Turk. J. Math. Comput. Sci.
9(2018) 14–24
© MatDer
http://dergipark.gov.tr/tjmcs
http://tjmcs.matder.org.tr

MATDER

Binary States Cellular Automata with Reflexive and Periodic Boundaries and Image problem

Shovkat Redjepov[a], Selman Uguz[b,*]

[a] *Department of Information Technologies, Faculty of Computer Engineering, Tashkent University of Information Technologies, Tashkent 100202, Uzbekistan*
[b] *SU Mathematics and Science Academy, Sanliurfa, 63120, Turkey*

Received: 31-12-2017 • Accepted: 29-09-2018

Abstract. The present paper focuses on the theory of two-dimensional (2D) linear cellular automata (CA) with respect to uniform reflexive and periodic boundary conditions. It is investigated the theoretical aspects of 2D linear CA over binary states field with image problem. We consider geometrical and visual aspects of images generated by these CA transition rules. Multiple copies of any arbitrary images corresponding to CA can be studied further by considering theses transition rules of von Neumann and Moore CAs. An important note that these special types of CAs can be applied many different special problems e.g. computability theory, applied mathematics, theoretical chemistry and biology, DNA and genetics research, image science, textile design.

2010 AMS Classification: 37B15, 68Q80.

Keywords: Cellular automata, binary states, reflexive and periodic boundary, CA and image problem.

1. Introduction

Cellular automata theory (CA for brevity) introduced by Ulam and von Neumann [27] in the early 1950's, have been systematically studied by Hedlund from purely mathematical point of view. One-dimensional (1D) CA has been investigated to a large extent. However, a little interest has been given to two-dimensional cellular automata (2D CA). Von Neumann [27] showed that a cellular automaton can be universal. Due to its complexity, von Neumann rules were never implemented on a computer. In the beginning of the eighties, Wolfram [28] has studied in much detail a family of simple 1D CA rules and showed that even these simplest rules are capable of emulating complex behavior. Some basic and precise mathematical models using matrix algebra over the binary field which characterize the behavior of 2D nearest neighborhood linear CA with null and periodic boundary conditions have been seen in the literature [13–16,20]. CA has received remarkable attention in the last few decades [20–25]. Due to its structure CA has given the opportunity to model and understand many behaviors in nature easier. Most of the work for CA is done for one-dimensional (1D) case. Here we study the theory of two dimensional uniform reflexive and periodic boundary CA (2D RB CA, 2D PB CA) of the all linear rules (e.g. von Neumann, Moore neighborhood and the others) and problem of image science for self copied patterns (see Figs. 6-7). We present some illustrative examples and figures to explain the method in detail.

*Corresponding Author
Email addresses: redjepov@mail.ru (S. Redjepov), drselmanuguz@gmail.com (S. Uguz)

In the literature, there are many contributions to 2D cellular automata and its applications. In [1, 17, 18], the theory and applications of two-dimensional, Null-boundary, nine-neighborhood, cellular automata linear rules and their relations are studied in details. Application of two-dimensional periodic cellular automata with basic image processing is investigated in [8,10,12]. In [6,7,9,11], sweeper's Algorithm and its application on image Clustering and hexagonal neighborhood CA are studied. Characterization of any non-linear Boolean function using a set of linear operators and investigation of the global dynamics of cellular automata using Boolean derivatives are presented in [2,19]. Nowadays many contributions on this direction still continues.

In this paper, we concentrate a special family of 2D finite linear CA with reflexive and periodic boundary condition over the binary states field $\mathbb{Z}_2$. Here, we set up a specific relation between the structure of these CA and transition matrix rules of 2D linear CA with reflexive and periodic boundary condition. It is determined of the transition rule matrices of this special CA by means of the matrix algebra theory. Since CAs nature is very simple to allow mathematical studies and to obtain very complicated and complex behaviors in dynamical systems, we believe that these linear CAs could be obtained many different kind of applications. The present results can give further to the algebraic consequences of these 2D CA and relates some elegant applications found by the authors in the literature (i.e. [20,21,24–26]).

2. Preliminaries

We investigate CA transition rules over the field $\mathbb{Z}_2$, and deal with the general case of the primary transition rule matrices.

2.1. **Technical Details.** In this section, we introduce 2D CA over the field Z_2 by using all primary local rules. We recall the definition of a CA. We consider the 2D integer lattice Z^2 and the configuration space $\Omega = \{0,1\}^{Z^2}$ with elements $\sigma : Z^2 \to \{0,1\}$. (see details in [25,26]).

