MANAS Journal of Engineering
ISSN 1694-7398 | e-ISSN 1694-7398

Volume 6 (Issue 2) (2018) Pages 177-192

Solutions of the Rational Difference Equations

$$x_{n+1}=\frac{x_{n-15}}{1+x_{n-3}x_{n-7}x_{n-11}}$$

Dağıstan Şimşek[*1,2], Peyil Esengul kyzy[1]

[1]Kyrgyz – Turkish Manas University, Department of Applied Mathematics and Informatics, Bishkek, Kyrgyzstan
[2]Konya Technical University, Faculty of Engineering and Natural Sciences, Konya, Turkey
dagistan.simsek@manas.edu.kg , dsimsek@selcuk.edu.tr

Received: 12.10.2018; Accepted: 13.12.2018

Abstract: *In this paper we study the solution behavior of the difference equation of the form*

$$x_{n+1}=\frac{x_{n-15}}{1+x_{n-3}x_{n-7}x_{n-11}},\quad n=0,1,2,3,...,$$

is examined. The initial conditions of the equation are arbitrary positive real numbers. Also, we discuss and illustrate the stability of the solutions in the neighborhood of the critical points and the periodicity of the considered equations.

Keywords: *Difference equation, period 16 solution*

1. INTRODUCTION

In recent times, the study of the periodic nature of nonlinear difference equations attracts a lot of interest. For some latest results that focus, among other problems, on the periodic nature of scalar nonlinear difference equations see, [3-19, 21-27]. Such problems often find applications in various fields of applied and theoretical mathematics, engineering, medicine, etc. In particular, they can be used to study the structural properties of various classes of functions, the behavior of the rational and algebraic polynomials and their derivatives and etc. (see, for example, [1, 2, 20])

Çinar, studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-1}}{1+ax_nx_{n-1}}$$

$$x_{n+1} = \frac{x_{n-1}}{-1+ax_nx_{n-1}}$$

$$x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}$$

for $n=0,1,2,...$ in [4, 5, 6], respectively.

In [21] Steviç assumed that $\beta=1$ and solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1+x_n} \qquad \text{for} \quad n=0,1,2,...,$$

where $x_{-1}, x_0 \in (0,\infty)$. Also**,** this results was generalized to the equation of the form:

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)} \qquad \text{for} \quad n=0,1,2,...,$$

where $x_{-1}, x_0 \in (0,\infty)$.

Şimşek et. al., studied the following problems

$$x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}}$$

with positive initial values for $n=0,1,2,...$ in [19,20], respectively.

In this paper we investigated the nonlinear difference equation of the form

$$x_{n+1} = \frac{x_{n-15}}{1 + x_{n-3}x_{n-7}x_{n-11}}, \quad n = 0,1,2,3,..., \tag{1}$$

where $x_{-15}, x_{-14}, x_{-13}, x_{-12}, x_{-11}, x_{-10}, x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (0,\infty)$

2. MAIN RESULT

Let $\bar{x}$ be the unique positive equilibrum of equation (1). Then clearly

$$\bar{x} = \frac{\bar{x}}{1 + \bar{x}\bar{x}\bar{x}} \Rightarrow \bar{x} + \bar{x}^{4} = \bar{x} \Rightarrow \bar{x}^{4} = 0 \Rightarrow \bar{x} = 0.$$

We can obtain $\bar{x} = 0$.

Theorem 1. Consider the difference equation (1). Then the following statements are true:

a) The sequences $(x_{16n-15}), (x_{16n-14}), (x_{16n-13}), (x_{16n-12}), (x_{16n-11}), (x_{16n-10}), (x_{16n-9}), (x_{16n-8}), (x_{16n-7}), (x_{16n-6}), (x_{16n-5}), (x_{16n-4}), (x_{16n-3}), (x_{16n-2}), (x_{16n-1}), (x_{16n})$ are decreasing and there exist $a_1, a_2, ..., a_{16} \geq 0$ such that

$$\lim_{n\to\infty} x_{16n-15} = a_1, \lim_{n\to\infty} x_{16n-14} = a_2, \lim_{n\to\infty} x_{16n-13} = a_3, \lim_{n\to\infty} x_{16n-12} = a_4, \lim_{n\to\infty} x_{16n-11} = a_5, \lim_{n\to\infty} x_{16n-10} = a_6$$
$$\lim_{n\to\infty} x_{16n-9} = a_7, \lim_{n\to\infty} x_{16n-8} = a_8, \lim_{n\to\infty} x_{16n-7} = a_9, \lim_{n\to\infty} x_{16n-6} = a_{10}, \lim_{n\to\infty} x_{16n-5} = a_{11}, \lim_{n\to\infty} x_{16n-4} = a_{12},$$
$$\lim_{n\to\infty} x_{16n-3} = a_{13}, \lim_{n\to\infty} x_{16n-2} = a_{14}, \lim_{n\to\infty} x_{16n-1} = a_{15}, \lim_{n\to\infty} x_{16n} = a_{16}$$

b) $(a_1, a_2, ..., a_{16}, a_1, a_2, ..., a_{16}, ...)$ is a solution of equation (1) of period 16.

