Available online: June 18, 2018

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 1030–1044 (2019)
DOI: 10.31801/cfsuasmas.501455
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr/index.php?series=A1

ESTIMATION OF THE LOCATION AND THE SCALE PARAMETERS OF BURR TYPE XII DISTRIBUTION

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ABSTRACT. The aim of this paper is to estimate the location and the scale parameters of Burr Type XII distribution. For this purpose, different estimation methods, namely, maximum likelihood (ML), modified maximum likelihood (MML), least squares (LS) and method of moments (MM) are used. The performances of these estimation methods are compared via Monte-Carlo simulation study under different sample sizes and parameter settings. At the end of the study, the wind speed data set and the annual flow data sets are analyzed for illustration of the modeling performance of Burr Type XII distribution.

1. INTRODUCTION

The Burr Type XII distribution was first introduced by [12] as one of the Burr system of distributions. Since then, it has gained significant attention due to the potential of using it in practical studies [22]. Therefore, several authors have been applied Burr XII distribution to different areas such as engineering [13, 42], reliability [2, 25], survival analysis [40, 41], hydrology [23, 27], wind energy [11], actuarial science [19] and so forth.

In literature, there have been considerable number of studies concerning with the estimation of the unknown parameters of Burr Type XII distribution. For example, Hossain and Nath [16] considered the estimation of the shape parameters of Burr Type XII distribution using the least squares (LS), maximum likelihood (ML) and maximum product spacing (MPS) methodologies. Furthermore, they investigated the performances of these estimators when the data include outliers. Watkins [39] proposed an algorithm for maximum likelihood estimation in three-parameter Burr Type XII distribution. Abbasi et al. [1] used a neural network approach to estimate the location, the scale and the shape parameters of Burr XII distribution. Dogru and Arslan [14] obtained the optimal B-robust estimators for the shape parameters of the Burr Type XII distribution. In addition to these studies, inference for

Received by the editors: November 03, 2017; Accepted: June 12, 2018.
2010 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.
Key words and phrases. Burr XII distribution, maximum likelihood, modified maximum likelihood, least squares, method of moments, efficiency.

the Burr Type XII distribution has been considered under different scenarios such as type-II censored data [41], progressively censored data [29], multiple censored data [38], record data [21], progressive first-failure censored data [30] and middle-censored data [4]. Furthermore, many authors are interested in the generalized version of Burr Type XII distribution in recent years. For example, [15] proposed McDonald Burr Type XII distribution, [20] studied the Kumaraswamy exponentiated Burr Type XII distribution and [5] investigated the Weibull Burr Type XII distribution.

In this study, we consider the estimation of the location and the scale parameters of Burr Type XII distribution, see also [6]. To do this, firstly we use the well-known and widely used ML methodology. However, since the ML estimators of the parameters cannot be obtained explicitly, we here use Tiku's [34, 35] modified maximum likelihood (MML) methodology. We also obtain the LS and the method of moments (MM) estimators of the parameters of Burr Type XII distribution. The performances of the proposed estimators are compared via Monte-Carlo simulation study with respect to bias, mean square error (MSE) and deficiency (Def) criteria.

The rest of the paper is organized as follows. In Section 2, we introduce the Burr Type XII distribution. The detailed information about the parameter estimation methods used in this study is provided in Section 3. Section 4 includes an extensive Monte-Carlo simulation study. Two real life data are analyzed in Section 5. The paper is ended with some concluding remarks.

2. Burr Type XII distribution

The probability density function (pdf) and the cumulative density function (cdf) of the Burr Type XII distribution are given as follows

$$f(x)=\frac{kc}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{c-1}\left(1+\left(\frac{x-\mu}{\sigma}\right)^{c}\right)^{-(k+1)},\ x>\mu,\ \ c,k,\sigma>0,\ -\infty<\mu<\infty \tag{1}$$

and

$$F(x)=1-\left(1+\left(\frac{x-\mu}{\sigma}\right)^{c}\right)^{-k},\ \ x>\mu,\ \ c,k,\sigma>0,\ -\infty<\mu<\infty \tag{2}$$

respectively. Here μ is the location, σ is the scale, c and k are the shape parameters. Burr Type XII distribution with parameters μ, σ, c and k is shortly denoted by $Burr\,(\mu,\sigma,c,k)$. It should also be noted that Burr Type XII distribution is also known as Singh-Maddala distribution, see [28].