The 2D finite CA consists of $m \times n$ cells arranged in m rows and n columns, where each cell takes one of the values of 0 or 1. A configuration of the system is an assignment of states to all the cells. Every configuration determines a next configuration via a linear transition rule that is local in the sense that the state of a cell at time $(t+1)$ depends only on the states of some of its neighbors at time t using modulo 2. For 2D CA nearest neighbors (see Figs. 1-3), there are nine cells arranged in a 3×3 matrix centering that particular cell (see [S6, S9, S10] for the details). For 2D CA there are some classic types of rules, but in this work only we restrict ourselves to primary linear rules (briefly, from Rule 1 to Rule 256). So, we can define (for an example Rule 170) the $(t+1)^{th}$ state of the $(i,j)^{th}$ cell as follows;

$$x_{(i,j)}^{(t+1)} = x_{(i-1,j)}^{(t)} + x_{(i,j+1)}^{(t)} + x_{(i+1,j)}^{(t)} + x_{(i,j-1)}^{(t)} (mod\ 2) \quad (Rule\ 170).$$

Any other rules can be written the linear combination of primary rules of Fig. 2. The dependence will be restricted to the case of being zero or nonzero, in other words if the coefficients in the above equation equal to 0 or 1, then this case will be assumed to be the same. This approach is adopted in this paper though these cases may be further distinguished. The linear combination of the neighboring cells on which each cell value is dependent is called the rule number of the 2D CA over the field Z_2.

In this paper, we will only consider a 2D finite CA generated by the primary rules with reflexive (RB) and periodic boundary (PB). It is well known that these CAs are discrete dynamical systems formed by a finite 2D array $m \times n$ composed by identical objects called cells. Let $\Phi : M_{m\times n}(Z_2) \to Z_2^{mn}$. Φ takes the t^{th} state $[X_t]$ given by

$$\begin{pmatrix} x_{11} & x_{12} & \dots & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & \dots & x_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & \dots & x_{mn} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} & \dots & x_{m1} & \dots & x_{mn} \end{pmatrix}^T,$$

where T is the transpose of the matrix. Therefore, the local rules will be assumed to act on Z_2^{mn} rather than $\Phi : M_{m\times n}(Z_2)$. Suppose binary information matrix is $[X_t]_{m\times n}$, of order $m \times n$:

$$[X_t]_{m\times n} = \begin{pmatrix} x_{11}^{(t)} & \dots & \dots & x_{1n}^{(t)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ x_{m1}^{(t)} & \dots & \dots & x_{mn}^{(t)} \end{pmatrix}$$

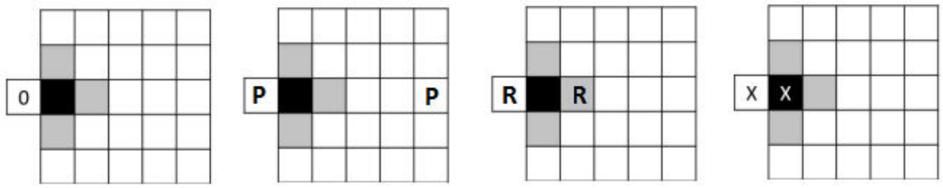

FIGURE 1. Null, periodic, reflexive and adiabatic boundary configurations on 2D CA respectively.

	Rule 64	Rule 128	Rule 256	
	Rule 32	Rule 1	Rule 2	
	Rule 16	Rule 8	Rule 4	

FIGURE 2. Rule convention chart of 2D linear CA over binary field. Any other rules can be written the linear combination of these primary rules.

is called the configuration of the 2D finite CA at time t. From the above equations, we can define as follows;

$$(T_{Rule})_{mn\times mn}\cdot\begin{pmatrix} x_{11}^{(t)} \\ \cdot \\ \cdot \\ x_{1n}^{(t)} \\ \cdot \\ \cdot \\ x_{m1}^{(t)} \\ \cdot \\ \cdot \\ x_{mn}^{(t)} \end{pmatrix} = \begin{pmatrix} x_{11}^{(t+1)} \\ \cdot \\ \cdot \\ x_{1n}^{(t+1)} \\ \cdot \\ \cdot \\ x_{m1}^{(t+1)} \\ \cdot \\ \cdot \\ x_{mn}^{(t+1)} \end{pmatrix}$$

The matrix $(T_{Rule})_{mn\times mn}$ is called the rule matrix with respect to the 2D finite $CA_{m\times n}$ with the transition rule (see [6] for details).

