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A Generalization of Ibn al-Haytham Recursive Formula for Sums of Powers

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Abstract. In this paper, we give a sequential generalization of Ibn al-Haytham recursive formula for sums of powers of any integer sequence. Then, we obtain a higher order dimensional generalization of the generalized Ibn al-Haytham formula. As by-products, we also show that how our recursive formulas imply other interesting integer sequences identities like Karaji L-summing equation and Abel's summation by parts lemma. Finally, as an application, we prove several identities related to Fibonnaci and harmonic numbers.

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1. Introduction

Ibn al-Haytham [6], known in Europe as Alhazen, developed a recursive formula for the sums of 4-th powers of the first n positive integers [5]. His proof was by induction on n and based on the following recursive step:

$$\begin{aligned}&(k+1)(1^3+2^3+\cdots+(k-1)^3+k^3)=\\&1^3(1+1+\cdots+1)+2^3(2+1+\cdots+1)+\cdots+k^3(k+1)=\\&(1^4+1^3+\cdots+1^3)+(2^4+2^3+\cdots+2^3)+\cdots+(k^4+k^3),\end{aligned}$$

after rearrangement, he got the following interesting recursive formula

$$(k+1)\left(\sum_{i=1}^{k} i^3\right)=\sum_{i=1}^{k} i^4+\sum_{i=1}^{k}\left(\sum_{j=1}^{i} j^3\right).$$

This clearly gives a recursive way to find $\sum_{i=1}^{n} i^m$, for any positive integer m, as follows

$$(n+1)\left(\sum_{k=1}^{n} k^{m-1}\right)=\sum_{k=1}^{n} k^m+\sum_{k=1}^{n}\left(\sum_{i=1}^{k} i^{m-1}\right). \tag{1.1}$$

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One may ask about the analogous formula for the sums of powers of reciprocal of the first n positive integers; i.e., $\sum_{i=1}^{n}\frac{1}{i^m}$. As you can see, the similar trick of Ibn al-Haytham also works here:

$$\begin{aligned}&(k+1)(\frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{(k-1)^3}+\frac{1}{k^3})=\\&\frac{1}{1^3}(1+1+\cdots+1)+\frac{1}{2^3}(2+1+\cdots+1)+\cdots+\frac{1}{k^3}(k+1)=\\&(\frac{1}{1^2}+\frac{1}{1^3}+\cdots+\frac{1}{1^3})+(\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^3})+\cdots+(\frac{1}{k^2}+\frac{1}{k^3}),\end{aligned}$$

and in general, one can prove by induction on n, for any fixed positive integer m, that

$$(n+1)\left(\sum_{k=1}^{n}\frac{1}{k^m}\right)=\sum_{k=1}^{n}\frac{1}{k^{m-1}}+\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{1}{i^m}\right). \tag{1.2}$$

It seems that we can go further and find a more general formula of recursive nature than those of (1.1) and (1.2) for sums of powers of general sequences of integer numbers. The main purpose of this paper is to obtain such a generalization and its applications in the area of integer sequences identities.

2. The Generalized Ibn al-Haytham Formula

In this section we are going to give a generalization of Ibn al-Haytham sums of powers formula for any arbitrary sequence.

For pedagogical reasons, let us try the main idea behind Ibn al-Haytham formula for finding the sums of powers of the first n positive odd integers; that is, $\sum_{i=1}^{n}(2i-1)^m$. As before, we first try the case $m=3$. After a moment of thought, we come up with the following recursive formula:

$$\begin{aligned}&(2k+1)(1^3+3^3+\cdots+(2k-1)^3)=\\&1^3(1+\overbrace{1+\cdots+1}^{2k})+3^3(3+\overbrace{1+\cdots+1}^{2k-2})+\cdots+(2k-1)^3(2k-1+\overbrace{1+1}^{2})=\\&\left[(1^4+1^3+\cdots+1^3)+(3^4+3^3+\cdots+3^3)+\cdots+((2k-1)^4+(2k-1)^3\right](2),\end{aligned}$$

or equivalently, we obtain

$$(2k+1)\left(\sum_{i=1}^{k}(2i-1)^3\right)=\sum_{i=1}^{k}(2i-1)^4+\sum_{i=1}^{k}\left(\sum_{j=1}^{i}(2j-1)^3\right)(2).$$