If $c\leq 1$, Burr Type XII distribution is L-shaped and if $c\leq 1$, it is unimodal. The combinations of c and k cover a wide range of skewness and kurtosis coefficients of some statistical distributions such as normal, Weibull, Lomax, logistic, Kappa and several Pearson-Type distributions etc. For example, Burr Type XII distribution reduces to normal distribution when c =4.8544 and k =6.2266. Also, the limiting

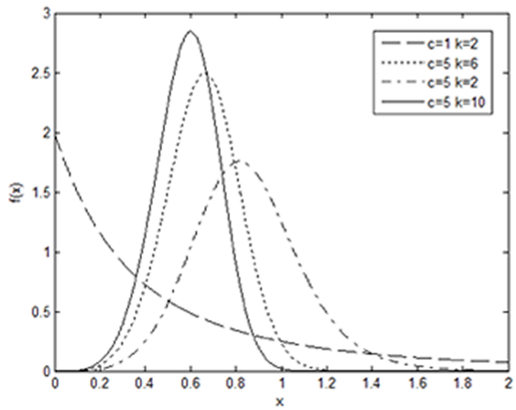

Figure 1. The density plots of Burr Type XII distribution for several values of shape parameters c and k.

distribution of (1) when $k \to \infty$ is Weibull, for more detailed information see [26, 33, 43].

The density plots of Burr Type XII distribution for several values of shape parameters c and k are illustrated in Figure 1. See also the following skewness $\left(\sqrt{\beta_1}\right)$ and kurtosis (β_2) values for better understanding the shape of Burr Type XII distribution:

(c, k)	(4,6)	(5,6)	(6,6)	(5,2)	(10,5)	(5,10)	(4,8)
$\sqrt{\beta_1}$	0.1779	-0.0135	-0.1468	0.6353	-0.3914	-0.1149	0.1062
β_2	3.0451	3.0099	3.0653	4.6304	3.4016	2.9280	2.9362

It should be noted that for existences of $\sqrt{\beta_1}$ and β_2, ck >3 and ck >4 are required, respectively [33]. It is clear from the skewness and the kurtosis values that Burr Type XII distribution may have short $(\beta_2 < 3)$ or long $(\beta_2 > 3)$ tails besides being symmetric or skewed. This provides flexibility for modeling various type of data sets.

3. Parameter Estimation Methods

In this section, we give detailed information about the parameter estimation methods which are used to estimate the location and the scale parameters of Burr Type XII distribution. It should be noted that the shape parameters of Burr XII distribution is assumed to be known throughout the study. This is because of the fact that efficiency of the estimators for μ and σ are reduced when they are estimated along with the shape parameters especially for small sizes. See also [10, 18] in the context of parameter estimation of three-parameter Weibull distribution. Therefore, the shape parameters c and k are treated as known in this section.

3.1. **Maximum Likelihood (ML) Method.** Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be a random sample of size n from Burr Type XII distribution. To obtain the ML estimators of the unknown location and scale parameters the log-likelihood function is written as given below

$$\ln L = n \ln k + n \ln c - n \ln \sigma + (c-1) \sum_{i=1}^{n} \ln \left(\frac{x_i - \mu}{\sigma} \right) - (k+1) \sum_{i=1}^{n} \ln \left(1 + \left(\frac{x_i - \mu}{\sigma} \right)^{c} \right)$$

After taking the derivatives of $\ln L$ with respect to the parameters μ and σ and equating them to zero, we obtain the following likelihood equations:

$$\frac{\partial \ln L}{\partial \mu} = -\frac{(c-1)}{\sigma} \sum_{i=1}^{n} g_1(z_i) + \frac{(k+1)c}{\sigma} \sum_{i=1}^{n} g_2(z_i) = 0, \tag{3}$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} - \frac{(c-1)}{\sigma} \sum_{i=1}^{n} z_i g_1(z_i) + \frac{(k+1)c}{\sigma} \sum_{i=1}^{n} z_i g_2(z_i) = 0. \tag{4}$$

Here, $z_i = (x_i - \mu)/\sigma$, $g_1(z_i) = 1/z_i$ and $g_2(z_i) = z_i^{c-1}/(1+z_i^c)$. It is clear that the likelihood equations cannot be solved explicitly because of the non-linear functions $g_1(z_i)$ and $g_2(z_i)$ in (3) and (4). Therefore, we resort to iterative methods for solving them.