Let us give some background of 2D CA boundary conditions in the literature. Considering the neighbourhood of the information cells, there are two well-known studied boundary approaches in the literature (see Fig. 1).

- Null or 0-fixed boundary: A null boundary (NB) CA is the one whose extreme cells are connected to 0-state.
- Adiabatic boundary: An adiabatic boundary (AB) CA is duplicating the value of the cell in an extra virtual neighbor.
- Reflexive boundary (RB): This is designed for the value left and right neighbors that are the same with respect to the boundary cell.
- Periodic boundary (PB): This implies that the space is wrapped around each spatial axis, i.e., the cells at one edge of a finite space are connected to the cells at the opposite edge.

The linear combination of the neighboring cells on which each cell value is dependent is called the rule number of the 2D CA over the binary field Z_2.

3. All Primary Transition Rule Matrices

In this section, we will obtain all primary transition rule matrices corresponding to 2D linear, reflexive and periodic boundary CA over the binary field Z_2.

3.1. **Transition Rule Matrices for Reflexive Boundary.**

Theorem 3.1. (Reflexive Case). *The transition rule matrices of 2D linear CA of all primary rules (1, 2, 4, 8, 16, 32, 64, 128, and 256) under the reflexive boundary condition are given in the following way*

$$T_{Rule1RB} = \begin{pmatrix} I & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & I & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & I & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & ... & I & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & I \end{pmatrix} \quad T_{Rule2RB} = \begin{pmatrix} K & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & K & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & K & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & K & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & K \end{pmatrix}$$

$$T_{Rule4RB} = \begin{pmatrix} 0 & K & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & K & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & K & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 & K \\ 0 & 0 & 0 & 0 & ... & K & 0 \end{pmatrix} \quad T_{Rule8RB} = \begin{pmatrix} 0 & I & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & I & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & I & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 & I \\ 0 & 0 & 0 & 0 & ... & I & 0 \end{pmatrix}$$

$$T_{Rule16RB} = \begin{pmatrix} 0 & L & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & L & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & L & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 & L \\ 0 & 0 & 0 & 0 & ... & L & 0 \end{pmatrix} \quad T_{Rule32RB} = \begin{pmatrix} L & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & L & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & L & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & L & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & L \end{pmatrix}$$

$$T_{Rule64RB} = \begin{pmatrix} 0 & L & 0 & ... & 0 & 0 & 0 \\ L & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & L & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & L & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & L & 0 \end{pmatrix} \quad T_{Rule128RB} = \begin{pmatrix} 0 & I & 0 & ... & 0 & 0 & 0 \\ I & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & I & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & I & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & I & 0 \end{pmatrix}$$

$$T_{Rule256RB} = \begin{pmatrix} 0 & K & 0 & ... & 0 & 0 & 0 \\ K & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & K & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & K & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & K & 0 \end{pmatrix}_{mn \times mn}$$

where I is the identity $n \times n$ matrix and the sub-matrices K, L are given as

$$K = \begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & ... & 0 & 1 \\ 0 & 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}_{n \times n} \quad L = \begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}_{n \times n}.$$

Proof. To construct for each primary transition rule matrix $T_{Rule\ RB}$, it is needed to determine the action of $T_{Rule\ RB}$ on the standard bases vectors. Firstly, we consider the linear transformation Φ from $m \times n$ matrix space to itself. Next, we relate the transformation Φ with rule matrix $T_{Rule\ RB}$. Let e_{ij} denote the matrix of size $m \times n$ where the (i, j) position is equal to one and the rest of the entries equal to zero. It is well-known that these vectors give the standard basis for this space (see [27,28]). Consider e_{ij}, the image of e_{ij} which is $\Psi(e_{ij})$ is related to the transition rules neighbors. $\Phi(e_{ij}))$ equals to a linear combination of its non-zero nearest neighbors in the following way

$$\Phi(e_{ij}) = e_{i+1,j} + e_{i,j-1} + e_{i-1,j} + e_{i,j+1},$$

with a care on the nearest bordering components of the matrix. Due to the neighboring linear relations that govern the rule, we obtain the transition rule matrices given in the theorem.

