Research article

ANALYTICAL ALGORITHM FOR COMPUTING GAIN AND PHASE MARGINS FOR DISCRETE TIME SYSTEM

Esmat Bekir

PhD Consulting Engineers, 5805 Serrania Avenue, Woodland Hills, 91367, California, USA

Abstract

This note describes an algorithm for computing the gain and phase margins for single input single output discrete time systems. It converts the complex computations into real computations. It results into Chebyshev polynomials from which we compute the desired margins. The algorithm is simple, easy to implement and devoid of complex computations.

Keywords: *Gain margin; phase margin; control system stability; discrete time system; Chebyshev polynomials.*

1. Introduction

The gain and phase margins of a control system play a central role in determining its stability [1,2]. The gain margin is the amount of amplification that the closed loop system can accommodate before becoming unstable and likewise for the phase margin. Analytical computation of these two margins for continuous time systems is a direct process as shown in references [3-5]. However, this process is a little more elaborate for discrete time systems. The intent here is to describe a simple algorithm for computing these margins for these systems.

A causal single input single output discrete time system is generally described by the proper rational polynomial:

$$G(z) = \frac{B(z)}{A(z)} \tag{1}$$

*Corresponding author: Esmat Bekir
E-mail: ebekir@hotmail.com

where $A(z)$ and $B(z)$ are polynomials of the time delay z, and causality mandates that the degree of $A(z)$ be at least as large as that of $B(z)$. With no loss of generality, we assume this system is imbedded in the forward part of a feedback control loop. In such case the closed loop system must be analyzed to ensure its stability.

Because instability implies that the roots of the characteristic equations leave the unit circle, z is confined to $z = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$, θ is in radians. The gain margin, G_m, is defined as the real value providing,

$$G_m G(z_{cg} = \exp i\theta_{cg}) = -1 \tag{2}$$

where θ_{cg} is the angle at which $\angle G(z_{cg} = \exp i\theta_{cg}) = -\pi$ is for the smallest normalized frequency. (Normalized frequency is frequency normalized by the sampling rate.) Likewise the phase margin, P_m, is defined as the phase lag that can be tolerated before the control system becomes unstable, *i.e*:

$$P_m = \angle G(z_{cp} = \exp i\theta_{cp}) + \pi \tag{3}$$

where θ_{cp} is the angle at which $|G(z_{cp} = \exp i\theta_{cp})| = 1$ is for the smallest normalized frequency. One approach to compute these margins is to evaluate $G(z)$ for many values of z and generate two lookup tables; z vs. $|G(z)|$ and z vs. $\angle G(z)$. From the 2nd table, we lookup z_{cg}. From the 1st table, we lookup $|z_{cg}|$ and use Eq. (2) to compute G_m. Similarly from the 1st table, we lookup z_{cp}. From the 2nd table we lookup $\angle G(z_{cp})$ and use Eq. (3) to compute P_m. Evidently, this process uses complex computations and gives approximate margins, but it may be adequate for speedy computations. Matlab uses the H$_\infty$ theory and other methods to compute these margins [6]. Herein, we introduce a simplified analytical method that is free of complex computations to yield the same results.

To start, we introduce some mathematical background in which we define symmetric and asymmetric polynomials, describe their properties and provide alternative algebraic means for representing them. These types of polynomials play a central role in the computations of the gain and phase margins.

2. Symmetric and asymmetric polynomials

The polynomial,

$$P(z) = \frac{1}{2}(p_1 z^n + p_2 z^{n-1} + \dots + p_{n+1-k} z^k + \dots + p_{-2} z^{-n+1} + p_{-1} z^{-n}) \tag{4}$$

is symmetric if $P(z) = P(1/z) \Rightarrow p_k = p_{-k}$; $k = 1, \dots, n$. Thus Eq. (4) becomes as follows:

$$P(z) = \frac{1}{2}[p_1 (z^n + z^{-n}) + p_2 (z^{n-1} + z^{-n+1}) + \dots + p_{n+1}] \tag{5}$$