c) $\lim_{n\to\infty} x_{16n-15} . \lim_{n\to\infty} x_{16n-11} . \lim_{n\to\infty} x_{16n-7} . \lim_{n\to\infty} x_{16n-3} = 0;$
$\lim_{n\to\infty} x_{16n-14} . \lim_{n\to\infty} x_{16n-10} . \lim_{n\to\infty} x_{16n-6} . \lim_{n\to\infty} x_{16n-2} = 0;$
$\lim_{n\to\infty} x_{16n-13} . \lim_{n\to\infty} x_{16n-9} . \lim_{n\to\infty} x_{16n-5} . \lim_{n\to\infty} x_{16n-1} = 0;$
$\lim_{n\to\infty} x_{16n-12} . \lim_{n\to\infty} x_{16n-8} . \lim_{n\to\infty} x_{16n-4} . \lim_{n\to\infty} x_{16n} = 0.$

or $a_1.a_5.a_9.a_{13} = 0$, $a_2.a_6.a_{10}.a_{14} = 0$, $a_3.a_7.a_{11}.a_{15} = 0$, $a_4.a_8.a_{12}.a_{16} = 0$.

d) If there exist $n_0 \in \mathbb{N}$ such that $x_{n+1} \leq x_{n-11}$ for all $n \geq n_0$, then $\lim_{n\to\infty} x_n = 0$.

e) The following formulas hold:

$$x_{16n+1}=x_{-15}\left(1-\frac{x_{-3}x_{-7}x_{-11}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\right);$$

$$x_{16n+2}=x_{-14}\left(1-\frac{x_{-2}x_{-6}x_{-10}}{1+x_{-2}x_{-6}x_{-10}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}\right);$$

$$x_{16n+3}=x_{-13}\left(1-\frac{x_{-1}x_{-5}x_{-9}}{1+x_{-1}x_{-5}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}\right);$$

$$x_{16n+4}=x_{-12}\left(1-\frac{x_{0}x_{-4}x_{-8}}{1+x_{0}x_{-4}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}\right);$$

$$x_{16n+5}=x_{-11}\left(1-\frac{x_{-3}x_{-7}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\right);$$

$$x_{16n+6}=x_{-10}\left(1-\frac{x_{-2}x_{-6}x_{-14}}{1+x_{-2}x_{-6}x_{-10}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}\right);$$

$$x_{16n+7}=x_{-9}\left(1-\frac{x_{-1}x_{-5}x_{-13}}{1+x_{-1}x_{-5}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}\right);$$

$$x_{16n+8}=x_{-8}\left(1-\frac{x_{0}x_{-4}x_{-12}}{1+x_{0}x_{-4}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}\right);$$

$$x_{16n+9}=x_{-7}\left(1-\frac{x_{-3}x_{-11}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\right);$$

$$x_{16n+10}=x_{-6}\left(1-\frac{x_{-2}x_{-10}x_{-14}}{1+x_{-2}x_{-6}x_{-10}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}\right);$$

$$x_{16n+11}=x_{-5}\left(1-\frac{x_{-1}x_{-9}x_{-13}}{1+x_{-1}x_{-5}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}\right);$$

$$x_{16n+12}=x_{-4}\left(1-\frac{x_{0}x_{-8}x_{-12}}{1+x_{0}x_{-4}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}\right);$$

$$x_{16n+13}=x_{-3}\left(1-\frac{x_{-7}x_{-11}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\right);$$

$$x_{16n+14}=x_{-2}\left(1-\frac{x_{-6}x_{-10}x_{-14}}{1+x_{-2}x_{-6}x_{-10}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}\right);$$

$$x_{16n+15}=x_{-1}\left(1-\frac{x_{-5}x_{-9}x_{-13}}{1+x_{-1}x_{-5}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}\right);$$

$$x_{16n+16}=x_{0}\left(1-\frac{x_{-4}x_{-8}x_{-12}}{1+x_{0}x_{-4}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}\right).$$

f) If $x_{16n+1}\to a_1\neq 0$, $x_{16n+5}\to a_5\neq 0$ and $x_{16n+9}\to a_9\neq 0$ then $x_{16n+13}\to 0$ as $n\to\infty$.

If $x_{16n+2}\to a_2\neq 0$, $x_{16n+6}\to a_6\neq 0$ and $x_{16n+10}\to a_{10}\neq 0$ then $x_{16n+14}\to 0$ as $n\to\infty$. İf $x_{16n+3}\to a_3\neq 0$, $x_{16n+7}\to a_7\neq 0$ and $x_{16n+11}\to a_{11}\neq 0$ then $x_{16n+15}\to 0$ as $n\to\infty$.

İf $x_{16n+4}\to a_4\neq 0$, $x_{16n+8}\to a_8\neq 0$ and $x_{16n+12}\to a_{12}\neq 0$ then $x_{16n+16}\to 0$ as $n\to\infty$.