	$x_{i-1,j-1}$	$x_{i-1,j}$	$x_{i-1,j+1}$	
	$x_{i,j-1}$	$x_{i,j}$	$x_{i,j+1}$	
	$x_{i+1,j-1}$	$x_{i+1,j}$	$x_{i+1,j+1}$	

FIGURE 3. Two dimensional 3×3 linear cellular automata with center $x_{i,j}$.

Theorem 3.2. (von Neumann CA with Reflexive Boundary). *Let us consider von Neumann CA with reflexive boundary. Then the transitions matrix rules corresponding to von Neumann neighborhood CA with reflexive boundary is equal to Rule 170RB. Hence the transition rule matrix is*

$$T_{170RB} = \begin{pmatrix} K+L & 0 & 0 & \dots & 0 & 0 & 0 \\ I & K+L & I & \dots & 0 & 0 & 0 \\ 0 & I & K+L & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & K+L & I \\ 0 & 0 & 0 & 0 & \dots & 0 & K+L \end{pmatrix}.$$

Proof. Considering the linearity of the rule numbers, we get

$$Rule\ 170RB = Rule\ 2RB + Rule\ 8RB + Rule\ 32RB + Rule\ 128RB.$$

Then

$$\begin{aligned} T_{Rule170RB} &= \begin{pmatrix} K & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & K & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & K & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & K & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & K \end{pmatrix} + \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & I \\ 0 & 0 & 0 & 0 & \dots & I & 0 \end{pmatrix} \\ &+ \begin{pmatrix} L & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & L & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & L & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & L & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & L \end{pmatrix} + \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 & 0 \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & I & 0 \end{pmatrix} \\ &= \begin{pmatrix} K+L & 0 & 0 & \dots & 0 & 0 & 0 \\ I & K+L & I & \dots & 0 & 0 & 0 \\ 0 & I & K+L & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & K+L & I \\ 0 & 0 & 0 & 0 & \dots & 0 & K+L \end{pmatrix}. \end{aligned}$$

Theorem 3.3. (Moore CA with Reflexive Boundary). *Let us consider Moore CA with reflexive boundary. Then the transitions matrix rules corresponding to Moore neighborhood CA with reflexive boundary is equal to Rule 510RB. So the transition rule matrix is*

$$T_{510RB} = \begin{pmatrix} K+L & 0 & 0 & \dots & 0 & 0 & 0 \\ I+K+L & K+L & I+K+L & \dots & 0 & 0 & 0 \\ 0 & I+K+L & K+L & I+K+L & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I+K+L & K+L & I+K+L \\ 0 & 0 & 0 & 0 & \dots & 0 & K+L \end{pmatrix}.$$

Proof. Using the linearity of the rule numbers, it is obtained

$$\begin{aligned} Rule\ 510RB &= Rule\ 2RB + Rule\ 4RB + Rule\ 8RB + Rule\ 16RB \\ &+ Rule\ 32RB + Rule\ 64RB + Rule\ 128RB + Rule\ 256RB. \end{aligned}$$

Also it can be written as

$$Rule\ 510RB = Rule\ 170RB + Rule\ 4RB + Rule\ 16RB + Rule\ 64RB + Rule\ 256RB.$$

Then

$$\begin{aligned} T_{510RB} &= \begin{pmatrix} K+L & 0 & 0 & ... & 0 & 0 & 0 \\ I & K+L & I & ... & 0 & 0 & 0 \\ 0 & I & K+L & I & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & I & K+L & I \\ 0 & 0 & 0 & 0 & ... & 0 & K+L \end{pmatrix} + \begin{pmatrix} 0 & K & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & K & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & K & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 & K \\ 0 & 0 & 0 & 0 & ... & K & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & L & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & L & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & L & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 & L \\ 0 & 0 & 0 & 0 & ... & L & 0 \end{pmatrix} + \begin{pmatrix} 0 & L & 0 & ... & 0 & 0 & 0 \\ L & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & L & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & L & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & L & 0 \end{pmatrix} + \begin{pmatrix} 0 & K & 0 & ... & 0 & 0 & 0 \\ K & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & K & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & K & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & K & 0 \end{pmatrix} \\ &= \begin{pmatrix} K+L & 0 & 0 & ... & 0 & 0 & 0 \\ I+K+L & K+L & I+K+L & ... & 0 & 0 & 0 \\ 0 & I+K+L & K+L & I+K+L & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & I+K+L & K+L & I+K+L \\ 0 & 0 & 0 & 0 & ... & 0 & K+L \end{pmatrix}. \end{aligned}$$

