Fundamental Journal of Mathematics and Applications, 1 (2) (2018) 205-211
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845

Existence and Iteration of Monotone Positive Solution for a Fourth-Order Nonlinear Boundary Value Problem

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Article Info

Keywords: *Boundary value problem, Green's function, Positive solution, Iterative method, Sign-changing*
2010 AMS: *34B10,34B18*
Received: *27 April 2018*
Accepted: *17 December 2018*
Available online: *25 December 2018*

Abstract

This paper is concerned with the following fourth-order three-point boundary value problem BVP

$$u^{(4)}(t) = f(t,u(t)), \quad t \in [0,1],$$
$$u'(0) = u''(0) = u(1) = 0,\ u'''(\eta) + \alpha u(0) = 0,$$

where $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $\alpha \in [0,6)$ and $\eta \in \left[\frac{2}{3},1\right)$. Although corresponding Green's function is sign-changing, we still obtain the existence of monotone positive solution under some suitable conditions on f by applying iterative method. An example is also given to illustrate the main results.

1. Introduction

Fourth-order ordinary differential equations have attracted a lot of attention due to their applications in engineering, physics, material mechanics, fluid mechanics and so on. Many approaches, such as the Leray– Schauder nonlinear alternative, fixed point index theory in cones, the method of upper and lower solutions, degree theory, Guo-Krasnoselskii's fixed point theorem, Leggett-Williams fixed-point theorem, are used to study the existence of single or multiple positive solutions to some fourth-order boundary value problem, see [1]-[13]. However, all the above-mentioned papers are achieved when corresponding Green's functions are nonnegative, which is a very important condition.
Recently, the existence of positive solutions of the boundary value problems with sign-changing Green's function has received increasing interest.
In 2008, Palamides and Smyrlis [14] studied the existence of at least one positive solution to the singular third-order three-point BVP with an indefinitely signed Green's function

$$u'''(t) = a(t) f(t,u(t)) = 0, \quad t \in (0,1),$$
$$u(0) = u(1) = u''(\eta) = 0,$$

where $\eta \in \left(\frac{17}{24},1\right)$. Their technique was a combination of the Guo-Krasnoselskii's fixed point theorem [15, 16] and properties of the corresponding vector field.
In 2018, Zhang et al [17] studied the existence of at least $n-1$ decreasing positive solutions of the problem

$$u^{(4)}(t) = f(t,u(t)) = 0, \quad t \in [0,1],$$
$$u(0) = u(1) = u''(\eta) = 0,$$

their main tool is the fixed point index theory.
It is worth mentioning that there are other types of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases; see [18]-[22].

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Motivated and inspired by the above-mentioned works, in this paper we will study the following nonlinear fourth-order three-point BVP with sign-changing Green's function

$$\begin{gathered} u^{(4)}(t)=f(t,u(t)) \quad t\in[0,1], \\ u'(0)=u''(0)=u(1)=0,\ u'''(\eta)+\alpha u(0)=0, \end{gathered} \tag{1.1}$$

by applying iterative method. Throughout this paper, we always assume that $f\in C([0,1]\times[0,+\infty),[0,+\infty))$, $\alpha\in[0,6)$ and $\eta\in\left[\frac{2}{3},1\right)$. By imposing some suitable conditions on f and η, we obtain the existence of monotone positive solution for the BVP (1.1). Moreover, our iterative scheme starts off with zero function, which implies that the iterative scheme is feasible.

2. Main results

Let Banach space $E=C[0,1]$ be equipped with the norm $\|u\|=\max_{t\in[0,1]}|u(t)|$.

Lemma 2.1. *The BVP*

$$\begin{gathered} u^{(4)}(t)=0 \quad t\in[0,1], \\ u'(0)=u''(0)=u(1)=0,\ u'''(\eta)+\alpha u(0)=0 \end{gathered}$$

has only trivial solution.