Proof. a) Firstly, we consider the equation (1). From this equation we obtain:

$$x_{n+1}(1+x_{n-3}x_{n-7}x_{n-11})=x_{n-15}.$$

If $x_{n-11},x_{n-7},x_{n-3}\in(0,+\infty)$, then $(1+x_{n-3}x_{n-7}x_{n-11})\in(1,+\infty)$. Since $x_{n-15}>x_{n+1}$ $n_0\in\mathbb{N}$, we obtain

$$\lim_{n\to\infty}x_{16n-15}=a_1,\lim_{n\to\infty}x_{16n-14}=a_2,\lim_{n\to\infty}x_{16n-13}=a_3,\lim_{n\to\infty}x_{16n-12}=a_4,\lim_{n\to\infty}x_{16n-11}=a_5,\lim_{n\to\infty}x_{16n-10}=a_6$$
$$\lim_{n\to\infty}x_{16n-9}=a_7,\lim_{n\to\infty}x_{16n-8}=a_8,\lim_{n\to\infty}x_{16n-7}=a_9,\lim_{n\to\infty}x_{16n-6}=a_{10},\lim_{n\to\infty}x_{16n-5}=a_{11},\lim_{n\to\infty}x_{16n-4}=a_{12},$$
$$\lim_{n\to\infty}x_{16n-3}=a_{13},\lim_{n\to\infty}x_{16n-2}=a_{14},\lim_{n\to\infty}x_{16n-1}=a_{15},\lim_{n\to\infty}x_{16n}=a_{16}.$$

b) $(a_1,a_2,...,a_{16},a_1,a_2,...,a_{16},...)$ is a solution of equation (1) of period 16.

c) Taking into account the equation (1), we obtain:

$$x_{16n+1}=\frac{x_{16n-15}}{1+x_{16n-3}x_{16n-7}x_{16n-11}}.$$

Taking limit as $n\to\infty$ on both sides of the above equality, we get:

$$\lim_{n\to\infty}x_{16n+1}=\lim_{n\to\infty}\frac{x_{16n-15}}{1+x_{16n-3}x_{16n-7}x_{16n-11}}.$$

Then

$$\lim_{n\to\infty}x_{16n+1}.\lim_{n\to\infty}x_{16n-3}.\lim_{n\to\infty}x_{16n-7}.\lim_{n\to\infty}x_{16n-11}=0 \text{ or } a_1.a_5.a_9.a_{13}=0.$$

Similarly,

$$\lim_{n\to\infty}x_{16n+2}.\lim_{n\to\infty}x_{16n-10}.\lim_{n\to\infty}x_{16n-6}.\lim_{n\to\infty}x_{16n-2}=0 \text{ or } a_2.a_6.a_{10}.a_{14}=0.$$

$$\lim_{n\to\infty}x_{16n+3}.\lim_{n\to\infty}x_{16n-9}.\lim_{n\to\infty}x_{16n-5}.\lim_{n\to\infty}x_{16n-1}=0 \text{ or } a_3.a_7.a_{11}.a_{15}=0.$$

$$\lim_{n\to\infty} x_{16n+4}.\lim_{n\to\infty} x_{16n-8}.\lim_{n\to\infty} x_{16n-4}.\lim_{n\to\infty} x_{16n} = 0 \text{ or } a_4.a_8.a_{12}.a_{16} = 0.$$

d) If there exist $n_0 \in \mathbb{N}$ such that $x_{n-11} \ge x_{n+1}$ for all $n \ge n_0$, then
$a_2 \le a_6 \le a_{10} \le a_{14} \le a_2, a_3 \le a_7 \le a_{11} \le a_{15} \le a_3, a_4 \le a_8 \le a_{12} \le a_{16} \le a_4, a_5 \le a_9 \le a_{13} \le a_1 \le a_5$
Since $a_{16}^2 = 0,\ a_{15}^2 = 0,\ a_{14}^2 = 0,\ a_{13}^2 = 0,\ a_{12}^2 = 0,\ a_{11}^2 = 0,\ a_{10}^2 = 0,\ a_9^2 = 0,\ a_8^2 = 0,$
$a_7^2 = 0,\ a_6^2 = 0,\ a_5^2 = 0,\ a_4^2 = 0,\ a_3^2 = 0,\ a_2^2 = 0,\ a_1^2 = 0$ we obtain the result.

e) Subracting x_{n-15} from the left and right-hand sides of equation (1) we obtain:

$$x_{n+1} - x_{n-15} = \frac{1}{1 + x_{n-3}x_{n-7}x_{n-11}}(x_{n-3} - x_{n-19})$$

and the following formula holds:

$$n \ge 4 \text{ for } \begin{cases} x_{4n-15} - x_{4n-31} = (x_1 - x_{-11})\prod_{i=1}^{n-4}\frac{1}{1 + x_{4i-3}x_{4i-7}x_{4i-11}} \\ x_{4n-14} - x_{4n-30} = (x_2 - x_{-10})\prod_{i=1}^{n-4}\frac{1}{1 + x_{4i-2}x_{4i-6}x_{4i-10}} \\ x_{4n-13} - x_{4n-29} = (x_1 - x_{-11})\prod_{i=1}^{n-4}\frac{1}{1 + x_{4i-1}x_{4i-5}x_{4i-9}} \\ x_{4n-12} - x_{4n-28} = (x_2 - x_{-10})\prod_{i=1}^{n-4}\frac{1}{1 + x_{4i}x_{4i-4}x_{4i-8}} \end{cases} \qquad (2)$$