3.2. **Transition Rule Matrices under Periodic Boundary.**

Theorem 3.4. (Periodic Case). *The transition rule matrices of 2D linear CA of all primary rules (1, 2, 4, 8, 16, 32, 64, 128, and 256) under the periodic boundary condition are given in the following way*

$$T_{Rule1PB} = \begin{pmatrix} I & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & I & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & I & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & ... & I & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & I \end{pmatrix} \qquad T_{Rule2PB} = \begin{pmatrix} M & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & M & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & M & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & ... & M & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & M \end{pmatrix}$$

$$T_{Rule4PB} = \begin{pmatrix} 0 & M & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & M & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & ... & 0 & M \\ M & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix} \qquad T_{Rule8PB} = \begin{pmatrix} 0 & I & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & I & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & ... & 0 & I \\ I & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix}$$

$$T_{Rule16PB} = \begin{pmatrix} 0 & N & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & N & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & ... & 0 & N \\ N & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix} \qquad T_{Rule32PB} = \begin{pmatrix} N & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & N & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & N & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & ... & N & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & N \end{pmatrix}$$

$$T_{Rule64PB}=\begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & N \\ N & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & N & 0 & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & N & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & N & 0 \end{pmatrix} \qquad T_{Rule128PB}=\begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & I \\ I & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & I & 0 & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & I & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & I & 0 \end{pmatrix}$$

$$T_{Rule256PB}=\begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & M \\ M & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & M & 0 & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & M & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & M & 0 \end{pmatrix}_{mn\times mn}$$

where I is the identity $n\times n$ matrix and the sub-matrices M,N are given as

$$M=\begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & ... & 0 & 1 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix}_{n\times n} \quad N=\begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & 1 \\ 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}_{n\times n}.$$

Proof. To obtain for each transition rule matrix $T_{Rule\ PB}$, it is needed to determine the action of $T_{Rule\ PB}$ on the bases vectors as given in the reflexive case. Consider the linear transformation ψ from $m\times n$ matrix space to itself. Next, we relate the transformation ψ with rule matrix $T_{Rule\ PB}$. Given e_{ij}, the image of e_{ij} which is $\psi(e_{ij})$ is related to the transition rules neighbors. $\psi(e_{ij}))$ equals to a linear combination of its non-zero neighbors as in the reflexive case. Then it is found the periodic transition rule matrices given in the theorem.

Theorem 3.5. (von Neumann CA with Periodic Boundary). *Let us consider von Neumann CA with periodic boundary. Then the transitions matrix rules corresponding to von Neumann neighborhood CA with periodic boundary is equal to Rule 170PB. Hence the transition rule is*

$$T_{170PB}=\begin{pmatrix} M+N & I & 0 & ... & 0 & 0 & I \\ I & M+N & I & ... & 0 & 0 & 0 \\ 0 & I & M+N & I & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & I & M+N & I \\ I & 0 & 0 & 0 & ... & I & M+N \end{pmatrix}.$$

Proof. By using the linearity of the rule numbers, we write

$$Rule\ 170PB = Rule\ 2PB + Rule\ 8PB + Rule\ 32PB + Rule\ 128PB.$$

Then

$$\begin{aligned} T_{Rule170PB} \;=\; & \begin{pmatrix} M & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & M & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & M & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & ... & M & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & M \end{pmatrix} + \begin{pmatrix} 0 & I & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & I & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & ... & 0 & I \\ I & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix} \\ & + \begin{pmatrix} N & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & N & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & N & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & ... & N & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & N \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & I \\ I & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & I & 0 & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & I & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & I & 0 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} M+N & I & 0 & \dots & 0 & 0 & I \\ I & M+N & I & \dots & 0 & 0 & 0 \\ 0 & I & M+N & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & M+N & I \\ I & 0 & 0 & 0 & \dots & I & M+N \end{pmatrix}.$$

Theorem 3.6. (Moore CA with Periodic Boundary). *Let us consider Moore CA with periodic boundary. Then the transitions matrix rules corresponding to Moore neighborhood CA with periodic boundary is equal to Rule 510PB. Hence the transition rule is*

$$T_{510PB} = \begin{pmatrix} M+N & I+M+N & 0 & \dots & 0 & 0 & M+N \\ I+M+N & M+N & I+M+N & \dots & 0 & 0 & 0 \\ 0 & I+M+N & M+N & I+M+N & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I+M+N & M+N & I+M+N \\ M+N & 0 & 0 & 0 & \dots & I+M+N & M+N \end{pmatrix}.$$