3.2. **Modified Maximum Likelihood (MML) Method.** In previous subsection, it is mentioned that the ML estimators of the parameters are obtained by using the iterative methods. However, using iterative methods may cause problematic situations such as (i) convergence to wrong root, (ii) convergence to multiple root and (iii) nonconvergence of iterations [24, 36]. To avoid the difficulties encountered in iterative methods and to obtain the explicit estimators of the parameters, we use the MML methodology proposed by Tiku [34, 35].

The MML methodology proceeds as follows: Let $z_{(1)} \le z_{(2)} \le \cdots \le z_{(n)}$ be the order statistics obtained by arranging z_i, $i = 1, 2, \ldots, n$ in ascending order. Since complete sums are invariant to ordering (i.e. $\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} z_{(i)}$), the likelihood equations (3) and (4) are rewritten in terms of the order statistics. Then, the non-linear functions $g_1(.)$ and $g_2(.)$ are linearized by using Taylor series expansion around the expected values of standardized order statistics $t_{(i)} = E\left(z_{(i)}\right)$, $i = 1, 2 \ldots, n$ as given below

$$g_1\left(z_{(i)}\right) \cong \alpha_{1i} - \beta_{1i} z_{(i)} \quad \text{and} \quad g_2\left(z_{(i)}\right) \cong \alpha_{2i} + \beta_{2i} z_{(i)}, \quad i = 1, 2, \ldots, n. \tag{5}$$

Finally, incorporating the linearized functions in (5) into the likelihood equations in (3) and (4), we obtain the modified likelihood equations as shown below:

$$\frac{\partial \ln L^*}{\partial \mu} = -\frac{(c-1)}{\sigma}\sum_{i=1}^{n}\left(\alpha_{1i}-\beta_{1i}z_{(i)}\right)+\frac{(k+1)\,c}{\sigma}\sum_{i=1}^{n}\left(\alpha_{2i}+\beta_{2i}z_{(i)}\right)=0, \quad (6)$$

$$\frac{\partial \ln L^*}{\partial \sigma} = -\frac{n}{\sigma}-\frac{(c-1)}{\sigma}\sum_{i=1}^{n}z_{(i)}\left(\alpha_{1i}-\beta_{1i}z_{(i)}\right)+ \frac{(k+1)\,c}{\sigma}\sum_{i=1}^{n}z_{(i)}\left(\alpha_{1i}-\beta_{1i}z_{(i)}\right)=0. \quad (7)$$

The solutions of these equations are the following MML estimators of μ and σ:

$$\widehat{\mu}=K+D\widehat{\sigma} \quad \text{and} \quad \widehat{\sigma}=\frac{B+\sqrt{B^2+4nC}}{2\sqrt{n(n-1)}}, \quad (8)$$

where

$$t_{(i)}=\left(\left(1-\frac{i}{n+1}\right)^{-1/k}-1\right)^{1/c},$$

$$\alpha_{1i}=2/t_{(i)}, \quad \beta_{1i}=1/t_{(i)}^2,$$

$$\alpha_{2i}=\frac{t_{(i)}^{c-1}\left(2+2t_{(i)}^c-c\right)}{\left(1+t_{(i)}^c\right)^2}, \quad \beta_{2i}=\frac{(c-1)\,t_{(i)}^{c-2}-t_{(i)}^{2c-2}}{\left(1+t_{(i)}^c\right)^2},$$

$$\delta_i=(c-1)\,\beta_{1i}+c\,(k+1)\,\beta_{2i}, \quad m=\sum_{i=1}^{n}\delta_i, \quad K=\sum_{i=1}^{n}\delta_i x_{(i)}/m,$$

$$\Delta_i=c\,(k+1)\,\alpha_{2i}-(c-1)\,\alpha_{1i}, \quad D=\sum_{i=1}^{n}\Delta_i/m,$$

$$B=\sum_{i=1}^{n}\Delta_i\left(x_{(i)}-K\right), \quad C=\sum_{i=1}^{n}\delta_i\left(x_{(i)}-K\right)^2. \quad (9)$$