Replacing n by $4j$ in (2) and summing from $j = 0$ to $j = n$ we obtain:

$$\left.\begin{aligned} x_{16n+1} - x_{-15} &= (x_1 - x_{-15})\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1 + x_{4i-3}x_{4i-7}x_{4i-11}} \\ x_{16n+2} - x_{-14} &= (x_2 - x_{-14})\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1 + x_{4i-2}x_{4i-6}x_{4i-10}} \\ x_{12n+3} - x_{-13} &= (x_1 - x_{-13})\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1 + x_{4i-1}x_{4i-5}x_{4i-9}} \\ x_{12n+4} - x_{-12} &= (x_2 - x_{-12})\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1 + x_{4i}x_{4i-4}x_{4i-8}} \end{aligned}\right\}(n = 0, 1, 2, ...,) \qquad (3)$$

Also, replacing n by $4j+1$ in (2) and summing up elements from $j=0$ to $j=n$ we obtain:

$$\left.\begin{aligned} x_{16n+5}-x_{-11} &= (x_1-x_{-15})\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} \\ x_{16n+6}-x_{-10} &= (x_2-x_{-14})\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}} \\ x_{16n+7}-x_{-9} &= (x_1-x_{-13})\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}} \\ x_{16n+8}-x_{-8} &= (x_2-x_{-12})\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}} \end{aligned}\right\}(n=0,1,2,...,) \tag{4}$$

Also, replacing n by $4j+2$ in (2) and summing up elements from $j=0$ to $j=n$ we obtain:

$$\left.\begin{aligned} x_{16n+9}-x_{-7} &= (x_1-x_{-15})\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} \\ x_{16n+10}-x_{-6} &= (x_2-x_{-14})\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}} \\ x_{16n+11}-x_{-5} &= (x_1-x_{-13})\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}} \\ x_{16n+12}-x_{-4} &= (x_2-x_{-12})\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}} \end{aligned}\right\}(n=0,1,2,...,) \tag{5}$$

Replacing n by $4j+3$ in (2) in a similar way and summing up elements from $j=0$ to $j=n$ we obtain:

$$\left.\begin{aligned} x_{16n+13}-x_{-3}&=(x_1-x_{-15})\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\\ x_{16n+14}-x_{-2}&=(x_2-x_{-14})\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}\\ x_{16n+15}-x_{-1}&=(x_3-x_{-13})\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}\\ x_{16n+16}-x_{0}&=(x_4-x_{-12})\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}} \end{aligned}\right\}(n=0,1,2,...,) \qquad (6)$$

Thus, we attained the formulas below:

$$\begin{aligned} x_{16n+1}&=x_{-15}(1-\frac{x_{-3}x_{-7}x_{-11}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}});\\ x_{16n+2}&=x_{-14}(1-\frac{x_{-2}x_{-6}x_{-10}}{1+x_{-2}x_{-6}x_{-10}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}});\\ x_{16n+3}&=x_{-13}(1-\frac{x_{-1}x_{-5}x_{-9}}{1+x_{-1}x_{-5}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}});\\ x_{16n+4}&=x_{-12}(1-\frac{x_{0}x_{-4}x_{-8}}{1+x_{0}x_{-4}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}); \end{aligned} \qquad (7)$$

$$\begin{aligned} x_{16n+5}&=x_{-11}(1-\frac{x_{-3}x_{-7}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}});\\ x_{16n+6}&=x_{-10}(1-\frac{x_{-2}x_{-6}x_{-14}}{1+x_{-2}x_{-6}x_{-10}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}});\\ x_{16n+7}&=x_{-9}(1-\frac{x_{-1}x_{-5}x_{-13}}{1+x_{-1}x_{-5}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}});\\ x_{16n+8}&=x_{-8}(1-\frac{x_{0}x_{-4}x_{-12}}{1+x_{0}x_{-4}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}); \end{aligned} \qquad (8)$$

$$
\begin{aligned}
x_{16n+9} &= x_{-7}\Big(1-\frac{x_{-3}x_{-11}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big);\\
x_{16n+10} &= x_{-6}\Big(1-\frac{x_{-2}x_{-10}x_{-14}}{1+x_{-2}x_{-6}x_{-10}}\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}\Big);\\
x_{16n+11} &= x_{-5}\Big(1-\frac{x_{-1}x_{-9}x_{-13}}{1+x_{-1}x_{-5}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}\Big);\\
x_{16n+12} &= x_{-4}\Big(1-\frac{x_{0}x_{-8}x_{-12}}{1+x_{0}x_{-4}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{4j+2}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}\Big);
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
x_{16n+13} &= x_{-3}\Big(1-\frac{x_{-7}x_{-11}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big);\\
x_{16n+14} &= x_{-2}\Big(1-\frac{x_{-6}x_{-10}x_{-14}}{1+x_{-2}x_{-6}x_{-10}}\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}\Big);\\
x_{16n+15} &= x_{-1}\Big(1-\frac{x_{-5}x_{-9}x_{-13}}{1+x_{-1}x_{-5}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}\Big);\\
x_{16n+16} &= x_{0}\Big(1-\frac{x_{-4}x_{-8}x_{-12}}{1+x_{0}x_{-4}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{4j+3}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}\Big).
\end{aligned}
\tag{10}
$$

