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Minimum Degree and Size Conditions for Hamiltonian and Traceable Graphs

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Abstract

A graph is called Hamiltonian (resp. traceable) if the graph has a Hamiltonian cycle (resp. path), a cycle (resp. path) containing all the vertices of the graph. In this note, we present sufficient conditions involving minimum degree and size for Hamiltonian and traceable graphs. One of the sufficient conditions strengthens the result obtained by Nikoghosyan in [1].

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. The complement of a graph G is denoted by G^c. we use G_r to denote any graph of order r. A graph G is empty if the graph G does not have any edge. We use $G_1 \vee G_2$ to denote the the join of two disjoint graphs G_1 and G_2. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. We define

$$\mathscr{A}(n) := \{G : G \text{ is } G_{\frac{n-2}{2}} \vee (K^c_{\frac{n-2}{2}} \cup K_2)\},$$

$$\mathscr{B}(n) := \{G : G \text{ is } G_{\frac{n-2}{2}} \vee K^c_{\frac{n+2}{2}}\},$$

$$\mathscr{C}(n) := \{G : G \text{ is } G_{\frac{n-1}{2}} \vee K^c_{\frac{n+1}{2}}\}$$

and

$$\mathscr{D}(n) := \mathscr{S}(n) \cup \mathscr{T}(n),$$

where $\mathscr{S}(n) := \{G : G \text{ is } w \vee (P \cup Q)$, where w is a vertex cut such that $G - \{w\}$ has exactly two components of P and Q which are complete graphs of order $\frac{n-1}{2}\}$,
$\mathscr{T}(n) := \{G : G$ has a vertex cut w such that $G - \{w\}$ has exactly two components of P and Q, where P is a complete graph of order $\frac{n-2}{2}$ and w is adjacent to each vertex in P, Q is a graph of order $\frac{n}{2}$ with $\delta(Q) \geq \frac{n-4}{2}$, and $\delta(G) \geq \frac{n-2}{2}\}$.

$$\mathscr{X}(n) := \{G : G \text{ is } K_{\frac{n}{2}} \cup K_{\frac{n}{2}}\}.$$

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$$\mathscr{Y}(n) := \{G : G \text{ is } K_{\frac{n-1}{2}} \cup H, \text{ where } H \text{ is a } \left(\frac{n-3}{2}\right) - \text{ regular graph of order } \frac{n+1}{2}\}.$$

Nikoghosyan obtained the following sufficient condition for Hamiltonian graphs in [1] (also see [3]).

Theorem 1.1. *Let G be a graph of order $n \geq 3$, size e, and minimum degree δ. If $\delta^2 + \delta \geq e+1$, then G is Hamiltonian.*

Motivated by Nikoghosyan's result above, we in this note strengthen Theorem 1.1 to the following Theorem 1.2 and present an analogous sufficient condition for the traceable graphs.

Theorem 1.2. *Let G be a graph of order $n \geq 3$, size e, and minimum degree δ. If $\delta^2 + \delta \geq e$, then G is empty or G is Hamiltonian or $G \in \mathscr{A}(n) \cup \mathscr{B}(n) \cup \mathscr{C}(n) \cup \mathscr{D}(n) \cup \mathscr{X}(n)$.*

Theorem 1.3. *Let G be a graph of order $n \geq 2$, size e, and minimum degree δ. If $\delta^2 + \frac{3\delta}{2} \geq e$, then G is empty or G is traceable or $G \in \mathscr{X}(n) \cup \mathscr{Y}(n)$.*

2. Lemmas

In order to prove Theorem 1.1 and Theorem 1.2, we need the following results as our lemmas. The first one follows from Theorem 2 proved by Zhao in [4].

Lemma 2.1. *If G is a connected graph of order $n \geq 3$ and $\delta \geq \frac{n-2}{2}$, then G is Hamiltonian or $G \in \mathscr{A}(n) \cup \mathscr{B}(n) \cup \mathscr{C}(n) \cup \mathscr{D}(n)$.*

Notice that the statements in Lemma 2.1 are slightly different from the statements in Theorem 2 in [4]. The reason for this is the convenience when we use Lemma 2.1 in our proofs.
The second one is Theorem 2.5 proved by Cranston and O in [5].