Proof. The linearity of the rule numbers can be written as

$$\begin{aligned} Rule\ 510PB &= Rule\ 2PB + Rule\ 4PB + Rule\ 8PB + Rule\ 16PB \\ &+ Rule\ 32PB + Rule\ 64PB + Rule\ 128PB + Rule\ 256PB. \end{aligned}$$

Also we get

$$Rule\ 510PB = Rule\ 170PB + Rule\ 4PB + Rule\ 16PB + Rule\ 64PB + Rule\ 256PB.$$

Then

$$\begin{aligned} T_{510PB} &= \begin{pmatrix} M+N & I & 0 & \dots & 0 & 0 & I \\ I & M+N & I & \dots & 0 & 0 & 0 \\ 0 & I & M+N & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & M+N & I \\ I & 0 & 0 & 0 & \dots & I & M+N \end{pmatrix} + \begin{pmatrix} 0 & M & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & M & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 0 & M \\ M & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & N & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & N & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 0 & N \\ N & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & N \\ N & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & N & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & N & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & N & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & M \\ M & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & M & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & M & 0 \end{pmatrix} \\ &= \begin{pmatrix} M+N & I+M+N & 0 & \dots & 0 & 0 & M+N \\ I+M+N & M+N & I+M+N & \dots & 0 & 0 & 0 \\ 0 & I+M+N & M+N & I+M+N & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I+M+N & M+N & I+M+N \\ M+N & 0 & 0 & 0 & \dots & I+M+N & M+N \end{pmatrix}. \end{aligned}$$

4. Image Problems and Cellular Automata

Self copied pattern producing is one of the most interesting topic and research area in nonlinear science. Pattern generation is the process of transforming copies of the motif about the array (1D), plane (2D) or space (3D) in order to create the whole repeating pattern with no overlaps and blank [14, 15, 21–25]. These patterns have some mathematical properties which make generating algorithm possible. A cellular automaton is a good candidate algorithmic approach used for pattern generation.

In the present paper it is used von Neumann and Moore CA with all the nearest neighborhoods (i.e. all primary rules and their combinations and Theorems 3.1-3.6 (von Neumann and Moore CA with Reflexive-Periodic Boundaries)) to

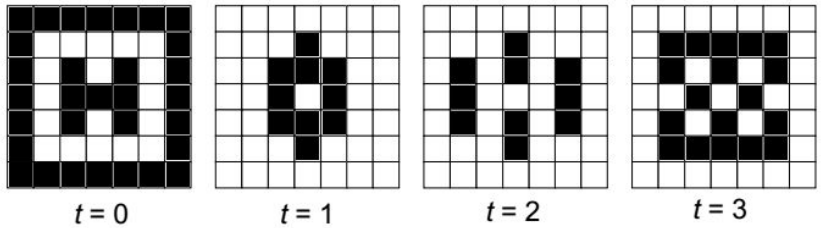

Figure 4. Image transition of 7×7 von Neumann CA (i.e. Rule 170) with reflexive boundary after 1,2,3 iterations of the first image.

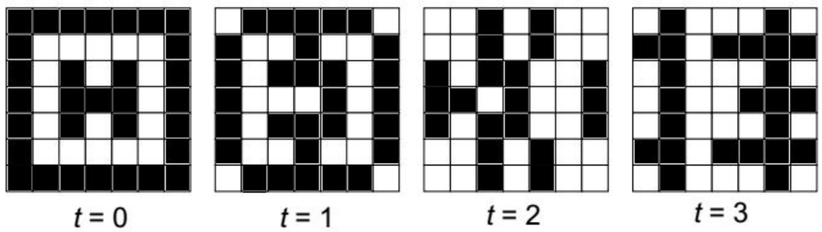

Figure 5. Image transition of 7×7 von Neumann CA (i.e. Rule 170) with periodic boundary after 1,2,3 iterations of the first image.

generate some image patterns of two states or digital images. For applying 2D reflexive and periodic CA linear rules in image problems, we study a binary matrix of size (7×7) and (100×100) respectively, and these are due to computational limitations. It is mapped each element of the matrix to a unique pixel on the screen (using MATLAB codes) and colored a pixel white for 0, black for 1 for the matrix elements (see Figs. 4-5). Then we take another image (see Figs. 6-7) whose size is less than (30×30) for which patterns are to be generated and put it in the center of the binary matrix. This is the way how the image is drawn within an area of (100×100) pixels. We see from Figs. 6-7 that the self copied patterns can be generated only when number of repetition is 16-iterations. Also behaviors for different boundaries produce different shapes when t = 16. This interesting results should be investigated as a image problem for CA in the next studies.