f) Suppose that $a_1 = a_5 = a_9 = a_{13} = 0$. By **e)** we have:

$$
\begin{aligned}
\lim_{n\to\infty} x_{16n+1} &= \lim_{n\to\infty} x_{-15}\Big(1-\frac{x_{-3}x_{-7}x_{-11}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big)\\
a_1 &= x_{-15}\Big(1-\frac{x_{-3}x_{-7}x_{-11}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big)\\
a_1 = 0 &\Rightarrow \frac{1+x_{-3}x_{-7}x_{-11}}{x_{-3}x_{-7}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}
\end{aligned}
\tag{11}
$$

Similarly,

$$
\begin{aligned}
\lim_{n\to\infty} x_{16n+5} &= \lim_{n\to\infty} x_{-11}\Big(1-\frac{x_{-3}x_{-7}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big)\\
a_5 &= x_{-11}\Big(1-\frac{x_{-3}x_{-7}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big)
\end{aligned}
$$

$$a_5 = 0 \Rightarrow \frac{1+x_{-3}x_{-7}x_{-11}}{x_{-3}x_{-7}x_{-15}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} \quad (12)$$

From equations (11) and (12),

$$\frac{1+x_{-3}x_{-7}x_{-11}}{x_{-3}x_{-7}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} > \frac{1+x_{-3}x_{-7}x_{-11}}{x_{-3}x_{-7}x_{-15}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} \quad (13)$$

thus, $x_{-11} > x_{-15}$.

Similarly,

$$\lim_{n\to\infty} x_{16n+9} = \lim_{n\to\infty} x_{-7}\Big(1-\frac{x_{-3}x_{-11}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big)$$

$$a_9 = x_{-7}\Big(1-\frac{x_{-3}x_{-11}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big)$$

$$a_9 = 0 \Rightarrow \frac{1+x_{-3}x_{-7}x_{-11}}{x_{-3}x_{-11}x_{-15}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} \quad (14)$$

From (12) and (14),

$$\frac{1+x_{-3}x_{-7}x_{-11}}{x_{-3}x_{-7}x_{-15}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} > \frac{1+x_{-3}x_{-7}x_{-11}}{x_{-3}x_{-11}x_{-15}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} \quad (15)$$

thus, $x_{-7} > x_{-11}$.

Similarly,

$$\lim_{n\to\infty} x_{16n+13} = \lim_{n\to\infty} x_{-3}\Big(1-\frac{x_{-7}x_{-11}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{n}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big)$$

$$a_{13} = x_{-3}\Big(1-\frac{x_{-7}x_{-11}x_{-15}}{1+x_{-3}x_{-7}x_{-11}}\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}\Big)$$

$$a_{13} = 0 \Rightarrow \frac{1+x_{-3}x_{-7}x_{-11}}{x_{-7}x_{-11}x_{-15}} = \sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} \quad (16)$$

From (14) and (16),

$$\frac{1+x_{-3}x_{-7}x_{-11}}{x_{-3}x_{-11}x_{-15}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}}>\frac{1+x_{-3}x_{-7}x_{-11}}{x_{-7}x_{-11}x_{-15}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-3}x_{4i-7}x_{4i-11}} \quad (17)$$

thus, $x_{-3}>x_{-7}$.

Hence we obtain $x_{-3}>x_{-7}>x_{-11}>x_{-15}$.

Suppose that $a_2=a_6=a_{10}=a_{14}=0$. The proofs of the following equations' correctness can be done similarly to the proofs of (13), (15), (17) and therefore, will be omitted here.

$$\frac{1+x_{-2}x_{-6}x_{-10}}{x_{-2}x_{-6}x_{-10}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}>\frac{1+x_{-2}x_{-6}x_{-10}}{x_{-2}x_{-6}x_{-14}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}} \quad (18)$$

$$\frac{1+x_{-2}x_{-6}x_{-10}}{x_{-2}x_{-6}x_{-14}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}>\frac{1+x_{-2}x_{-6}x_{-10}}{x_{-2}x_{-10}x_{-14}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}} \quad (19)$$

$$\frac{1+x_{-2}x_{-6}x_{-10}}{x_{-2}x_{-10}x_{-14}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}}>\frac{1+x_{-2}x_{-6}x_{-10}}{x_{-6}x_{-10}x_{-14}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-2}x_{4i-6}x_{4i-10}} \quad (20)$$

thus, $x_{-2}>x_{-6}>x_{-10}>x_{-14}$.