Proof. It is simple to check. □

Now, for any $y\in E$, we consider the BVP

$$\begin{gathered} u^{(4)}(t)=y(t) \quad t\in[0,1], \\ u'(0)=u''(0)=u(1)=0,\ u'''(\eta)+\alpha u(0)=0. \end{gathered}$$

After a direct computation, one may obtain the expression of Green's function $G(t,s)$ of the BVP as follows: for $s\geq\eta$,

$$G(t,s)=\begin{cases} -\frac{(6-\alpha t^3)(1-s)^3}{6(6-\alpha)}, & 0\leq t\leq s\leq 1 \\ \frac{(t-s)^3}{6}-\frac{(6-\alpha t^3)(1-s)^3}{6(6-\alpha)}, & 0\leq s\leq t\leq 1 \end{cases}$$

and for $s<\eta$,

$$G(t,s)=\begin{cases} -\frac{(6-\alpha t^3)(1-s)^3}{6(6-\alpha)}+\frac{1-t^3}{6-\alpha}, & 0\leq t\leq s\leq 1 \\ \frac{(t-s)^3}{6}-\frac{(6-\alpha t^3)(1-s)^3}{6(6-\alpha)}+\frac{1-t^3}{6-\alpha}, & 0\leq s\leq t\leq 1. \end{cases}$$

Remark 2.2. *$G(t,s)$ has the following properties:*

$$G(t,s)\geq 0 \quad for \quad 0\leq s<\eta \qquad and \quad G(t,s)\leq 0 \quad for \quad \eta\leq s\leq 1.$$

Moreover, for $s\geq\eta$,

$$\max\{G(t,s):t\in[0,1]\}=G(1,s)=0,$$

$$\min\{G(t,s):t\in[0,1]\}=G(0,s)=-\frac{(1-s)^3}{6-\alpha}\geq-\frac{(1-\eta)^3}{6-\alpha}$$

and for $s<\eta$,

$$\max\{G(t,s):t\in[0,1]\}=G(0,s)=\frac{s^3+3s-3s^2}{6-\alpha}\leq\frac{\eta^3+3\eta-3\eta^2}{6-\alpha},$$

$$\min\{G(t,s):t\in[0,1]\}=G(1,s)=0.$$

So, if we let $M=\max\{|G(t,s)|:t,s\in[0,1]\}$, then

$$M=\max\left\{\frac{(1-\eta)^3}{6-\alpha},\frac{\eta^3+3\eta-3\eta^2}{6-\alpha}\right\}<\frac{1}{6-\alpha}.$$

Let

$$K=\{y\in E:\ y(t)\ \textit{is nonnegative and decreasing on}\ [0,1]\}.$$

Then K is a cone in E. Note that this induces an order relation "$\precsim$" in E by defining $u\precsim v$ if and only if $v-u\in K$. In the remainder of this paper, we always assume that f satisfies the following two conditions:

(H_1) for each $u\in[0,+\infty)$, the mapping $t\longmapsto f(t,u)$ is decreasing;

(H_2) for each $t\in[0,1]$, the mapping $u\longmapsto f(t,u)$ is increasing.

Now, we define an operator T as follows:

$$(Tu)(t)=\int_0^1 G(t,s)f(s,u(s))\,ds, \quad u\in K,\ t\in[0,1].$$

Obviously, if u is a fixed point of T in K, then u is a nonnegative and decreasing solution of the BVP (1.1).

Lemma 2.3. *$T:K\to K$ is completely continuous.*

Proof. Let $u\in K$. Then, for $t\in[0,\eta]$, we have

$$(Tu)(t)=\int_0^t\left[\frac{(t-s)^3}{6}+\frac{1-t^3}{6-\alpha}-\frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right]f(s,u(s))\,ds$$
$$+\int_t^\eta\left[\frac{1-t^3}{6-\alpha}-\frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right]f(s,u(s))\,ds+\int_\eta^1\left[-\frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right]f(s,u(s))\,ds,$$