It should be noted that the divisor n in expression for $\widehat{\sigma}$ is replaced by $\sqrt{n\,(n-1)}$ as a bias correction.

Remark 1. *For some c and k, the values of the β_{2i} coefficients can be negative. This situation may cause $C<0$. This may yield negative or nonreal estimates of $\widehat{\sigma}$. To overcome this problem, the coefficients β_{2i} and α_{2i} are replaced by β_{2i}^* and α_{2i}^* as follows:*

$$\beta_{2i}^*=\frac{1+t_{(i)}^{c-2}}{\left(1+t_{(i)}^c\right)^2}, \quad \alpha_{2i}^*=\frac{t_{(i)}^{2c-1}-t_{(i)}}{\left(1+t_{(i)}^c\right)^2}, i=1,\ldots,n,$$

respectively. This alternative representation does not alter the asymptotic properties of the estimators since $z_{(i)} - t_{(i)} \cong 0$ *and consequently*

$$\alpha_{2i} + \beta_{2i} z_{(i)} \cong \alpha^*_{2(i)} + \beta^*_{2i} z_{(i)} \, (i = 1, \ldots, n) \, ,$$

see i.e. [17, 31].

3.3. **Least Squares (LS) Method.** The LS estimators are obtained by minimizing the following equation

$$\sum_{i=1}^{n} \left(F\left(X_{(i)}\right) - \frac{i}{n+1} \right)^2 \tag{10}$$

with respect to the parameters of interest, i.e. μ and σ [32]. Here, $F(\cdot)$ is the cdf, $X_{(i)}$ is the $i-$th ordered observation, i.e. $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ and $i/(n+1)$ are the expected values of $F\left(X_{(i)}\right)$. Therefore, in the context of Burr Type XII distribution, (10) reduces to

$$\sum_{i=1}^{n} \left(1 - \left(1 + \left(\frac{x_{(i)} - \mu}{\sigma} \right)^c \right)^{-k} - \frac{i}{n+1} \right)^2 . \tag{11}$$

Then, the LS estimators of the parameters are obtained by minimizing the equation (11) with respect to the parameters of interest. It is clear that numerical or iterative methods should also be performed to obtain the LS estimators.

3.4. **Method of Moments (MM) estimators.** The MM estimators of the location and the scale parameters of Burr Type XII distribution are obtained by equating the theoretical moments to the corresponding sample moments as shown below

$$\overline{x} = \mu + \frac{k\Gamma\left(k - 1/c\right)\Gamma\left(1/c + 1\right)}{\Gamma\left(k+1\right)}\sigma$$

and

$$s^2 = \frac{k\Gamma\left(k - 2/c\right)\Gamma\left(2k+1\right)\Gamma\left(k+1\right) - k^2\Gamma^2\left(k - 1/c\right)\Gamma^2\left(1/c+1\right)}{\Gamma^2\left(k+1\right)}\sigma^2.$$

Therefore, the MM estimators of the parameters μ and σ are obtained as given below

$$\widetilde{\mu} = \overline{x} - \frac{k\Gamma\left(k - 1/c\right)\Gamma\left(1/c + 1\right)}{\Gamma\left(k+1\right)}\widetilde{\sigma} \tag{12}$$

and

$$\widetilde{\sigma} = \sqrt{\frac{s^2\Gamma^2\left(k+1\right)}{k\Gamma\left(k - 2/c\right)\Gamma\left(2k+1\right)\Gamma\left(k+1\right) - k^2\Gamma^2\left(k - 1/c\right)\Gamma^2\left(1/c+1\right)}} \tag{13}$$

respectively. Here, $\overline{x}$ denotes the sample mean and s^2 stands for the sample standard deviation.