For the case where $a_3=a_7=a_{11}=a_{15}=0$, the proofs is similar to the proofs of the (13), (15), (17) and therefore, will be omitted here.

$$\frac{1+x_{-1}x_{-5}x_{-9}}{x_{-1}x_{-5}x_{-9}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}>\frac{1+x_{-1}x_{-5}x_{-9}}{x_{-1}x_{-5}x_{-13}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}} \quad (21)$$

$$\frac{1+x_{-1}x_{-5}x_{-9}}{x_{-1}x_{-5}x_{-13}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}>\frac{1+x_{-1}x_{-5}x_{-9}}{x_{-1}x_{-9}x_{-13}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}} \quad (22)$$

$$\frac{1+x_{-1}x_{-5}x_{-9}}{x_{-1}x_{-9}x_{-13}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}}>\frac{1+x_{-1}x_{-5}x_{-9}}{x_{-5}x_{-9}x_{-13}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i-1}x_{4i-5}x_{4i-9}} \quad (23)$$

thus, $x_{-1}>x_{-5}>x_{-9}>x_{-13}$.

Suppose that $a_4=a_8=a_{12}=a_{16}=0$. As in the cases above, the proofs of the following equations is similar to the proofs of (13), (15), (17) and therefore, will be omitted here as well.

$$\frac{1+x_0x_{-4}x_{-8}}{x_0x_{-4}x_{-8}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}>\frac{1+x_0x_{-4}x_{-8}}{x_0x_{-4}x_{-12}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}} \quad (24)$$

$$\frac{1+x_0x_{-4}x_{-8}}{x_0x_{-4}x_{-12}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}>\frac{1+x_0x_{-4}x_{-8}}{x_0x_{-8}x_{-12}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}} \quad (25)$$

$$\frac{1+x_0x_{-4}x_{-8}}{x_0x_{-8}x_{-12}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}}>\frac{1+x_0x_{-4}x_{-8}}{x_{-4}x_{-8}x_{-12}}=\sum_{j=0}^{\infty}\prod_{i=1}^{4j+1}\frac{1}{1+x_{4i}x_{4i-4}x_{4i-8}} \quad (26)$$

thus, $x_0 > x_{-4} > x_{-8} > x_{-12}$.

Hence we obtain $x_0 > x_{-4} > x_{-8} > x_{-12}$, $x_{-1} > x_{-5} > x_{-9} > x_{-13}$, $x_{-2} > x_{-6} > x_{-10} > x_{-14}$, $x_{-3} > x_{-7} > x_{-11} > x_{-15}$. Thus, we face a contradiction which completes the proof of theorem.

3. EXAMPLES

Example 3.1: Consider the following equation $x_{n+1}=\dfrac{x_{n-15}}{1+x_{n-3}x_{n-7}x_{n-11}}$.

If the initial conditions are selected as:

$x[-15]=2; x[-14]=3; x[-13]=4; x[-12]=5; x[-11]=6; x[-10]=7; x[-9]=8;$
$x[-8]=9; x[-7]=10; x[-6]=11; x[-5]=12; x[-4]=13; x[-3]=14; x[-2]=15;$
$x[-1]=16; x[0]=17;$

the following solutions are obtained:

x(n) = {0.0023781212841854932,0.0025951557093425604,0.002602472348731295,
0.002512562814070352,4.5013380909901874,4.901271956390066,5.334490,
5.86752827140549,8.6966640806827,9.237538148524923,9.8189563365282,
10.42357512953368,12.807665010645849,13.422849340009103,14.080599812,
14.76265466816648,0.000004733739666817867,0.000004263238483558193,
0.0000035238568955208630,0.0000028184698278112774,4.498965954184857,
4.898682432783714,5.331892555703612,5.784244170813406,…}

The graph of the solutions is given in Fig.1.