which together with (H_1) and (H_2) implies that

$$(Tu)'(t)=\int_0^t\left[\frac{(t-s)^2}{2}-\frac{3t^2}{6-\alpha}+\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right]f(s,u(s))\,ds+\int_t^\eta\left[-\frac{3t^2}{6-\alpha}+\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right]f(s,u(s))\,ds$$
$$+\int_\eta^1\left[\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right]f(s,u(s))\,ds$$
$$=\int_0^t\left[\frac{t^2}{2}+\frac{s^2-2ts}{2}-\frac{3t^2}{6-\alpha}+\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right]f(s,u(s))\,ds+\int_t^\eta\left[-\frac{3t^2}{6-\alpha}+\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right]f(s,u(s))\,ds$$
$$+\int_\eta^1\left[\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right]f(s,u(s))\,ds$$
$$=\int_0^\eta\frac{\alpha t^2\left(-3s+3s^2-s^3\right)}{2(6-\alpha)}f(s,u(s))\,ds-\frac{t^2}{2}\int_t^\eta f(s,u(s))\,ds+\int_0^t\frac{s^2-2ts}{2}f(s,u(s))\,ds$$
$$-\frac{t^2}{2}\int_t^\eta f(s,u(s))\,ds+\int_0^t\frac{s^2-2ts}{2}f(s,u(s))\,ds$$
$$\le f(\eta,u(\eta))\left[\frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left(-3s+3s^2-s^3\right)ds-\frac{t^2}{2}\int_t^\eta ds+\int_0^t\frac{s^2-2ts}{2}ds+\frac{\alpha t^2}{2(6-\alpha)}\int_\eta^1(1-s)^3\,ds\right]$$
$$=\frac{t^2}{2}f(\eta,u(\eta))\left[\frac{\alpha t^2}{(6-\alpha)}\left(\frac{1}{4}-\eta\right)-\eta+\frac{t}{3}\right]$$
$$\le\frac{t^2}{2}f(\eta,u(\eta))\left[\frac{\alpha t^2(1-4\eta)}{(6-\alpha)}-\frac{2\eta}{3}\right]$$
$$\le 0.$$

For $t\in[\eta,1]$, we have

$$(Tu)(t)=\int_0^\eta\left[\frac{(t-s)^3}{6}+\frac{1-t^3}{6-\alpha}-\frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right]f(s,u(s))$$
$$+\int_\eta^t\left[\frac{(t-s)^3}{6}-\frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right]f(s,u(s))+\int_t^1\left[-\frac{\left(6-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\right]f(s,u(s)),$$

which together with (H_1) and (H_2) implies that

$$(Tu)'(t)=\int_0^\eta\left[\frac{(t-s)^2}{2}-\frac{3t^2}{6-\alpha}+\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right]f(s,u(s))\,ds+\int_\eta^t\left[\frac{(t-s)^2}{2}+\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\right]f(s,u(s))\,ds$$
$$+\int_t^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}f(s,u(s))\,ds$$
$$=\frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left(-3s+3s^2-s^3\right)f(s,u(s))\,ds+\int_0^\eta\left(\frac{s^2-ts}{2}\right)f(s,u(s))\,ds$$
$$+\int_\eta^t\frac{(t-s)^2}{2}f(s,u(s))\,ds+\int_\eta^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}f(s,u(s))\,ds$$
$$\le\frac{\alpha t^2}{2(6-\alpha)}f(\eta,u(\eta))\left[\int_0^\eta\left(-3s+3s^2-s^3\right)ds+\int_0^\eta\left(\frac{s^2-ts}{2}\right)ds+\int_\eta^t\frac{(t-s)^2}{2}ds+\int_\eta^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}ds\right]$$
$$=\frac{t^2}{2}f(\eta,u(\eta))\left[\frac{\alpha t^2(1-4\eta)}{(6-\alpha)}+\frac{t-3\eta}{3}\right]$$
$$=\frac{t^2}{2}f(\eta,u(\eta))\left[\frac{\alpha t^2(1-4\eta)}{(6-\alpha)}+\frac{1-3\eta}{3}\right]$$
$$\le 0.$$

So, $(Tu)(t)$ is decreasing on $[0,1]$. At the same time, since $(Tu)(1)=0$, we know that $(Tu)(t)$ is nonnegative on $[0,1]$. This indicates that $Tu\in K$.

Now, we assume that $D \subset K$ is a bounded set. Then there exists a constant $C_1 > 0$ such that $\|u\| \leq C_1$ for any $u \in D$. In what follows, we will prove that $T(D)$ is relatively compact.
Let

$$C_2 = \sup\{f(t,u) : (t,u) \in [0,1] \times [0,C_1]\}.$$

Then for any $y \in T(D)$, there exists $u \in D$ such that $y = Tu$, and so,

$$\begin{aligned} |y(t)| = |(Tu)(t)| &= \left| \int_0^1 G(t,s) f(s,u(s)) ds \right| \\ &\leq \int_0^1 |G(t,s)| f(s(,u(s))) ds \\ &\leq M \int_0^1 f(s,u(s)) ds \leq MC_2, \quad t \in [0,1], \end{aligned}$$

which implies that $T(D)$ is uniformly bounded. On the other hand, when $\varepsilon > 0$, if we choose $0 < \tau < \min\left\{1-\eta, \frac{\varepsilon}{12C_2(M+1)}\right\}$, then, for any $u \in D$,

$$\int_{\eta-\tau}^{\eta+\tau} f(s,u(s)) ds \leq 2C_2\tau < \frac{\varepsilon}{6(M+1)}. \tag{2.1}$$