4. Simulation Study

In this section, we perform an extensive Monte Carlo simulation study to compare the performances of the LS, the MM, the ML and the MML estimators of the location parameter μ and the scale parameter σ of Burr Type XII distribution with respect to bias and MSE criteria. We also use the *Def* criterion for joint efficiencies of $\widehat{\mu}$ and $\widehat{\sigma}$ shown below

$$Def\left(\widehat{\mu},\widehat{\sigma}\right) = MSE\left(\widehat{\mu}\right) + MSE\left(\widehat{\sigma}\right). \tag{14}$$

The simulation study is performed for different sample sizes and different shape parameter settings. The sample sizes are considered as n =20, 50 and 100. Here, they are categorized as small (n =20), moderate (n =50) and large (n =100). To investigate the effect of the shape parameters on the efficiencies of the different parameter estimators, we consider the following cases.

(c, k) = (5,6) $\Rightarrow$ symmetric,

(c, k) = (5,2) $\Rightarrow$ long tailed positively skewed,

(c, k) = (4,8) $\Rightarrow$ short tailed positively skewed.

Without loss of generality, μ and σ are taken to be 0 and 1, respectively. It should also be noted that `"fminsearch"` function, available in the optimization toolbox of MATLAB software, is used for the numerical computations of ML, LS and MM estimators. The results of the simulation study are reported in Table 1. It should be stated that the smallest MSE and *Def* values for each setup are shown by bold face in this table.

It is clear from Table 1 that the performances of the ML, the MML and the MM estimators of μ and σ are more or less the same for all sample sizes when the shape of the distribution is symmetric, i.e., c =5, k =6. In this case, the LS estimator demonstrates the weakest performance with the highest bias, MSE and *Def* values.

In the long tailed positively skewed case, i.e. c =5, k =2, the MML estimators of μ and σ have the smallest bias and they are followed by the MM estimators. In the context of efficiency, the ML estimator demonstrates the strongest performances with the lowest deficiency for all sample sizes. It should be noted that the ML and the MML estimators are close to each other for the moderate and the large sample sizes, as expected. This is because of the fact that the MML estimator is asymptotically equivalent to the ML estimator, see [37]. On the other hand, the MM estimator performs better than the LS estimator. Similar to the symmetric case, the performances of the LS estimators are quite poor for all sample sizes.

When the shape parameters c =4, k =8, the biases of the MM estimators of μ and σ are lower than the other estimators for all sample sizes. They are followed by the MML estimators. The LS estimator underestimates μ for all sample sizes. In view of the MSE, the ML and the MML estimators perform better than the MM estimator does. Similar to other shape parameter settings, the LS estimator shows the worst performances. According to the *Def* values, the ML and the MML

TABLE 1. Simulated Mean, MSE and Def values for parameters μ and σ.