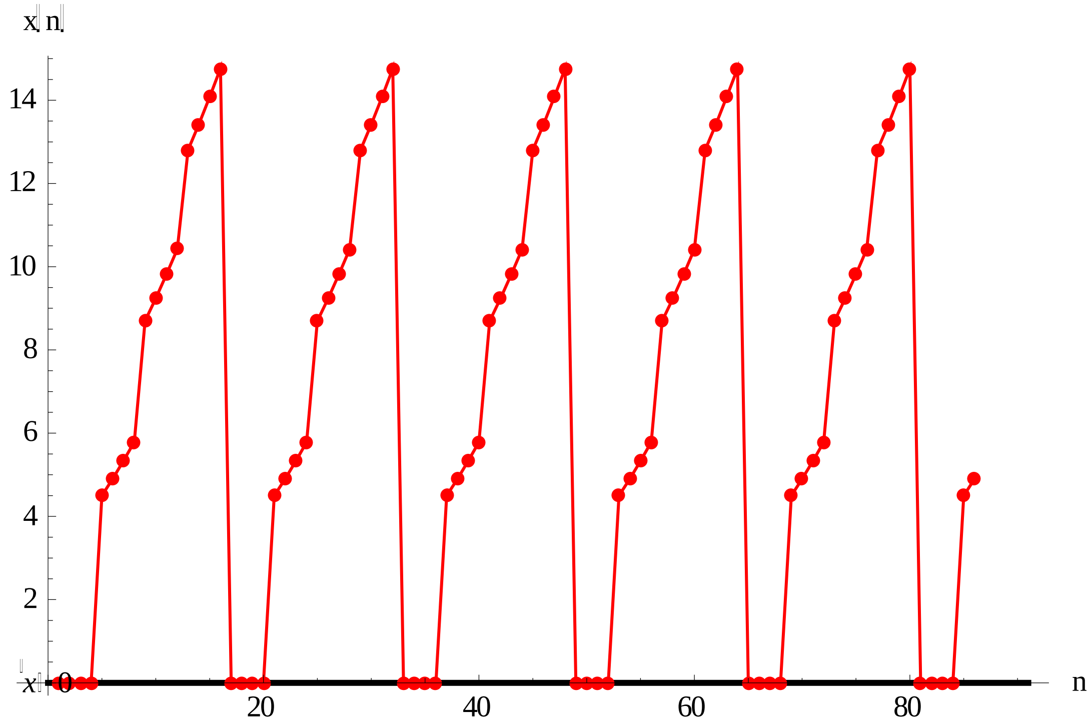

Figure 1. *x(n) graph of the solutions.*

Example 3.2: Consider the following equation $x_{n+1} = \dfrac{x_{n-15}}{1+x_{n-3}x_{n-7}x_{n-11}}$.

If the initial conditions are selected as follows:

$x[-15] = 0.999; x[-14] = 0.998; x[-13] = 0.997; x[-12] = 0.996; x[-11] = 0.995;$
$x[-10] = 0.994; x[-9] = 0.993; x[-8] = 0.992; x[-7] = 0.991; x[-6] = 0.99; x[-5] = 0.989;$
$x[-4] = 0.988; x[-3] = 0.987; x[-2] = .0986; x[-1] = 0.985; x[0] = 0.984;$

the following solutions are obtained:

x(n)={0.5062774309150935,0.9097303920457783,0.5067741406697103,0.5070213125858963,
0.6654634543918772,0.9129296366989845,0.6648003025279128,0.6644687113750755,
0.74369846352699,0.9150659317837468,0.7425760891664093,0.7420150386695299,
0.7892471743846864,0.05602335091853418,0.7878887127855291,0.7872096248661782,
0.36407076922682924,0.8690573002926931,0.3648605575249685,0.365254820148283,…}

The graph of the solutions is given in Fig.2.

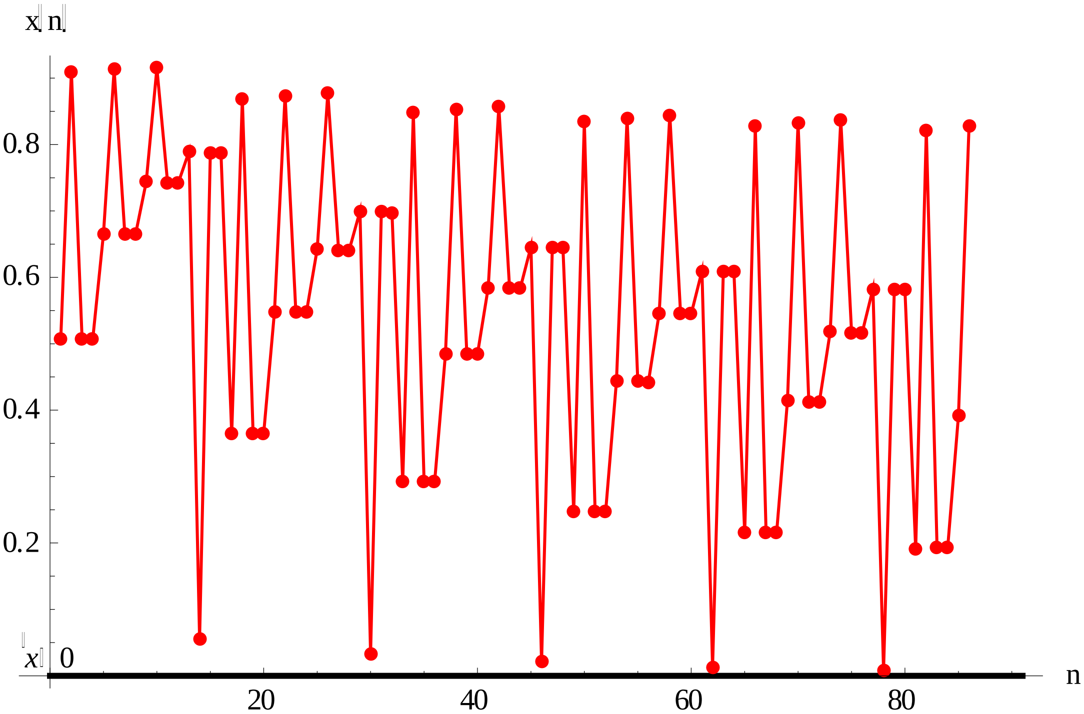

Figure 2. *x*(*n*) *graph of the solutions.*

REFERENCES:

[1] Abdullayev F.G., Ozkartepe N.P., On the ımprovement of the rate of convergence of the generalızed Bıeberbach polynomıals ın regıons wıth zero angles, *Ukr.Math. J.,* 64 (5), (2012), 653-671.