Lemma 2.2. *Every connected k-regular graph with at most $3k+3$ vertices has a Hamiltonian path.*

3. Proofs

Proof of Theorem 1.2 Let G be a graph satisfying the conditions in Theorem 1.2. If $\delta = 0$, then G is empty. From now on, we assume that $\delta \geq 1$. Suppose that G is not Hamiltonian. Then, from the conditions in Theorem 1.2, we have that

$$\delta^2 + \delta \geq e \geq \frac{\sum_{v \in V(G)} d(v)}{2} \geq \frac{n\delta}{2}.$$

Therefore $\delta \geq \frac{n-2}{2}$.

Case 1 G is disconnected.

Suppose G consists of k ($k \geq 2$) components G_1 of order n_1, G_2 of order n_2, $\cdots$, G_k of order n_k. Without loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. Then we have $2n_1 \leq \sum_{i=1}^{k} n_i = n$. Thus $n_1 \leq \frac{n}{2}$. Therefore $\frac{n-2}{2} \leq \delta \leq d(x) \leq n_1 - 1 \leq \frac{n-2}{2}$, where x is any vertex in G_1. Hence $\frac{n-2}{2} = \delta = n_1 - 1 = \frac{n-2}{2}$. So $\delta^2 = \frac{n-2}{2}\delta$ and $\delta^2 + \delta = \frac{n\delta}{2}$. Now we have

$$\frac{n\delta}{2} \leq \frac{\sum_{v \in V(G)}}{2} \leq e \leq \delta^2 + \delta = \frac{n\delta}{2}.$$

Thus G is δ-regular graph with $\delta = \frac{n-2}{2}$ and $e = \delta^2 + \delta$. Notice that $\frac{n}{2} = n_1 \leq n_2 \leq \cdots \leq n_k$. We must have $k = 2$, $n_2 = \frac{n}{2}$, and G_1 and G_2 are complete graphs of order $\frac{n}{2}$. Therefore $G \in \mathscr{X}(n)$.

Case 2 G is connected.

From Lemma 2.1, we have $G \in \mathscr{A}(n) \cup \mathscr{B}(n) \cup \mathscr{C}(n) \cup \mathscr{D}(n)$.

Hence, the proof of Theorem 1.2 is complete.

Proof of Theorem 1.3 Let G be a graph satisfying the conditions in Theorem 1.3. Notice that G is empty when $\delta = 0$ and G is empty or traceable when $n = 2$ or 3. From now on, we assume that $\delta \geq 1$ and $n \geq 4$. Suppose that G is not traceable. Then, from the conditions in Theorem 1.2, we have that

$$\delta^2 + \frac{3\delta}{2} \geq e \geq \frac{\sum_{v \in V(G)} d(v)}{2} \geq \frac{n\delta}{2}.$$

Therefore $\delta \geq \frac{n-3}{2}$.

Case 1 G is disconnected.

Suppose G consists of k ($k \geq 2$) components G_1 of order n_1, G_2 of order n_2, $\cdots$, G_k of order n_k. Without loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. Then we have $2n_1 \leq \sum_{i=1}^{k} n_i = n$. Thus $n_1 \leq \frac{n}{2}$. Therefore $\delta \leq d(x) \leq n_1 - 1 \leq \frac{n-2}{2}$, where x is any vertex in G_1.

Case 1.1 $\delta = \frac{n-2}{2}$.

Thus $\frac{n-2}{2} \le \delta \le d(x) \le n_1 - 1 \le \frac{n-2}{2}$, where x is any vertex in G_1. Therefore $\frac{n-2}{2} = \delta = d(x) = n_1 - 1 = \frac{n-2}{2}$, where x is any vertex in G_1. Hence G_1 is a complete graph of order $\frac{n}{2}$. Notice that $\frac{n}{2} = n_1 \le n_2 \le \cdots \le n_k$. We must have $k = 2$ and $n_2 = \frac{n}{2}$. Since $n_2 = \frac{n}{2}$ and $\frac{n-2}{2} = n_2 - 1 \ge d(y) \ge \delta = \frac{n-2}{2}$ for any vertex y in G_2, G_2 is a complete graph of order $\frac{n}{2}$. So $G \in \mathscr{X}(n)$.