		Estimator	μ		σ		Def
			Mean	MSE	Mean	MSE	
20	$c=5,k=6$	ML	0.0261	**0.0120**	0.9613	**0.0252**	**0.0372**
		MML	0.0106	0.012	0.9858	0.0253	0.0373
		LS	-0.0331	0.0209	1.05	0.0451	0.0661
		MM	0.0077	0.012	0.9883	0.0253	0.0373
	$c=5,k=2$	ML	0.0324	**0.0211**	0.9635	**0.0306**	**0.0517**
		MML	0.007	0.0217	1.0003	0.0322	0.054
		LS	-0.0462	0.036	1.0569	0.0528	0.0889
		MM	0.0181	0.0257	0.9794	0.0387	0.0645
	$c=4,k=8$	ML	0.0246	**0.0082**	0.9564	0.0245	**0.0327**
		MML	0.015	0.0082	0.9767	**0.0244**	0.0326
		LS	-0.0254	0.0142	1.0449	0.0443	0.0586
		MM	0.0068	0.0083	0.9868	0.0248	0.0332
50	$c=5,k=6$	ML	0.0105	**0.0050**	0.9846	**0.0098**	**0.0148**
		MML	0.0053	0.005	0.9933	0.0099	0.015
		LS	-0.012	0.0077	1.0182	0.0158	0.0236
		MM	0.0027	0.0051	0.996	0.01	0.0151
	$c=5,k=2$	ML	0.0143	**0.0080**	0.983	**0.0118**	**0.0197**
		MML	0.0058	0.0081	0.9961	0.012	0.0201
		LS	-0.0145	0.0123	1.0169	0.0178	0.0301
		MM	0.007	0.011	0.9914	0.0167	0.0277
	$c=4,k=8$	ML	0.0111	**0.0034**	0.9817	**0.0097**	**0.0132**
		MML	0.0089	0.0035	0.9876	0.0099	0.0133
		LS	-0.0094	0.0055	1.0183	0.016	0.0215
		MM	0.0036	0.0036	0.9944	0.0102	0.0138
100	$c=5,k=6$	ML	0.0062	**0.0024**	0.9911	**0.005**	**0.0075**
		MML	0.0041	0.0024	0.995	0.0051	0.0075
		LS	-0.0051	0.0039	1.0078	0.0082	0.0121
		MM	0.002	0.0025	0.9973	0.0051	0.0076
	$c=5,k=2$	ML	0.0100	**0.0040**	0.989	**0.0061**	**0.0102**
		MML	0.0071	0.0041	0.9941	0.0062	0.0103
		LS	-0.005	0.0064	1.0068	0.0092	0.0156
		MM	0.0083	0.0056	0.9906	0.0088	0.0144
	$c=4,k=8$	ML	0.0038	**0.0014**	0.9946	**0.0042**	**0.0056**
		MML	0.0036	0.0014	0.9962	0.0042	0.0057
		LS	-0.0058	0.0025	1.0121	0.0074	0.01
		MM	0.0003	0.0015	1.0005	0.0045	0.006

estimators demonstrate the strongest performances and the LS estimator shows the worst performances among the others.

The results of the simulation study show that MML estimators can be preferred for the following two reasons: First, they are more efficient than the MM and the LS estimators and are as efficient as the ML estimators. Second, the MML estimators are explicitly formulated, i.e. they are expressed as the functions of sample observations. In other words, the MML estimators are easy to obtain since they do not require any iterations unlike the other methods used in this study. Therefore, we use the MML estimators in the rest of the paper for having high efficiency together with the computational simplicity.

5. Application

In this part of the study, we implement the considered estimation methods for modelling wind speed data and annual flow data sets.

5.1. **Wind speed data.** Now, we use the annual and seasonal wind speed data collected at Eskişehir, Turkey in 2009, see also [7, 8]. We model wind speed by using four parameter Burr Type XII distribution. The descriptive statistics for the data set are tabulated in Table 2.

Table 2. Descriptive statistics for the wind speed data.

Period	$\overline{x}$	s^2	$\sqrt{\beta_1}$	β_2	n
Annual	1.7264	0.6308	0.9727	4.5405	8759
Summer	1.8379	0.5009	0.8447	4.3239	2208
Spring	1.7945	0.6800	0.8781	3.8146	2207
Autumn	1.4423	0.4524	0.8848	3.8199	2184
Winter	1.8303	0.7846	1.0768	4.9018	2160

As mentioned in Section 3, the shape parameters c and k are assumed to be known and the simulation study is carried out under this assumption. However, the shape parameters should be estimated in real life problems in contrast to the simulation study, see for example [3, 9]. In this study, we therefore use the methodology known as profile likelihood to find the estimates of the shape parameters. The profile likelihood methodology is explained step by step below:

Step 1. Calculate $\widehat{\mu}_{MML}$ and $\widehat{\sigma}_{MML}$ for given c and k.