Since $G(t,s)$ is uniformly continuous on $[0,1] \times [0,\eta-\tau]$ and $[0,1] \times [\eta+\tau,1]$, there exists $\delta > 0$ such that for any $t_1,t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$,

$$|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon}{3(C_2+1)(\eta-\tau)}, \ s \in [0,\eta-\tau] \tag{2.2}$$

and

$$|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon}{3(C_2+1)(1-\eta-\tau)}, \quad s \in [\eta+\tau,1]. \tag{2.3}$$

In view of (2.1), (2.2) and (2.3), for any $y \in T(D)$ and $t_1,t_2 \in [0,1]$ with
$|t_1 - t_2| < \delta$,

$$\begin{aligned} |y(t_1) - y(t_2)| &= |T(t_1) - T(t_2)| \\ &= \left| \int_0^1 (G(t_1,s) - G(t_2,s)) f(s,u(s)) ds \right| \\ &\leq \int_0^1 |(G(t_1,s) - G(t_2,s))| f(s,u(s)) ds \\ &= \int_0^{\eta-\tau} |(G(t_1,s) - G(t_2,s))| f(s,u(s)) ds + \int_{\eta-\tau}^{\eta+\tau} |(G(t_1,s) - G(t_2,s))| f(s,u(s)) ds \\ &\quad + \int_{\eta+\tau}^1 |(G(t_1,s) - G(t_2,s))| f(s,u(s)) ds \\ &\leq C_2 \frac{\varepsilon}{3(C_2+1)(\eta-\tau)}(\eta-\tau) + \frac{\varepsilon}{3(M+1)} M + C_2 \frac{\varepsilon}{3(C_2+1)(1-\eta-\tau)}(1-\eta-\tau) \\ &= \frac{C_2\varepsilon}{3(C_2+1)} + \frac{M\varepsilon}{3(M+1)} + \frac{C_2\varepsilon}{3(C_2+1)} = \varepsilon, \end{aligned}$$

which implies that $T(D)$ is equicontinuous. By Arzela-Ascoli theorem, we know that $T(D)$ is relatively compact. Thus, we have shown that T is a compact operator.
Finally, we prove that T is continuous. Suppose that $u_n \ (n = 1,2,\ldots)$, $u_0 \in K$ and $\|u_n - u_0\| \to 0 \ (n \to 0)$. Then there exists $C_3 > 0$ such that for any n, $\|u_n\| \leq C_3$.
Let

$$C_4 = \sup\{f(t,u) : (t,u) \in [0,1] \times [0,C_3]\}.$$

Then for any n and $t \in [0,1]$, we have

$$G(t,s) f(s,u_n(s)) \leq MC_4, \quad s \in [0,1].$$

By applying Lebesgue Dominated Convergence theorem, we obtain

$$\begin{aligned} \lim_{n\to\infty} (Tu_n)(t) &= \lim_{n\to\infty} \int_0^1 G(t,s) f(s,u_n(s)) ds \\ &= \int_0^1 G(t,s) \lim_{n\to\infty} f(s,u_n(s)) ds \\ &= \int_0^1 G(t,s) f(s,u_0(s)) ds = T(u_0)(t), \ t \in [0,1], \end{aligned}$$

which indicates that T is continuous.Therefore, $T : K \to K$ is completely continuous. □