In general, we can prove by induction that

$$(2n+1)\left(\sum_{i=1}^{n}(2i-1)^{m-1}\right)=\sum_{i=1}^{n}(2i-1)^m+\sum_{i=1}^{n}\left(\sum_{j=1}^{i}(2j-1)^{m-1}\right)(2).$$

As a last illustrative example, we use similar technique for calculating the sums of powers of the first n positive integer numbers congruent with 1 modulo 3; that is, $\sum_{i=1}^{n}(3i-2)^m$. Hence, we get the following recursive formula:

$$\begin{aligned}&(3k+1)(1^3+4^3+\cdots+(3k-2)^3)=\\&1^3(1+\overbrace{1+\cdots+1}^{3k})+4^3(4+\overbrace{1+\cdots+1}^{3k-3})+\cdots+(3k-2)^3(3k-2+\overbrace{1+1+1}^{3})=\\&\left[(1^4+1^3+\cdots+1^3)+(4^4+4^3+\cdots+4^3)+\cdots+((3k-2)^4+(3k-2)^3\right](3),\end{aligned}$$

or equivalently,

$$(3n+1)\left(\sum_{i=1}^{n}(3i-2)^3\right)=\sum_{i=1}^{n}(3i-2)^4+\sum_{i=1}^{n}\left(\sum_{j=1}^{i}(3j-2)^3\right)(3).$$

And again one can prove by induction that

$$(3n+1)\left(\sum_{i=1}^{n}(3i-2)^{m-1}\right)=\sum_{i=1}^{n}(3i-2)^m+\sum_{i=1}^{n}\left(\sum_{j=1}^{i}(3j-2)^{m-1}\right)(3).$$

The above examples lead us to the following conjectures, as a generalization of Ibn al-Haytham formula for sums of powers of any arbitrary sequence and also it's reciprocal.

Conjecture 1. For any sequence of real numbers $(a_i)_{i\geq 1}$, we have the following identity

$$a_{n+1}\left(\sum_{k=1}^{n} a_k^{m-1}\right) = \sum_{k=1}^{n} a_k^m + \sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_i^{m-1}\right)(a_{k+1}-a_k).$$

Conjecture 2. For any sequence of real numbers $(a_i)_{i\geq 1}$, we have the following identity

$$a_{n+1}\left(\sum_{k=1}^{n} \frac{1}{a_k^m}\right) = \sum_{k=1}^{n} \frac{1}{a_k^{m-1}} + \sum_{k=1}^{n}\left(\sum_{i=1}^{k} \frac{1}{a_i^{m}}\right)(a_{k+1}-a_k).$$

Indeed, we prove something stronger which is one of the main results of this paper. The above conjectures can be seen as the intermediate corollaries of the following theorem.

Theorem 2.1 (The Generalized Ibn al-Haytham Formula)**.** *For any two sequences of real numbers $(a_i)_{i\geq 0}$ and $(b_i)_{i\geq 0}$, we have*

$$a_{n+1}\left(\sum_{k=m}^{n} b_k\right) = \sum_{k=m}^{n} b_k a_k + \sum_{k=m}^{n}\left(\sum_{j=m}^{k} b_j\right)(a_{k+1}-a_k), \quad (0\leq m\leq n). \tag{2.1}$$

Proof. We can rewrite the left-hand side of (2.1), as follows

$$\begin{aligned} a_{n+1}\left(\sum_{k=m}^{n} b_k\right) &= \sum_{k=m}^{n} b_k(a_{n+1}) = \sum_{k=m}^{n} b_k\left[(a_{n+1}-a_k)+a_k\right] \\ &= \sum_{k=m}^{n} b_k a_k + \sum_{k=m}^{n} b_k(a_{n+1}-a_k), \end{aligned}$$

therefore, we only need to prove the following identity

$$\sum_{k=m}^{n} b_k(a_{n+1}-a_k) = \sum_{k=m}^{n}\left(\sum_{j=m}^{k} b_j\right)(a_{k+1}-a_k).$$