The polynomial,

$$Q(z) = \frac{1}{2i}\left(q_1 z^n + q_2 z^{n-1} + \dots + q_{n+1-k} z^k + \dots + q_{-2} z^{-n+1} + q_{-1} z^{-n}\right) \tag{6}$$

is asymmetric if $Q(z) = -Q(1/z) \Rightarrow q_k = -q_{-k}$; $k = 1, \dots, n$. Thus Eq. (6) becomes:

$$Q(z) = \frac{1}{2i}[q_1(z^n - z^{-n}) + q_2(z^{n-1} - z^{-n+1}) + \dots + q_n(z - z_{-1})], \qquad q_{n+1} = 0 \tag{7}$$

Since we are confining our analysis to $z = e^{i\theta}$ one can see that:

$$z^k + z^{-k} = 2\cos k\theta \qquad\qquad k \text{ is integer} \tag{8}$$
$$z^k - z^{-k} = 2i\sin k\theta$$

which implies that $P(\theta)$ and $Q(\theta)$ take following trigonometric forms:

$$P(\theta) = p_1 \cos n\theta + p_2 \cos(n-1)\theta + \cdots + 0.5p_{n+1} \tag{9}$$

$$Q(\theta) = q_1 \sin n\theta + q_2 \sin(n-1)\theta + \cdots + q_n \sin \theta \tag{10}$$

Thus, on the unit circle the symmetric and asymmetric polynomials are purely real polynomials. Our ultimate goal is to represent $P(\theta)$ and $Q(\theta)$ in power series form of $x = \cos\theta$. First, we need to convert the sines in Eq. (10) to cosines as shown below. We observe that,

$$\sin k\theta = 2\sin\theta\cos(k-1)\theta + \sin(k-2)\theta \tag{11}$$

and substituting from Eq. (11) for the first term of $Q(\theta)$ in Eq. (10) gives:

$$Q(\theta) = 2q_1 \sin\theta\cos(n-1)\theta + q_2 \sin(n-1)\theta \tag{12}$$
$$+ (q_1 + q_3)\sin(n-2)\theta + \dots + q_n \sin\theta$$

Continuing the substitution from Eq. (11) into Eq. (12) results in,

$$Q(\theta) = 2\sin\theta\, [q'_1 \cos(n-1)\theta + q'_2 \cos(n-2)\theta + \cdots + q'_{n-1}\cos\theta + 0.5q'_n] \tag{13}$$

which makes use of following recursion conditions:

$$q'_j = q'_{j-2} + q_j, \quad j = 3,4,\dots,n, \quad q'_1 = q_1, \quad q'_2 = q_2 \tag{14}$$

The above polynomials take the more concise forms as follow:

$$P(\theta) = \mathbf{pc} \tag{15}$$

$$Q(\theta) = 2\sin\theta\,\mathbf{qc}_o \tag{16}$$

where $\mathbf{p}, \mathbf{q}, \mathbf{c}$ and $\mathbf{c_0}$ are vectors given below:

$$\mathbf{p} = [p_1 \ p_2 \ \dots \ 0.5p_{n+1}] \tag{17a}$$

$$\mathbf{q} = [q'_1 \ q'_2 \ \dots \ 0.5q'_n] \tag{17b}$$

$$\mathbf{c} = [\cos n\theta \ \cos(n-1)\theta \ \dots \ \cos\theta \ 1]^T \tag{18a}$$

$$\mathbf{c_0} = [\cos(n-1)\theta \ \cos(n-2)\,\theta \dots \cos\theta \ 1]^T \tag{18b}$$