Step 2. Calculate the value of log-likelihood function using the following equation:

$$\begin{aligned}\ln L(\hat{\mu}_{MML}, \hat{\sigma}_{MML}, c, k) \quad = \quad & n \ln k + n \ln c - n \ln \hat{\sigma}_{MML} + (c-1) \sum_{i=1}^{n} \ln \left(\frac{x_i - \hat{\mu}_{MML}}{\hat{\sigma}_{MML}} \right) \\ & -(k+1) \sum_{i=1}^{n} \ln \left(1 + \left(\frac{x_i - \hat{\mu}_{MML}}{\hat{\sigma}_{MML}} \right)^{c} \right).\end{aligned}$$

Step 3. Repeat Step 1 and Step 2 for serious values of c and k.

Step 4. c and k values maximizing the log-likelihood function among the others are chosen as a plausible values of the shape parameters. The estimates of c and k obtained at the end of this step are also denoted by $\hat{c}$ and $\hat{k}$, respectively.

Table 3. The MML estimates for the wind speed data.

Period	$\hat{c}$	$\hat{k}$	$\hat{\mu}$	$\hat{\sigma}$	R^2
Annual	2.11	5.96	0.3000	3.5341	0.9981
Summer	2.77	3.40	0.2940	2.4960	0.9962
Spring	2.13	6.18	0.2942	3.7680	0.9972
Autumn	1.85	13.79	0.3000	5.0704	0.9967
Winter	2.06	5.14	0.3000	3.5166	0.9968

Step 5. $\widehat{\mu}_{MML}$ and $\widehat{\sigma}_{MML}$ are taken as the estimates of the location and the scale parameters corresponding to $\hat{c}$ and $\hat{k}$ obtained in Step 4.

[3, 17] are recommended for the details of this procedure.

The results of the parameter estimates are given in Table 3. The modelling performance of Burr Type XII distribution for the wind speed data is measured by using R^2 criterion. These criteria is formulated as follows:

$$R^2 = 1 - \frac{\sum \left(\hat{F}\left(X_{(i)}\right) - \dfrac{i}{n+1} \right)^2}{\sum \left(\hat{F}\left(X_{(i)}\right) - \overline{\hat{F}}\left(X_{(i)}\right) \right)^2}.$$

Here, $\hat{F}$ is the estimated cdf, $X_{(i)}$ is the $i-$th order statistics, $\overline{\hat{F}}=\sum \hat{F}_i/n$. The higher values of R^2 implies the better fitting.

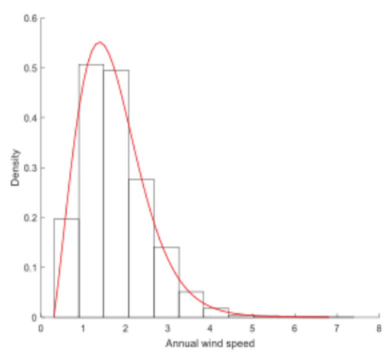

Figure 2. The histogram and the fitted Burr Type XII density for the annual wind speed data.

It is obvious from Table 3 that Burr Type XII distribution gives satisfactory results in terms of R^2 criterion since it is very close to 1. It should also be noted that Weibull distribution is commonly used for modelling the wind speed data, see

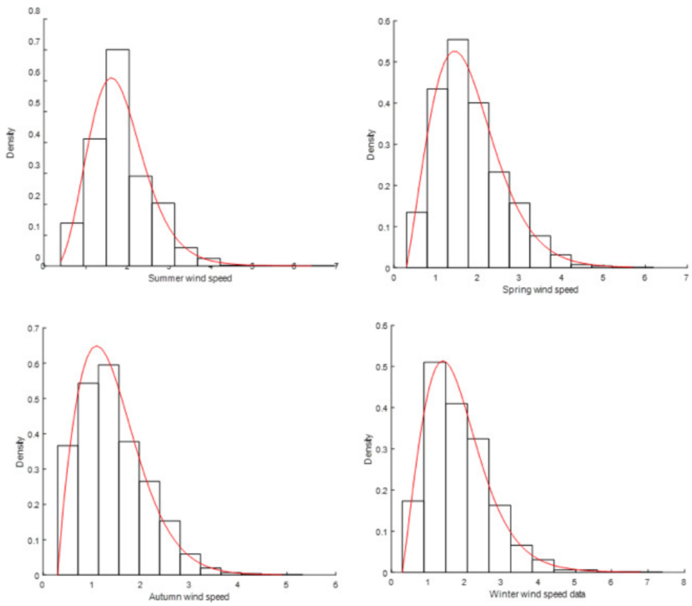

Figure 3. The histogram and the fitted Burr XII density for seasonal wind speed data.