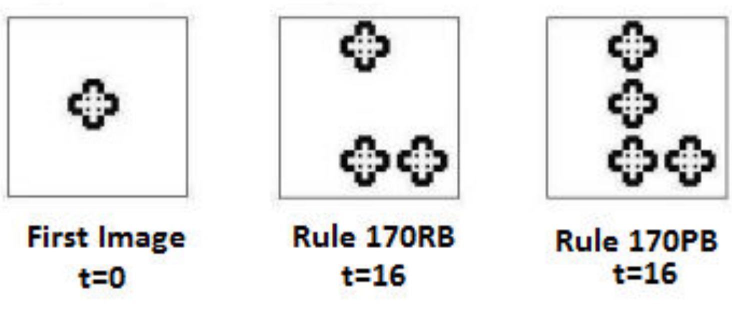

Figure 6. Image transition of 100×100 von Neumann CA (i.e. Rule 170) with reflexive and periodic boundaries after 16 iterations of the first image.

5. Conclusion

In the present paper it is studied the theory two dimensional, uniform reflexive and periodic boundary CA of linear primary rules and problem of image science. It is easily seen that 2D linear these CAs could be applied successfully in

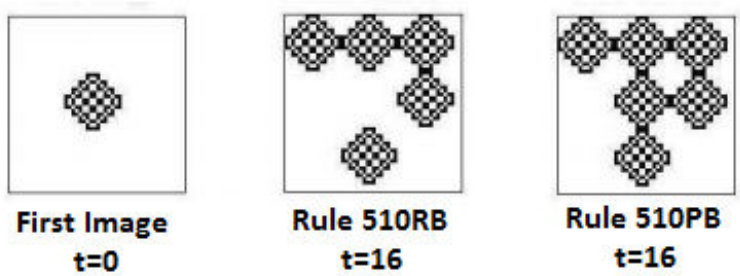

Figure 7. Image transition of 100×100 Moore CA (i.e. Rule 510) with reflexive and periodic boundaries after 16 iterations of the first image.

evolution patterns of image problems. The some characterizations and applications on 2D finite linear CAs considering matrix algebra built on any other states fields should be explored for the next studies. New interesting results and further connections on this direction wait to be explored in 2D CA and the other science branches [3–5, 22].

References

[1] Choudhury, P.P., Nayak, K., Sahoo, S., Rath, P., *Theory and applications of two-dimensional, null-boundary, nine-neighborhood, cellular automata linear rules*, Journal of Current Computer Science and Technology, **2**(2012), 50–54. arXiv:0804.2346. 1

[2] Choudhury, P.P., Sahoo, S., Chakraborty, M., Bhandari, S., Pal, A., *Investigation of the global dynamics of cellular automata using Boolean derivatives*, Computers and Mathematics with Applications, **57**(2009), 1337-1351. arXiv:0804.2346. 1

[3] Choudhury, P.P., Sahoo, S., Hassan, S. S., Basu, S., Ghosh, D., Kar, D., Ghosh, Ab., Ghosh, Av., Ghosh A.K., *Classification of cellular automata rules based on their properties*, Int. J. of Comp. Cogn., **8**(2010), 50–54. 5

[4] Chou H.H., Reggia J.A., *Emergence of self-replicating structures in a cellular automata space*, Phys. D, **110**(1997), 252–276. 5

[5] Dihidar K., Choudhury P.P., *Matrix algebraic formulae concerning some exceptional rules of two dimensional cellular automata*, Inf. Sci., **165**(2004), 91–101. 5

[6] Mohammed, J., Mohanty, B., Sahoo, S., *Two dimensional cellular automata and its reduced rule matrix*, Int. Journal of The Orissa Mathematical Society, **1**(2012), 97-104. 1

[7] Mohammed, J., Sahoo, S., *Design and analysis of matrices for two dimensional cellular automata linear rules in hexagonal neighborhood*, Mathematica Aeterna, **1**(2011), 537–545. 1