By creative telescoping trick [8], we get

$$a_{n+1}-a_k = \sum_{j=k}^{n}(a_{j+1}-a_j).$$

Hence, we have

$$\sum_{k=m}^{n} b_k(a_{n+1}-a_k) = \sum_{k=m}^{n} b_k\left(\sum_{j=k}^{n}(a_{j+1}-a_j)\right).$$

Now, interchanging the summation order yields

$$\sum_{k=m}^{n} b_k\left(\sum_{j=k}^{n}(a_{j+1}-a_j)\right) = \sum_{k=m}^{n}\left(\sum_{j=m}^{k} b_j\right)(a_{k+1}-a_k).$$

Thus, finally, we obtain

$$a_{n+1}\left(\sum_{k=m}^{n} b_k\right) = \sum_{k=m}^{n} b_k a_k + \sum_{k=m}^{n}\left(\sum_{j=m}^{k} b_j\right)(a_{k+1}-a_k),$$

as required. □

Remark 2.2. Note that by choosing $b_k = \frac{1}{a_k^m}$ in Theorem 2.1, we get a proof of the second conjecture. Moreover, by putting $b_k = a_k^{m-1}$ in formula (2.1) we obtain a proof of the first conjecture.

As an immediate consequence of our main result, we have the following well-known *Abel's summation by parts lemma* [1].

Corollary 2.3 (Abel's Summation By Parts)**.** *For two arbitrary sequences* $(x_i)_{i\geq 0}$ *and* $(y_i)_{i\geq 0}$*, we have*

$$\sum_{k=m}^{n-1} (x_{k+1} - x_k)\, y_k = \sum_{k=m}^{n-1} x_{k+1}\, (y_k - y_{k+1}) + x_n y_n - x_m y_m, \quad (m < n).$$

Proof. By choosing $a_k = x_k$ and $b_k = y_{k+1} - y_k$ in Theorem 2.1, we get the desired result. □

Recall that the *discrete derivative* operator Δ on sequences is defined, as follows

$$\Delta x_k = x_{k+1} - x_k.$$

More generally, the k-th discrete derivative of a sequence can be defined recursively by

$$\Delta^0 x_k = x_k, \quad \Delta^k x_k = \Delta\left(\Delta^{k-1}\right) x_k \quad (k \geq 1).$$

We also have the following well-known *Newton's Identity*

$$\Delta^n x_k = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} x_{k+n-l}.$$

Corollary 2.4. *For two arbitrary sequences* $(x_i)_{i\geq 0}$ *and* $(y_i)_{i\geq 0}$*, we have*

$$\Delta^r x_{n+1} \left(\sum_{k=m}^{n} y_k \right) = \sum_{k=m}^{n} y_k \Delta^r x_k + \sum_{k=m}^{n} \left(\sum_{j=m}^{k} y_j \right) \Delta^{r+1} x_k. \tag{2.2}$$

Proof. By choosing $a_k = \Delta^r x_k$ and $b_k = y_k$ in Theorem 2.1 and also considering the recursive definition of the operator Δ^r, we get the desired result. □

We also have the following *discrete integral* formula [2].

Corollary 2.5 (Discrete Integral Formula)**.** *For any discrete function* ϕ*, we have*

$$\sum_{k=m}^{n} (\phi(k) - \phi(k-1))\,(n + k - 1) = \sum_{k=m}^{n} \phi(k) - (n - m + 1)\phi(m - 1). \tag{2.3}$$

Proof. Put $a_k = k$ and $b_k = \phi(k) - \phi(k-1)$ in formula (2.1). □

Example 2.6. By choosing $m = 0$ and $\phi(k) = \binom{n}{k}$ in identity (2.3) and considering the binomial formula $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and the convention $\binom{n}{k} = 0 \ (k < 0)$, we obtain

$$\sum_{k=0}^{n} k \binom{n}{k} = n 2^{n-1}.$$

Example 2.7. letting $x_k = k^2, y_k = \frac{1}{k^2}$ and $r = 1$ in formula (2.2), we get the following identity for the generalized harmonic numbers

$$\sum_{k=1}^{n} H_k^{(2)} = (n+1) H_n^{(2)} - H_n,$$

where the generalized harmonic numbers of the s-th order is defined as $H_n^{(s)} = \sum\limits_{k=1}^{n} \frac{1}{k^s}$.