Theorem 2.4. *Assume that $f(t,0)\not\equiv 0$ for $t\in[0,1]$ and there exist two positive constants a and b such that the following conditions are satisfied:*
(H_3) $f(0,a)\le(6-\alpha)a$;
(H_4) $b(u_2-u_1)\le f(t,u_2)-f(t,u_1)\le 2b(u_2-u_1)$, $0\le t\le 1$,
$0\le u_1\le u_2\le a$. *If we construct an iterative sequence $v_{n+1}=Tv_n$, $n=0,1,2,...$, where $v_0(t)\equiv 0$ fot $t\in[0,1]$, then $\{v_n\}_{n=1}^{\infty}$ converges to v^* in E and v^* is a decreasing and positive solution of the BVP (1.1)*

Proof. Let $K_a=\{u\in K:\ \|u\|\le a\}$. Then it follows from Lemma 2.3 that $Tu\in K$. In view of (H_3) and $0\le u(s)\le 1$ for $s\in[0,1]$, we have

$$\begin{aligned}0\le (Tu)(t)&=\int_0^1 G(t,s)f(s,u(s))\,ds\\&\le\int_0^1|G(t,s)|f(0,a)\,ds\\&\le(6-\alpha)aM\le a,\quad t\in[0,1],\end{aligned}$$

which shows that $\|Tu\|\le a$. So, $T:K_a\to K_a$. Now, we prove that $\{v_n\}_{n=1}^{\infty}$ converges to v^*in E and v^* is a decreasing and positive solution of the BVP (1.1). Indeed, in view of $v_0\in K_a$ and $T:K_a\to K_a$, we have $v_n\in K_a$, $n=0,1,2,\ldots$. Since the set $\{v_n\}_{n=0}^{\infty}$ is bounded and T is completely continuous, we know that the set $\{v_n\}_{n=0}^{\infty}$ is relatively compact. In what follows, we prove that $\{v_n\}_{n=0}^{\infty}$ is monotone by induction. First, it is obvious that $v_1-v_0=v_1\in K$, which shows that $v_0\lesssim v_1$. Next, we assume that $v_{k-1}\lesssim v_k$. Then it follows from (H_4) that for $0\le t\le\eta$, we obtain

$$\begin{aligned}&v'_{k+1}(t)-v'_k(t)\\&=(Tv_k)'(t)-(Tv_{k-1})'(t)\\&=\textstyle\int_0^1\frac{\partial G(t,s)}{\partial t}[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds\\&=\textstyle\frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta(3s^2-3s-s^3)[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds\\&\quad\textstyle+\int_0^t\left(\frac{s^2-2ts}{2}\right)[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds-\frac{t^2}{2}\int_t^\eta[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds\\&\quad\textstyle+\frac{\alpha t^2}{2(6-\alpha)}\int_\eta^1(1-s)^3[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds\\&\le\textstyle\frac{b\alpha t^2}{2(6-\alpha)}\int_0^\eta(3s^2-3s-s^3)[v_k(s)-v_{k-1}(s)]\,ds+b\int_0^t\left(\frac{s^2-2ts}{2}\right)[v_k(s)-v_{k-1}(s)]\,ds\\&\quad\textstyle-\frac{bt^2}{2}\int_t^\eta[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds+\frac{2b\alpha t^2}{2(6-\alpha)}\int_\eta^1(1-s)^3[v_k(s)-v_{k-1}(s)]\,ds\\&\le\textstyle b[v_k(\eta)-v_{k-1}(\eta)][\frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta(3s^2-3s-s^3)\,ds+\int_0^t\left(\frac{s^2-2ts}{2}\right)ds-\frac{t^2}{2}\int_t^\eta ds+\frac{\alpha t^2}{(6-\alpha)}\int_\eta^1(1-s)^3\,ds]\\&=\textstyle\frac{t^2}{2}b[v_k(\eta)-v_{k-1}(\eta)]\left[\frac{\alpha(\eta^4-4\eta^3+6\eta^2-8\eta+2)}{4(6-\alpha)}-\eta+\frac{t}{3}\right]\\&\le\textstyle\frac{t^2}{2}b[v_k(\eta)-v_{k-1}(\eta)]\left[\frac{\alpha(-3\eta+2)}{4(6-\alpha)}-\frac{2\eta}{3}\right]\\&\le\textstyle\frac{t^2}{2}b[v_k(\eta)-v_{k-1}(\eta)]\left[\frac{\alpha(-3\eta+2)}{4(6-\alpha)}-\frac{2\eta}{3}\right]\le 0.\end{aligned}$$