[8] Mohammed, J., Nayak, D.R., *An efficient edge detection technique by two dimensional rectangular cellular automata*, Information Communication and Embedded Systems (ICICES) (2014). 1

[9] Nayak, D.R., Sahu, S.K., Mohammed, J., *A cellular automata based optimal edge detection technique using twenty-five neighborhood model*, arXiv:1402.1348. 1

[10] Nayak, D.R., Patra,P.K., Mahapatra, A., *A survey on two dimensional cellular automata and its application in image processing*, arXiv:1407.7626. 1

[11] Padhan, U., Tripathy, S. K., Sahoo, S., Nayak, B. K. Jena, O.P., *Sweeper's algorithm and its application on image clustering*, International Journal of Computer Applications, **119** (2015), 30–37. 1

[12] Pattanaik, S.R., Nayak, B.K., Sahoo, S., *Application of two-dimensional periodic cellular automata in image processing*, Int. Journal of Computer, Mathematical Sciences and Applications, **5**(2011), 49–55. 1

[13] Sahin, U., Sahin, F., Uguz, S., Hybridized fuzzy cellular automata thresholding algorithm for edge detection optimized by PSO, High Capacity Optical Networks and Enabling Technologies (HONET-CNS), 10th International Conference IEEE, (2013), 228–232. 1

[14] Sahin U., Uguz S., Akin H., *The transition rules of 2D linear cellular automata over ternary field and self- replicating patterns*, Int. J. Bif. and Chaos, **25**(2015), 1550011. 1, 4

[15] Sahin U., Uguz S., Akin H., Siap, I., *Three-state von Neumann cellular automata and pattern generation*, App. Math. Mod., **39**(2015), 2003–2024. 1, 4

[16] Sahin U., Uguz S., Sahin F., *Salt and pepper noise filtering with fuzzy-cellular automata*, Comp. and Elect. Eng., **40**(2014), 59–69. 1

[17] Sahoo, S., Choudhury, P.P., Pal, A., Nayak, B.K., *Solutions on 1-D and 2-D density classification problem using programmable cellular automata*, J. Cellular Automata, **9**(2014), 59–88. 1

[18] Sahoo, S., Sahoo, S.K., Nayak, B.K., Choudhury, P.P., *Encompression using two-dimensional cellular automata rules*, arXiv: 0808.1470. 1

[19] Sahoo, S., Choudhury, P.P., Chakraborty, M., Nayak, B.K., *Characterization of any non-linear Boolean function using a set of linear operators*, Journal of Orissa Mathematical Society, **29** (2010), 111–133. 1

[20] Siap I., Akin H., Uguz S., *Structure and reversibility of 2D hexagonal cellular automata*, Comput. Math. App., **62**(2011), 4161–4169. 1

[21] Uguz S., Akin H., Siap I., *Reversibility algorithms for 3-state hexagonal cellular automata with periodic boundaries*, Int. J. Bif. and Chaos, **23**(2013), 1350101-1-15. 1, 4

[22] Uguz S., Sahin , U., Akin H., Siap I., *Self-replicating patterns in 2D linear cellular automata*, Int. J. Bif. and Chaos, **24**(2014), 1430002. 1, 4, 5

[23] Uguz S., Sahin , U., Akin H., Siap I., *2D cellular automata with an image processing application*, Acta Physica Polonica A, **125**(2014), 435–438. 1, 4

[24] Uguz S., Sahin U., Sahin, F., *Edge detection with fuzzy cellular automata transition function optimized by PSO*, Comp. and Elect. Eng., **43**(2015), 180-ÔÇô192. 1, 4

[25] Uguz S., Akin H., Siap I., Sahin, U., *On the irreversibility of Moore cellular automata over the ternary field and image application*, App. Math. Mod., **40**(2016), 8017–8032. 1, 2.1, 4

[26] Uguz S., Redjepov S., Acar E., Akin H., *Structure and reversibility of 2D von Neumann cellular automata over triangular lattice*, Int. J. Bif. and Chaos, **27**(2017), 1750083. 1, 2.1

[27] von Neumann J., The Theory of Self-Reproducing Automata, (Edited by A. W. Burks), Univ. of Illinois Press, Urbana, 1966. 1, 3.1

[28] Wolfram S., *Cellular automata* , Rev. Mod. Phys., **55**(1983), 601–644. 1, 3.1