Case 1.2 $\delta = \frac{n-3}{2}$.

Thus $\delta^2 = \frac{n-3}{2}\delta$ and $\delta^2 + \frac{3\delta}{2} = \frac{n\delta}{2}$. Now we have

$$\frac{n\delta}{2} \le \frac{\sum_{v\in V(G)}}{2} \le e \le \delta^2 + \frac{3\delta}{2} = \frac{n\delta}{2}.$$

Thus G is δ-regular graph with $\delta = \frac{n-3}{2}$ and $e = \delta^2 + \frac{3\delta}{2}$. Notice now that n is odd. Then $n_1 \le \frac{n}{2}$ implies that $n_1 \le \frac{n-1}{2}$. Thus for any vertex x in G_1 we have $\frac{n-3}{2} = d(x) \le n_1 - 1 \le \frac{n-3}{2}$. Therefore G_1 is a complete graph of order $\frac{n-1}{2}$. Notice that $\frac{n-1}{2} = n_1 \le n_2 \le \cdots \le n_k$. We must have $k = 2$ and $n_2 = \frac{n+1}{2}$. Hence G_2 is a $\left(\frac{n-3}{2}\right)$-regular graph of order $\frac{n+1}{2}$. So $G \in \mathscr{Y}(n)$.

Case 2 G is connected.

Case 2.1 n is even.

Then $\delta \ge \frac{n-3}{2}$ implies that $\delta \ge \frac{n-2}{2}$. From Lemma 2.1, we have G is Hamiltonian or $G \in \mathscr{A}(n) \cup \mathscr{B}(n) \cup \mathscr{C}(n) \cup \mathscr{D}(n)$.

First, we prove that it is impossible that $G \in \mathscr{B}(n)$. Suppose, to the contrary, that $G \in \mathscr{B}(n)$. Then $\delta = \frac{n-2}{2}$. Clearly, $e \ge \frac{n^2-4}{4}$. Then we can get a contradiction from

$$\delta^2 + \frac{3\delta}{2} \ge e \ge \frac{n^2-4}{4}.$$

Obviously, G is traceable when G is Hamiltonian. It is easy to verify that G is traceable when $G \in \mathscr{A}(n) \cup \mathscr{C}(n) \cup \mathscr{S}(n)$. When $G \in \mathscr{T}(n)$, notice that $\delta(Q) \ge \frac{|V(Q)|}{2}$ when $n \ge 8$. Thus Q is Hamiltonian when $n \ge 8$. It is easy to verify that G is traceable when $n \ge 8$. When $n = 4$ or 6, we can also verify that G is traceable. Hence we arrive at a contradiction.

Case 2.2 n is odd.

Then $\delta \ge \frac{n-3}{2} + 1 = \frac{n-1}{2}$ or $\delta = \frac{n-3}{2}$.

When $\delta \ge \frac{n-3}{2} + 1 = \frac{n-1}{2}$, then $G \notin \mathscr{A}(n) \cup \mathscr{B}(n) \cup \mathscr{T}(n)$. From Lemma 2.1, we have G is Hamiltonian or $G \in \mathscr{C}(n) \cup \mathscr{S}(n)$. Obviously, G is traceable when G is Hamiltonian or $G \in \mathscr{C}(n) \cup \mathscr{S}(n)$. Hence we arrive at a contradiction.

When $\delta = \frac{n-3}{2}$, then $\delta^2 = \frac{n-3}{2}\delta$ and $\delta^2 + \frac{3\delta}{2} = \frac{n\delta}{2}$. Now we have

$$\frac{n\delta}{2} \le \frac{\sum_{v\in V(G)}}{2} \le e \le \delta^2 + \frac{3\delta}{2} = \frac{n\delta}{2}.$$

Thus G is δ-regular graph with $\delta = \frac{n-3}{2}$ and $e = \delta^2 + \frac{3\delta}{2}$. From Lemma 2.2, we have that G is traceable, a contradiction.

Hence, the proof of Theorem 1.3 is complete.

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