For $\eta\le t\le 1$, we get

$$\begin{aligned}&v'_{k+1}(t)-v'_k(t)\\&=(Tv_k)'(t)-(Tv_{k-1})'(t)\\&=\textstyle\int_0^1\frac{\partial G(t,s)}{\partial t}[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds\\&=\textstyle\frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta(3s^2-3s-s^3)[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds+\int_0^\eta\left(\frac{s^2-2ts}{2}\right)[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds\\&\quad\textstyle+\int_\eta^t\frac{(t-s)^2}{2}[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds+\int_\eta^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}[f(s,v_k(s))-f(s,v_{k-1}(s))]\,ds\\&\le\textstyle\frac{b\alpha t^2}{2(6-\alpha)}\int_0^\eta(3s^2-3s-s^3)[v_k(s)-v_{k-1}(s)]\,ds+b\int_0^\eta\left(\frac{s^2-2ts}{2}\right)[v_k(s)-v_{k-1}(s)]\,ds\\&\quad\textstyle+2b\int_\eta^t\frac{(t-s)^2}{2}[v_k(s)-v_{k-1}(s)]\,ds+2b\int_\eta^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}[v_k(s)-v_{k-1}(s)]\,ds]\\&\le\textstyle b\times[v_k(\eta)-v_{k-1}(\eta)]\times[\frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta(3s^2-3s-s^3)\,ds+\int_0^\eta\left(\frac{s^2-2ts}{2}\right)ds+2\int_\eta^t\frac{(t-s)^2}{2}ds+2b\int_\eta^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}ds]\\&=\textstyle b\times[v_k(\eta)-v_{k-1}(\eta)]\times[\frac{\alpha t^2(\eta^4-4\eta^3+6\eta^2-8\eta+2)}{8(6-\alpha)}-\frac{\eta^3}{6}+\frac{t\eta^2}{2}+\frac{t^3}{3}-t^2\eta]\\&\le\textstyle\frac{t^2}{2}b\times[v_k(\eta)-v_{k-1}(\eta)]\times[\frac{\alpha t^2(\eta^4-4\eta^3+6\eta^2-8\eta+2)}{4(6-\alpha)}+\frac{2t}{3}-\eta]\\&\le\textstyle\frac{t^2}{2}b\times[v_k(\eta)-v_{k-1}(\eta)]\times[\frac{\alpha t^2(-3\eta+2)}{4(6-\alpha)}+\frac{2-3\eta}{3}]\le 0,\end{aligned}$$

hence

$$v'_{k+1}(t) - v'_k(t) \leq 0,\ t \in [0,1], \tag{2.4}$$

that is $v_{k+1}(t) - v_k(t)$ is decreasing on $[0,1]$. At the same time, it is easy to see that

$$v_{k+1}(1) - v_k(1) = \int_0^1 G(1,s)\left[f(s, v_k(s) - v_{k-1}(s))\right] ds = 0,$$

the last equation implies that

$$v_{k+1}(t) - v_k(t) \geq 0,\ t \in [0,1]. \tag{2.5}$$

It follows from (2.4) and (2.5) that $v_{k+1} - v_k \in K$, which indicates that $v_{k+1} \lesssim v_k$. Thus, we have shown that $v_{k+1} \lesssim v_k$, $n = 0,1,2,...$ Since $\{v_n\}_{n=1}^{\infty}$ is relatively compact and monotone, there exists a $v^* \in K_a$ such that $\lim_{n\to\infty} v_n = v^*$, which together with the continuity of T and the fact that $v_{n+1} = Tv_n$ implies that $v^* = Tv^*$. This indicates that v^* is a decreasing nonnegative solution of (1.1). Moreover, in view of $f(t,0) \neq 0$ for $t \in [0,1]$, we know that zero function is not a solution of (1.1), which shows that is v^* a positive solution of (1.1). $\square$

3. An example

Consider the boundary value problem

$$\begin{gathered} u^{(4)}(t) = f(t,u(t)) \quad t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0,\ u'''(\eta) + \alpha u(0) = 0, \end{gathered} \tag{3.1}$$