[2] Abdullayev F.G., Ozkartepe N.P., Unıform Convergence of the p-Bıeberbach Polynomıals in Domaıns wıth Zero Angles, *Science China Math,* 58 (5), (2015), 1063-1078.

[3] Amleh A. M., Grove E. A., Ladas G., Georgiou D. A., On the recursive sequence $x_{n+1}=\alpha+\frac{x_{n-1}}{x_n}$, *J. Math. Anal. Appl.*, 233 (2), (1999), 790-798.

[4] Çinar C., On the positive solutions of the difference equation $x_{n+1}=\frac{x_{n-1}}{1+ax_nx_{n-1}}$, *Appl. Math. Comp.*, 158 (3), (2004), 809–812.

[5] Çinar C., On the positive solutions of the difference equation $x_{n+1}=\frac{x_{n-1}}{-1+ax_nx_{n-1}}$, *Appl. Math. Comp.*, 158 (3), (2004), 793–797.

[6] Çinar C., On the positive solutions of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}$, *Appl. Math. Comp*., 156 (3), (2004), 587–590.

[7] Elabbasy E. M., El-Metwally H., Elsayed E. M., On the difference equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$, *Advances in Difference Equation*, (2006), 1-10.

[8] Elabbasy E. M., El-Metwally H., Elsayed E. M., Qualitative behavior of higher order difference equation, *Soochow Journal of Mathematics*, 33 (4), (2007), 861-873.

[9] Elabbasy E. M., El-Metwally H., Elsayed E. M., Global attractivity and periodic character of a fractional difference equation of order three,*Yokohama Mathematical Journal*, 53, (2007), 89-100.

[10] Elabbasy E. M., El-Metwally H., Elsayed E. M., On the difference equation $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$, *J. Conc. Appl. Math*., 5 (2), (2007), 101-113.

[11] Elabbasy E. M. and Elsayed E. M., On the Global Attractivity of Difference Equation of Higher Order,*Carpathian Journal of Mathematics*, 24 (2), (2008), 45–53.

[12] Elsayed E. M., On the Solution of Recursive Sequence of Order Two, *Fasciculi Mathematici*, 40, (2008), 5–13.

[13] Elsayed E. M., Dynamics of a Recursive Sequence of Higher Order, *Communications on Applied Nonlinear Analysis*, 16 (2), (2009), 37–50.

[14] Elsayed E. M., Solution and atractivity for a rational recursive sequence, *Discrete Dynamics in Nature and Society*, (2011),17.

[15] Elsayed E. M., On the solution of some difference equation, *Europan Journal of Pure and Applied Mathematics*, 4 (3), (2011), 287–303.

[16] Elsayed E. M., On the Dynamics of a higher order rational recursive sequence, *Communications in Mathematical Analysis*, 12 (1), (2012), 117–133.

[17] Elsayed E. M., Solution of rational difference system of order two, *Mathematical and Computer Modelling*, 55, (2012), 378–384.

[18] Gibbons C. H., Kulenović M. R. S. and Ladas G., *On the recursive sequence* $x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\chi + x_n}$, *Math. Sci. Res. Hot-Line*, 4 (2), (2000), 1-11.

[19] Kulenović M.R.S., Ladas G.,Sizer W.S., On the recursive sequence $x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\chi x_n + \delta x_{n-1}}$ *Math. Sci. Res. Hot-Line*, 2, 5, (1998), 1-16.

[20] Özkartepe N.P., Abdullayev F.G., Interference of the weight and contour for algebraic polynomials in the weighted Lebesgue spaces I , *Ukr. Math, 68 (10), (2017),*1574–1590.

[21] Steviç S., On the recursive sequence $x_{n+1} = \frac{x_{n-1}}{g(x_n)}$, *Taiwan. J. Math.*, 6 (3), (2002), 405-414.

[22] Şimşek D., Çınar C. and Yalçınkaya İ., On the recursive sequence $x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$, *Int. J. Contemp. Math. Sci.*, 1 (9-12), (2006), 475-480.

[23] Şimşek D., Çınar C., Karataş R., Yalçınkaya İ., On the recursive sequence $x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}}$, *Int. J. Pure Appl. Math.*, 27, 4, (2006), 501-507.

[24] Şimşek D., Çınar C., Karataş R., Yalçınkaya İ., On the recursive sequence $x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}x_{n-3}}$ *Int. J. Pure Appl. Math.*, 28, 1, (2006), 117-124.

[25] Şimşek D., Çınar C., Yalçınkaya İ., On The Recursive Sequence *x(n+1) = x[n-(5k+9)] / 1+x(n-4)x(n-9) x[n-(5k+4)]*, *Taiwan.Journal of Mathematics*, Vol. 12, 5, (2008), 1087-1098.

[26] Şimşek D., Doğan A., On a Class of Recursive Sequence, *Manas Journal of Engineering*, 2, 1, (2014), 16-22.

[27] Şimşek D., Eröz M., Solutions of the Rational Difference Equations $x_{n+1} = \frac{x_{n-3}}{1+x_n x_{n-1} x_{n-2}}$, *Manas Journal of Engineering*, 4, 1, (2016), 12-20.