|\alpha_t|\,|f''(a_1+t\eta(a_2,a_1))|\,dt \\
&\leq \eta^2(a_2,a_1)\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |\alpha_t|^q\,|f''(a_1+t\eta(a_2,a_1))|^q\,dt\right)^{\frac{1}{q}} \\
&\leq \eta^2(a_2,a_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 |\alpha_t|^q \max\left\{|f''(a_1)|^q, |f''(a_2)|^q\right\}dt\right)^{\frac{1}{q}} \\
&= 2^{\frac{1}{q}}\eta^{1+\frac{1}{p}}(a_2,a_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{|f''(a_1)|^q, |f''(a_2)|^q\right\}\right)^{\frac{1}{q}}\left(\int_{a_2}^{\alpha}|x|^q\,dx\right)^{\frac{1}{q}} \\
&= 2^{\frac{1}{q}}\eta^{1+\frac{1}{p}}(a_2,a_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{|f''(a_1)|^q, |f''(a_2)|^q\right\}\right)^{\frac{1}{q}} C_\eta^{\frac{1}{q}}(q,a_1,a_2)
\end{aligned}$$

The proof is completed. □

Corollary 5. *Suppose that all the assumptions of Theorem 2.8 are satisfied. If we choose* $\eta(b,a_1)=a_2-a_1$ *then when* $|f''|^q$ *is prequasiinvex on* K *for* $q>1$ *we have*

$$\begin{aligned}
&\left|\frac{a_1+a_2}{2}f'(a_2)-\frac{f(a_2)a_2-f(a_1)a_1}{a_2-a_1}-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)dx\right| \\
&\leq 2^{\frac{1}{q}}\left(\frac{a_2-a_1}{p+1}\right)^{\frac{1}{p}}\left(\max\left\{|f''(a_1)|^q, |f''(a_2)|^q\right\}\right)^{\frac{1}{q}} C^{\frac{1}{q}}(q,a_1,a_2)
\end{aligned}$$

where

$$C(q,a_1,a_2) := \begin{cases} \frac{a_2-a_1}{2}L_q^q(A,a_1), & a_1>0, A>0, \\ \frac{2}{q+1}A\left[A^{q+1},(-a_1)^{q+1}\right], & a_1<0, A>0, \\ \frac{a_2-a_1}{2}L_q^q(-a_1,-A), & a_1<0, A<0. \end{cases}$$

Remark 2.9. If the mapping η satisfies condition C then by use of the prequasiinvexity of $|f''|^q$ we get

$$\begin{aligned}
|f''(a_1+t\eta(a_2,a_1))|^q &= |f''(a_1+\eta(a_2,a_1)+(1-t)\eta(a_1,a_1+\eta(a_2,a_1)))|^q \\
&\leq \max\left\{|f''(a_1)|^q, |f''(a_1+\eta(a_2,a_1))|^q\right\} \qquad (2.9)
\end{aligned}$$

for every $t\in[0,1]$.

If we use the inequality (2.9) in the proof of Theorem 2.8, then the inequality (2.8) becomes the following inequality:

$$\begin{aligned} |I_f(a_1,a_2,\eta)| &\leq 2^{\frac{1}{q}}\eta^{1+\frac{1}{p}}(a_2,a_1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} C_\eta^{\frac{1}{q}}(q,a_1,a_2) \\ &\quad \times \left(\max\left\{|f''(a_1)|^q, |f''(a_1+\eta(a_2,a_1))|^q\right\}\right)^{\frac{1}{q}} \end{aligned} \tag{2.10}$$

We note that by use of the prequasiinvexity of $|f''|^q$ we have

$$|f''(a_1+\eta(a_2,a_1))|^q \leq \max\left\{|f''(a_1)|^q, |f''(a_1+\eta(a_2,a_1))|^q\right\}.$$

Therefore, the inequality (2.10) is better than the inequality (2.8).