If we let $\eta = \frac{3}{4}$, $\alpha = 4$ and $f(t,u) = \frac{1}{2}u^2(t) + t$, $(t,u) \in [0,1] \times [0,+\infty)$, then all the hypotheses of Theorem 2.4 are fulfilled with $a = 3$ and $b = \frac{3}{4}$. Therefore, it follows from Theorem 2.4 that the BVP (3.1) has a decreasing and positive solution. Moreover, the iterative scheme is $v_0(t) \equiv 0$ for $t \in [0,1]$ and

$$v_{n+1}(t) = \begin{cases} \int_0^t \left[\frac{(t-s)^3}{6} + \frac{1-t^3}{2} - \frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \quad + \int_t^{\frac{3}{4}} \left[\frac{1-t^3}{2} - \frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \quad + \int_{\frac{3}{4}}^1 \left[-\frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \qquad if \quad t \in \left[0, \frac{3}{4}\right],\ n = 0,1,2... \\ \int_0^{\frac{3}{4}} \left[\frac{(t-s)^3}{6} + \frac{1-t^3}{2} - \frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \quad + \int_{\frac{3}{4}}^t \left[\frac{(t-s)^3}{6} - \frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \quad \int_t^1 \left[-\frac{(3-2t^3)(1-s)^3}{6}\right] \times \left[\frac{1}{2}(v_n(s))^2 + s\right] ds \\ \qquad if \quad t \in \left[\frac{3}{4}, 1\right],\ n = 0,1,2... \end{cases}$$

The first, second, third, and fourth terms of this scheme are as follows:

$$v_0(t) \equiv 0,$$

$$v_1(t) = \frac{7t^5}{120} - \frac{119t^3}{480} + \frac{37}{160}$$

$$v_2(t) = \frac{7t^{14}}{49420800} - \frac{833t^{12}}{342144000} - \frac{7427t^{11}}{20275200} - \frac{184253t^{10}}{165888000} + \frac{37t^9}{4147200}$$

$$-\frac{49069t^7}{102400} + \frac{t^5}{60} + \frac{1369t^4}{614400} - \frac{147553086840691879t^3}{298491637137408000} + \frac{143787255710603}{1554643943424000}$$

$$v_3(t) = \frac{49t^{32}}{210790224950722560000} - \frac{833t^{30}}{794386238570496000000} - \frac{7427t^{29}}{40798108054978560000}$$

$$-\frac{268461101t^{28}}{42732501109309440000000} + \frac{26846981t^{27}}{6330740905082880000000} + \frac{26815806199t^{26}}{6892640985415680000000}$$

$$+\frac{400171550569t^{25}}{1792086656208076800000000} + \frac{371462295299t^{24}}{77197579036655616000000} + \frac{114032891993t^{23}}{10453838827880448000000}$$

$$+\frac{3453761875703t^{22}}{17271559802585088000000} + \frac{78497989670046547290071t^{21}}{10594667708559944822292480000000} - \frac{272903089t^{20}}{15277249658880000000}$$

$$-\frac{1851000739420343895193t^{19}}{4750136730870832403841024000000}+\frac{3618768882947953403120 89t^{18}}{1155588818738167415203430400 00}$$

$$+\frac{2718808325190382897978 7t^{17}}{14141821208334216619622400000000}-\frac{34723371605213907361t^{15}}{51630934252241446502400000 0}$$

$$+\frac{977587338666778516044941t^{14}}{4956569688215178864230400000 0}+\frac{84063076723229553385124009617965 43t^{13}}{26729177232291014019801887539200000 0}$$

$$-\frac{29501725604687291t^{12}}{21276483895154442240000}-\frac{166598650952378994752 3t^{11}}{14524746339092099235840000 0}$$

$$+\frac{21771913436216758949940023416550641t^{10}}{4490501775024890355326717106585600000 00}$$

$$+\frac{143787255710603t^{9}}{141037298547425280000}+\frac{196844753067815507t^{8}}{802345520625352704000000}$$

$$-\frac{21216253428451373750458316293037t^{7}}{19490025065212197722772209664000000 0}+\frac{t^{5}}{60}$$

$$+\frac{2067477490478633503448662360 9t^{4}}{580060269797982075082506240000 00}$$

$$-\frac{229994247914657276716497149730890704260235815069 11t^{3}}{921310739875039011664903405515483824259072000000 00}$$

$$+\frac{310661312414757109061653185761538923825439093587}{133523295634063624878971508045722293370880000000 0}$$

Acknowledgement

The authors want to thank the anonymous referee for the throughout reading of the manuscript and several suggestions that help us improve the presentation of the paper.