There is also an interesting connection between the *generalized Ibn al-Haytham formula* and the following one - dimensional version of *Karaji's L-summing method* (see [4]).

Proposition 2.8 (One-Dimensional Karaji's L- Summing Formula)**.** *For any arbitrary discrete function* f*, we have*

$$\sum_{k=0}^{n} \left(\sum_{i=0}^{k} f(i) + k f(k) \right) = (n+1) \sum_{i=0}^{n} f(k).$$

Proof. The proof is straight forward by putting $a_k = k$ and $b_k = f(k)$, with $m = 0$, in Theorem 2.1. □

The *convolution product* of two sequences $\mathbf{a} = (a_n)_{n\geq 1}$ and $\mathbf{b} = (b_n)_{n\geq 1}$, denoted by $\mathbf{a} \star \mathbf{b}$, is defined by

$$\mathbf{a} \star \mathbf{b} = (c_n)_{n\geq 1}$$

where $c_n, (n \geq 1)$ is given, as follows

$$c_n = \sum_{k=1}^{n} a_{n+1-k} b_k.$$

Using similar argument as in the proof of our main theorem, we can obtain the following formula which we will call it the generalized Ibn al-Haytham formula of convolution-type:

Theorem 2.9 (The Generalized Ibn al-Haytham Formula of Convolution-Type)**.** *For any two sequences of real numbers* $(a_i)_{i\geq 0}$ *and* $(b_i)_{i\geq 0}$*, we have*

$$a_{n+1}\left(\sum_{k=m}^{n} b_k\right) = \sum_{k=m}^{n} b_k a_{n+m-k} + \sum_{k=m}^{n}\left(\sum_{j=m}^{k} b_{n+m-j}\right)(a_{k+1} - a_k), \quad (0 \leq m \leq n). \tag{2.4}$$

Example 2.10. By choosing $a_k = b_k = \frac{1}{k}$ in formula (2.4), we obtain

$$\sum_{k=1}^{n} \frac{1}{k(n+1-k)} - \sum_{k=1}^{n} \frac{1}{k(k+1)} H_{n-k} = H_n. \tag{2.5}$$

On the other hand, one can easily show that

$$\sum_{k=1}^{n} \frac{1}{k(n+1-k)} = \frac{2}{n+1} H_n. \tag{2.6}$$

Hence, from identities (2.5) and (2.6), we finally get

$$\sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k}\right) H_{n-k} = \frac{n-1}{n+1} H_n.$$

3. The Generalized Two-Dimensional Ibn al-Hyatham Formula

As we saw in the previous section, one of the consequences of our generalized formula was the one-dimensional Karaji's L-summing formula. Since Karaji's L-summing method is originally in two dimension, it is quite natural to seeking for a two-dimensional generalization of our main theorem.

Theorem 3.1 (The Generalized Two-Dimensional Ibn al-Haytham Formula)**.** *For any two-dimensional array* $A = (a_{i,j})_{i,j\geq 0}$*, we have*

$$\sum_{k=m}^{n} a_{n+1,k} = \sum_{k=m}^{n} a_{k,k} + \sum_{k=m}^{n}\sum_{j=m}^{k} \left(a_{k+1,j} - a_{k,j}\right), \qquad (0 \leq m \leq n). \tag{3.1}$$

Proof. We have

$$\sum_{k=m}^{n} a_{n+1,k} = \sum_{k=m}^{n} (a_{n+1,k} - a_{k,k}) + \sum_{k=m}^{n} a_{k,k}. \tag{3.2}$$

On the other hand, by creative telescoping, we get

$$a_{n+1,k} - a_{k,k} = \sum_{j=k}^{n} \left(a_{j+1,k} - a_{j,k}\right).$$

Hence, by the above identity and changing order of summations, we obtain

$$\sum_{k=m}^{n} (a_{n+1,k} - a_{k,k}) = \sum_{k=m}^{n}\sum_{j=k}^{n} \left(a_{j+1,k} - a_{j,k}\right) = \sum_{j=m}^{n}\sum_{k=m}^{j} \left(a_{j+1,k} - a_{j,k}\right). \tag{3.3}$$

Finally, considering the identities (3.2) and (3.3), we get the desired result. □

As an immediate consequence of the above theorem, we get the following two-dimensional Karaji's L-summing formula [4].