Even though Eqs. (15) and (16) are amenable to real number computations, they are inadequate for computing their roots. They need to be transformed into powers of $x = \cos\theta$. One may realize that the entries of $\mathbf{c}$ and $\mathbf{c_0}$ in Eq. (18) are nothing but Chebyshev polynomials, $T_n(x) = \cos n\theta$, can be transformed to the desired powers as given in references [7,8]. For example, in a matrix form, $\mathbf{c}$ in Eq. (18) can be represented by:

$$\mathbf{c} = V\mathbf{x} \tag{19}$$

where $\mathbf{x}$ is a column vector given as:

$$\mathbf{x} = [x^n \ x^{n-1} \ ... \ 1]^T \tag{20}$$

and V is the transformation matrix. Numerical values of V for few degrees are listed in references [7,8]. However this approach requires either to store the above matrix or to generate it in the computer; both are not very attractive. A more appealing approach is using the identity, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, $T_0(x) = 1$, $T_1(x) = x$, where $x = \cos\theta$. Applying this identity repetitively, degree of the T_n term goes down while power of x goes up. An in-place procedure [8],

$$for \ i = n - 1 \ down \ to \ 1 \ and \ for \ j = 1, ..., i$$
$$p(j + 2) = p(j + 2) - p(j); \tag{21}$$
$$p(j) = 2p(j);$$

to the nth degree polynomial whose coefficients are as given in Eq. (17) yields the desired transformation.

Now, we describe the method for computing the gain margin.

3. Procedure for computing the gain margin

To carry out the computation of G_m in real arithmetic, we multiply both terms of Eq. (2) by $A(1/z)$, which will now become as follows:

$$G_m B(z)A(1/z) + A(z)A(1/z) = 0 \tag{22}$$

Evidently,

$$D(z) = A(z)A(1/z) \tag{23}$$

is real and

$$N(z) = B(z)A(1/z) \tag{24}$$

is complex number for which the real and imaginary parts are as follow:

$$N_r(z) = [N(z) + N(1/z)]/2 \tag{25}$$

$$N_i(z) = [N(z) - N(1/z)]/2i \tag{26}$$

The modified characteristic equation now becomes:

$$G_m(N_r + iN_i) + D = 0 \tag{27}$$

Therefore, the real and imaginary parts of Eq. (27) are written as:

$$G_m N_r + D = 0 \tag{28}$$

$$G_m N_i = 0 \tag{29}$$

The above equation shows that the gain margin is determined at z_{cg} at which $N_i = 0$, and for this value of z, the gain margin is found by computing following expression.

$$G_m = -D(z_{cg})/N_r(z_{cg}) \tag{30}$$

The polynomial N_i, in Eq. (26), is asymmetric and its roots can be found as follows: Let n and m be the degrees of A and B. Eqs. (24) and (26) yield:

$$N_i(z)2/i = N(1/z) - N(z) = B(1/z)\,A(z) - B(z)\,A(1/z) \tag{31}$$

To make the above in positive powers of z, let $B(1/z) = z^{-m}\bar{B}(z)$, $A(1/z) = z^{-n}\bar{A}(z)$, $C(z) = \bar{B}(z)A(z)$, and $\bar{C}(z) = \bar{A}(z)B(z)$. (Note that the coefficients of $\bar{B}(z)$ and $\bar{A}(z)$ are the reverse of $B(z)$ and $A(z)$ respectively.) This implies Eq. (32),

$$N_i(z)2/i = z^{-m}C(z) - z^{-n}\bar{C}(z) \tag{32}$$

where $C(z)$ is given as:

$$C(z) = \bar{B}(z)A(z) = \sum_{j=1}^{n+m+1} c_j z^{m+n+1-j} \tag{33}$$

Coefficients of $\bar{C}(z)$ are the reverse of $C(z)$, $\bar{c}_j = c_{m+n+2-j}$, $j = 1, \ldots, m+n+1$. Thus:

$$\bar{C}(z) = \sum_{j=1}^{n+m+1} \bar{c}_j z^{m+n+1-j} = \sum_{j=1}^{n+m+1} c_{m+n+2-j} z^{m+n+1-j} = \sum_{k=1}^{n+m+1} c_k z^{-1+k} \tag{34}$$