References

[1] A. R. Aftabizadeh, *Existence and uniqueness theorems for fourth-order boundary value problems*, J. Math. Anal. Appl., **116** (1986), 415-426.
[2] A. Cabada, S. Tersian, *Existence and multiplicity of solutions to boundary value problems for fourth-order impulsive differential equations*, Bound. Value Probl., **105** (2014).
[3] D. R. Anderson, R. I. Avery, *A fourth-order four-point right focal boundary value problem*, Rocky Mountain J. Math., **36** (2006), 367-380.
[4] E. Alves, T. F. Ma, M. L. Pelicer, *Monotone positive solutions for a fourth order equation with nonlinear boundary conditions*, Nonlinear Anal., **71** (2009), 3834-3841.
[5] J. R. Graef, B. Yang, *Positive solutions for fourth-order focal boundary value problem*, Rocky mountain journal of mathematics, **44**(3) (2014), 937-951.
[6] N. Bouteraa, S. Benaicha, H. Djourdem, M. E. Benattia, *Positive solutions of nonlinear fourth-order two-point boundary value problem with a parameter*, Romanian J. Math. Comput. Sci., **8**(1) (2018), 17-30.
[7] N. Bouteraa, S. Benaicha, *Triple positive solutions of higher-order nonlinear boundary value problems*, J. Comput. Sci. Comput. Math., **7**(2) (2017).
[8] R. P. Agarwal, *On fourth-order boundary value problems arising in beam analysis*, Differ. Integral Equ., **2**(1) (1989), 91–110.
[9] S. Lia, X. Zhanga, *Existence and uniqueness of monotone positive solutions for an elastic beam equation with nonlinear boundary conditions*, Comput. Math. Appl., **63** (2012), 1355–1360.
[10] W. Wang, Y. Zheng, H. Yang, J. Wang, *Positive solutions for elastic beam equations with nonlinear boundary conditions and a parameter*, Bound. Value Probl., **80** (2014), 1-17.
[11] Y. Li, *Positive solutions of fourth-order boundary value problems with two parameters*, Journal of Mathematical Analysis and Applications, **281**(2) (2003), 477–484.
[12] Z. Bai, *The upper and lower solution method for some fourth-order boundary value problem*, Nonlinear Anal., **67** (2007), 1704- 1709.
[13] Z. Bekri, S. Benaicha, *Existence of positive of solution for a nonlinear three-point boundary value problem*, Sib. 'Elektron. Mat. Izv., **14** (2017), 1120–1134.
[14] A. P. Palamides, G. Smyrlis, *Positive solutions to a singular third-order three-point BVP with an indefinitely signed Green's function*, Nonlinear Anal., **68** (2008), 2104-2118.
[15] D. J. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, **5**, Academic Press, New York, NY, USA, 1988.
[16] M. A. Krasnoselskii, *Positive Solutions of Opearator Equations*, Noordhoff, Groningen, The Netherlands, 1964.
[17] Y. Zhang, J. P. Sun, J. Zhao, *Positive solutions for a fourth-order three-point BVP with sign-changing Green's function*, Electron. J. Qual. Theory Differ. Equ., **5** (2018), 1-11.
[18] A. Cabada, R. Enguica, L. Lopez-Somoza, *Positive solutions for second-order boundary value problems with sign changing Green's functions*, Electron. J. Differential Equations, **245** (2017), 1–17.
[19] G. Infante, J. R. L. Webb, *Three-point boundary value problems with solutions that change sign*, J. Integral Equations Appl., **15**(1) (2003), 37–57.
[20] J. P. Sun, X. Q. Wang, *Existence and iteration of monotone positive solution of BVP for an elastic beam equation*, Mathematical Problems in Engineering, **2011**, Article ID 705740, 10 pages.
[21] J. P. Sun, J. ZHAO, *Iterative technique for a third-order three-point BVP with sign-changing green's function*, Electron. J. Differential Equations, **2013**(215) (2013), 1-9.
[22] Y. H. Zhao, X. L. Li, *Iteration for a third-order three-point BVP with sign-changing green's function*, J. Appl. Math., (2014), Article ID 541234, 6 pages.