also [7, 8]. Therefore, we also model the same wind speed data using Weibull distribution and compare the modelling performance with Burr Type XII distribution. The results show that Burr Type XII distribution is more preferable than Weibull distribution in terms of R^2 criterion. Since this issue is out of scope of the study, we do not include the results here for the sake of brevity. However, they can be provided upon request.

Figure 2-3 illustrates the density plots of Burr Type XII distribution based on the MML estimates given in Table 3. It is clear from these figures that the Burr Type XII distribution provides substantially good fitting performance.

5.2. **Annual flow data.** Here, we use the annual flow data from the Pearl River basin in China. This data set was collected by [27] and modeled by using extended three-parameter Burr Type XII distribution. In this application, we model this data set by using four parameter Burr Type XII distribution. The descriptive statistics for the data set are tabulated in Table 4.

Similar as in Subsection 5.1, the estimates of the shape parameters c and k are obtained by using profile likelihood methodology. Then, based on these estimate values, the MML estimates of the parameters μ and σ and R^2 values are given in Table 5.

TABLE 4. Descriptive statistics for the flow data.

Period	$\overline{x}$	s^2	$\sqrt{\beta_1}$	β_2	n
Annual	2.3575	0.0746	0.3637	3.5393	98

TABLE 5. The MML estimates for the flow data.

Period	$\hat{c}$	$\hat{k}$	$\hat{\mu}$	$\hat{\sigma}$	R^2
Annual	5.35	2.31	1.3172	1.2531	0.9939

It is clear from Table 5 that the Burr Type XII distribution provides very good fit to annual flow data with R^2 values close to 1. Furthermore, in Figure 4, we draw the density plots of Burr Type XII distribution based on the estimate values given in Table 5. The modeling performances of Burr Type XII distribution is illustrated apparently by this figure.

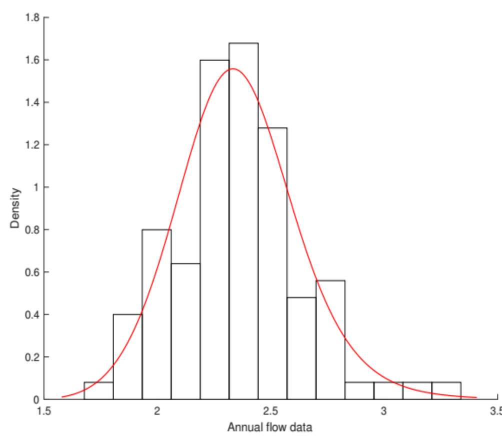

FIGURE 4. The histogram and the fitted Burr XII density for the annual flow data.

6. CONCLUSION

In this study, the location and the scale parameters of Burr Type XII distribution are estimated via ML, MML, LS and MM methods. Different than MML method, ML, LS and MM methods require iterative techniques in the estimation procedure. Therefore, MML estimators have closed forms and are easy to compute. The performances of the mentioned methods are evaluated using Monte-Carlo Simulation study under different sample sizes and parameter settings. The results of the simulation study demonstrate that the ML and the MML estimators are more preferable in terms of MSE and *Def* criteria. It should be noted that the ML estimators are the most efficient ones among the others, as expected. However, the MML estimators are also as good as the ML estimators and they do not require any iterative methods as mentioned above. Therefore, we conclude that if the concern is computational simplicity together with the efficiency, we propose to use the MML estimators as an alternative to ML estimators. Wind speed data set and annual flow data set are analyzed for illustration in the application part of the study. The results show that the modelling performance of Burr Type XII distribution based on the MML estimates is considerably good for these data sets.

In our future studies, we are planning to use the Burr Type XII distribution as an error terms distribution of a linear model since its flexible data-modeling characteristic. The model parameters will estimate by using MML methodology. The robustness properties of these estimators will also be investigated.

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