Substituting for the $C(z)$ and $\bar{C}(z)$ polynomials from Eqs. (33) and (34) into Eq. (32) yields:

$$N_i(z)2/i = z^{-m}\sum_{j=1}^{n+m+1} c_j z^{m+n+1-j} - z^{-n}\sum_{j=1}^{n+m+1} c_j z^{-1+j} = \sum_{j=1}^{n+m+1} c_j(z^{n+1-j} - z^{-n-1+j}) \tag{35}$$

The above summation can be simplified if it is split into three summations:

$$N_i(z)2/i = \sum_{j=1}^{n-m} c_j(z^{n+1-j} - z^{-n-1+j}) + \sum_{j=n-m+1}^{n} c_j(z^{n+1-j} - z^{-n-1+j})$$
$$+ \sum_{j=n+1}^{n+m+1} c_j(z^{n+1-j} - z^{-n-1+j}) \tag{36}$$

Applying $k = 2n+2-j$ to the last term, noting that $j = n+1$ term vanishes, gives:

$$\sum_{j=n+2}^{n+m+1} c_j(z^{n+1-j} - z^{-n-1+j}) = \sum_{k=n}^{n-m+1} c_{2n+2-k}(z^{-n-1+k} - z^{n+1-k}) \tag{37}$$

Substituting back into Eq. (36) gives:

$$N_i(z)2/i = \sum_{j=1}^{n-m} c_j(z^{n+1-j} - z^{-n-1+j}) + \sum_{j=n-m+1}^{n} (c_j - c_{2n+2-j})(z^{n+1-j} - z^{-n-1+j}) \tag{38}$$

Thus, Eq. (38) can be simplified to Eq. (39).

$$N_i(z) = -\sum_{j=1}^{n} q_j \sin(n+1-j)\,\theta$$
$$q_j = c_j, \qquad\qquad\quad j = 1, \ldots, n-m \tag{39}$$
$$q_j = c_j - c_{2n+2-j}, \quad j = n-m+1, \ldots, n$$

Precisely following the above steps for N_r, we reach to following solution.

$$N_r(z) = \sum_{j=1}^{n+1} q_j \cos(n+1-j)\,\theta$$

$$
\begin{aligned}
q_j &= c_j, & j &= 1, \dots, n-m \\
q_j &= c_j + c_{2n+2-j}, & j &= n-m+1, \dots, n \\
q_j &= c_{n+1}, & j &= n+1
\end{aligned}
$$

$$(40)$$

Therefore, the procedure for determining the gain margin is summarized below:

i. Compute the coefficients c_j of the C polynomial from the coefficients of A and B using Eq. (33).
ii. Construct polynomial N_i from Eq. (39) and solve for its roots to get z_{cg}.
iii. Construct polynomial D from Eq. (23) and N_r from Eq. (40), evaluate at z_{cg}.
iv. Determine the gain margin from Eq. (30).

Next, we describe the process for computing the phase margin.

4. Procedure for computing the phase margin

Computing the phase margin, P_m, is governed by Eq. (3). It shows that the amplitude of $G(z)$ equals to 1, at z_{cp} for which $G(z_{cp})G(1/z_{cp}) = 1$. Thus, we solve Eq. (41).

$$W(z_{cp}) = A(z_{cp})A(1/z_{cp}) - B(z_{cp})B(1/z_{cp}) = 0 \tag{41}$$

Similar to Eqs. (22)-(27), we see the equation given below.

$$G(z) = (N_r + iN_i)/D \tag{42}$$

Hence, $\angle G$ becomes,

$$\angle G = \arctan(N_i, N_r) \tag{43}$$

where N_r and N_i are computed at z_{cp}. The phase margin is then found as:

$$P_m = \angle G + \pi \tag{44}$$

The procedure for determining P_m is summarized here:
i. Construct W from Eq. (41) and solve for its roots to get z_{cp}.
ii. Evaluate N_r and N_i at z_{cp}.
iii. Compute the phase margin from Eq. (44).

One observes that polynomial, W, is symmetric. Again its roots can be found by converting it to a Chebyshev polynomial and determine its roots.