Theorem 2.10. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot) : K \times K \to \mathbb{R}^n$ and $a_1, a_2 \in K$ with $\eta(a_2,a_1) > 0$. Suppose that $f : K \to \mathbb{R}$ is a twice differentiable function on K such that $f'' \in L[a_1, a_1+\eta(a_2,a_1)]$. If $|f''|^q$ is prequasiinvex on K for $q \geq 1$, then the following inequality holds:*

$$|I_f(a_1,a_2,\eta)| \leq 4\left(\max\left\{|f''(a_1)|^q, |f''(a_2)|^q\right\}\right)^{\frac{1}{q}} D_{1,\eta}(a_1,a_2) \tag{2.11}$$

where

$$D_{1,\eta}(a_1,a_2) := \left\{ \begin{array}{ll} \frac{\eta(a_2,a_1)}{2}\left[L_2^2(\alpha,a_1) - a_1 L(\alpha,a_1)\right], & a_1>0, \alpha>0 \\ \frac{2}{3}A\left(\alpha^3,a_1^3\right) - aA\left(\alpha^2,a_1^2\right), & a_1<0, \quad \alpha>0 \\ -\frac{\eta(a_2,a_1)}{2}\left[L_2^2(\alpha,a_1) - aL(\alpha,a_1)\right], & a_1<0, \ \alpha<0 \end{array} \right. .$$

Proof. From Lemma 2.1 and Power-mean integral inequality, we obtain

$$\begin{aligned} |I_f(a_1,a_2,\eta)| &\leq \eta^2(a_2,a_1)\int_0^1 t\,|\alpha_t|\,|f''(a_1+t\eta(a_2,a_1))|\,dt \\ &\leq \eta^2(a_2,a_1)\left(\int_0^1 t\,|\alpha_t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|\alpha_t|\,|f''(a_1+t\eta(a_2,a_1))|^q\,dt\right)^{\frac{1}{q}} \\ &\leq \eta^2(a_2,a_1)\left(\int_0^1 t\,|\alpha_t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|\alpha_t|\max\left\{|f''(a_1)|^q, |f''(a_2)|^q\right\}dt\right)^{\frac{1}{q}} \\ &= \eta^2(a_2,a_1)\left(\max\left\{|f''(a)|^q, |f''(a_2)|^q\right\}\right)^{\frac{1}{q}}\int_0^1 t\,|\alpha_t|\,dt \\ &= 4\left(\max\left\{|f''(a_1)|^q, |f''(a_2)|^q\right\}\right)^{\frac{1}{q}}\int_{a_1}^{a_1+\frac{\eta(b,a_1)}{2}}(x-a_1)\,|x|\,dt \\ &= 4\left(\max\left\{|f''(a_1)|^q, |f''(a_2)|^q\right\}\right)^{\frac{1}{q}} D_{1,\eta}(a_1,a_2) \end{aligned}$$

The proof is completed. □

Corollary 6. *Suppose that all the assumptions of Theorem 2.10 are satisfied. If we choose $\eta(a_2,a_1) = a_2 - a_1$ then when $|f''|^q$ is prequasiinvex on K for $q \geq 1$ we* *have*

$$\left|\frac{a_1+a_2}{2}f'(a_2)-\frac{f(a_2)a_2-f(a_1)a_1}{a_2-a_1}-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)dx\right|$$
$$\leq \quad 4\frac{D_1(a_1,a_2)}{a_2-a_1}\left[\max\left\{|f''(a_1)|^q,|f''(a_2)|^q\right\}\right]^{\frac{1}{q}}$$

where

$$D_1(a_1,a_2)=\begin{cases} \frac{a_2-a_1}{2}\left[L_2^2(A,a_1)-a_1L(A,a_1)\right], & a_1>0,\ A>0 \\ \frac{2}{3}A\left(A^3,a_1^3\right)-a_1A\left(A^2,a_1^2\right), & a_1<0,\ A>0 \\ -\frac{a_2-a_1}{2}\left[L_2^2(A,a_1)-a_1L(A,a_1)\right], & a_1<0,\ A<0 \end{cases}.$$

Remark 2.11. If we use the inequality (2.9) in the proof of Theorem 2.10, then the inequality (2.11) becomes the following inequality:

$$|I_f(a_1,a_2,\eta)|\leq 4\left(\max\left\{|f''(a_1)|^q,|f''(a_1+\eta(a_2,a_1))|^q\right\}\right)^{\frac{1}{q}}D_{1,\eta}(a_1,a_2)$$

This inequality is better than the inequality (2.11).

Corollary 7. *If we take $q=1$ in Theorem 7, then we have the following inequality:*

$$|I_f(a,b,\eta)|\leq 4\max\{|f''(a)|,|f''(b)|\}D_{1,\eta}(a,b)$$

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