5. Application of the procedure

To expedite the derivation of this algorithm, we treated the discrete time transfer function (DTTF), from the outset, as a theoretical mathematical entity without regards to where it is originated from or how its results would be applicable. Now, we address some applicable practical issues. In practice, DTTF usually results from sampling linear continuous time system at a constant time interval T and transforming them to DTTF. There are various transformation methods, and one of them is the bilinear transformation. The Laplace variable $s = i\omega$ is now transformed into the delay $z = e^{i\theta} = e^{i\omega T}$. Hence, θ represents the

frequency ω normalized by the sampling rate, $1/T$; and the frequency, ω, is obtained by dividing θ by T. Now, from Eqs. (2) and (3), the gain and phase cross over frequencies are given by $\omega_{cg} = \theta_{cg}/T$ and $\omega_{cp} = \theta_{cp}/T$. To describe the procedure, we shall use following example.

$$G(z) = \frac{0.04798z + 0.0464}{z^3 - 1.41z^2 + 0.1808z + 0.36}, \qquad T = 0.1 \qquad (45)$$

The 4th element of denominator, $A(4)$, will be used as a variable that ranges from 0.3 to 0.36. The gain and phase margins are computed and plotted, against the variable $A(4)$, using the above analytical algorithm vs. the Matlab 'margin' function as shown in Figs. 1 and 2. Both figures show that, practically, differences between the results of methods are eye indistinguishable.
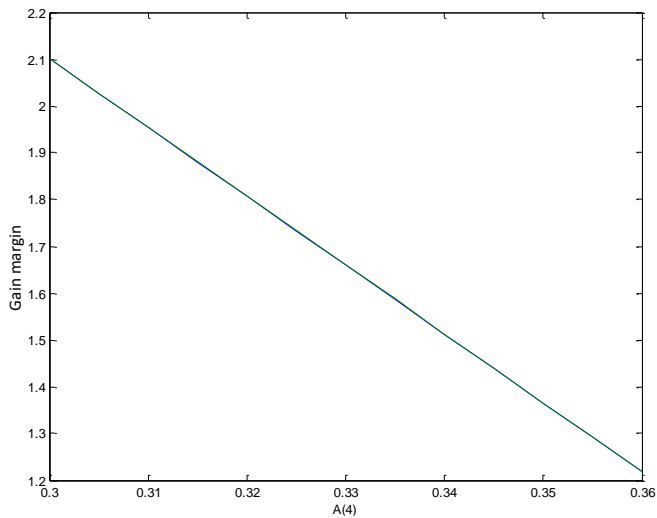

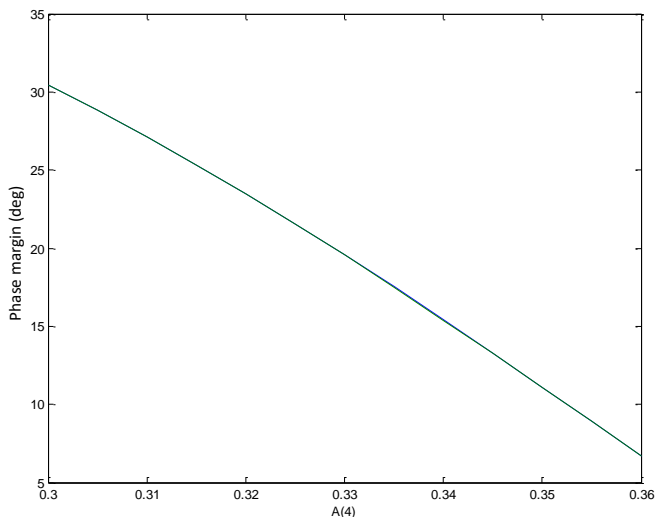**Fig. 1** Gain margin vs A(4) graph.

Fig. 2 Phase margin vs A(4) graph.

Figs. 3 and 4 depict the differences between these two solutions for margin parameters against the variable $A(4)$. Both figures show that these differences are minor. For example when $A(4) = 0.36$, the cross over frequencies and the gain and phase margins found by the analytical algorithm are,

$$\omega_{cg} = 4.1959 \text{ rad/s} \quad G_m = 1.2179$$
$$\omega_{cp} = 3.9846 \text{ rad/s} \quad P_m = 6.6869 \text{ deg}$$

while those given by the Matlab function '*margin*' are as follows:

$$\omega_{cg} = 4.1960 \text{ rad/s} \quad G_m = 1.2181$$
$$\omega_{cp} = 3.9846 \text{ rad/s} \quad P_m = 6.6861 \text{ deg}$$
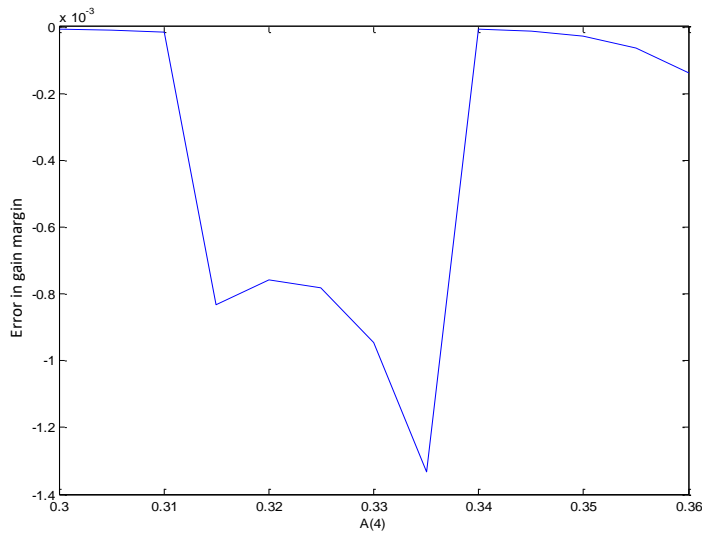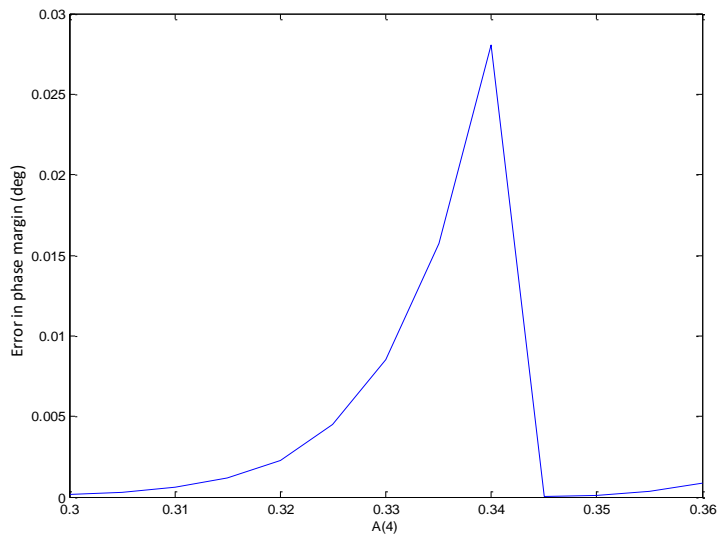

Fig. 3 Gain margin error vs A(4) graph.

Fig. 4 Phase margin error vs A(4) graph.

As seen, the results of both methods match very closely. However, this analytical method is programmable and can be implemented in any language and can be imbedded in any design. Despite the appearance of trigonometric functions everywhere, they are only used when computing the cross over frequencies and in computing the phase margin in Eq. (43).

6. Summary and conclusions

This paper has presented a numerical algorithm for computing the gain and phase margins for a discrete time system. The algorithm entirely uses real computations. Inputs to this algorithm are just the coefficients of the numerator and denominator of the polynomials of the discrete system. The method is programmable and non-iterative; therefore, it can be imbedded in procedures for computing the design and stability of discrete time systems. Computationally, when compared with the Matlab function '*margin*', it provides almost identical results.

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