Corollary 3.2 (Two-Dimensional Karaji's L-Summing Formula). *For any two-dimensional array $A = (a_{i,j})_{i,j\geq 1}$, we have*

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{i,k}\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_{i,k} + \sum_{j=1}^{k} a_{k,j} - a_{k,k}\right).$$

Proof. For a two-dimensional array $F = (f_{i,j})_{i,j\geq 1}$, by formula (3.1) for $m = 1$, we have

$$\sum_{k=1}^{n} f_{n+1,k} = \sum_{k=1}^{n} f_{k,k} + \sum_{k=1}^{n}\sum_{j=1}^{k}\left(f_{k+1,j} - f_{k,j}\right). \tag{3.4}$$

Letting $f_{i,j} = \sum_{l=1}^{i-1} a_{l,j}$ in identity (3.4), we get the desired result. □

Corollary 3.3. *For any two variable discrete function $f(x, y)$, we have*

$$\sum_{k=1}^{n}\sum_{i=1}^{k} f(n, i) = \sum_{k=1}^{n}\sum_{i=1}^{k} f(k-1, i) + \sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j}\left(f(k, i) - f(k-1, i)\right). \tag{3.5}$$

Proof. Put $a_{i,j} = \sum_{l=1}^{j} f(i-1, l)$ and $m = 1$ in formula (3.1). □

Remark 3.4. Note that by choosing $a_{i,j} = a_i.b_j$ in (3.1) , we get the generalized Ibn al-Haytham formula (2.1).

Example 3.5. Letting $f(i, j) = \binom{i+j}{i}$ in formula (3.5), we obtain the following *binomial identity*

$$\sum_{k=1}^{n}\binom{2k}{k+1} + \sum_{k=0}^{n}\binom{2k+1}{k+1} = \binom{2(n+1)}{n} + \binom{n+1}{2}.$$

4. Integer Sequences Identities

In this section we are going to give an application of our main theorem for proving some interesting integer sequences identities.

Fibonacci numbers probably are one of the most famous sequences in all of mathematical sciences. They are recursively defined as

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 1). \tag{4.1}$$

Using the method of *creative telescoping*, we can obtain the following identity regarding the sum of the first n Fibonacci numbers

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

Next, we are going to obtain several interesting combinatorial identities regarding sums of Fibonacci numbers.

By choosing $a_k = F_{k+1}$ and $b_k = 1$ in formula (2.1), considering (4.1), we get

$$\sum_{k=1}^{n} kF_k = nF_{n+2} - F_{n+3} + 2.$$

If we put $a_k = F_{k+1}$ and $b_k = F_k$, then we obtain

$$\sum_{k=1}^{n} F_k F_{k+3} = F_{n+2}^2 - 1. \tag{4.2}$$

But if we choose $a_k = F_{k+1}$ and $b_k = F_{k+1}$, then we get

$$\sum_{k=1}^{n} F_{k+1}^2 + \sum_{k=1}^{n} F_k F_{k+3} = F_{n+2}F_{n+3} - 2. \tag{4.3}$$

Therefore, from identities (4.2) and (4.3), we conclude that

$$\sum_{k=1}^{n} F_{k+1}^2 = F_{n+2}F_{n+1} - 1.$$

Now, letting $a_k = F_{k+1}$ and $b_k = \left(\frac{1}{2}\right)^k$, yields

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k F_{k-1} = 1 - \left(\frac{1}{2}\right)^n F_{n+2}. \tag{4.4}$$

As a last applied example for Fibonacci numbers identities let $I = \sum_{k=1}^{n}(-1)^k F_{2k}$ and $J = \sum_{k=1}^{n}(-1)^k F_{2k+1}$. Next, if we first put $a_k = (-1)^k$ and $b_k = F_{2k} = F_{2k+1} - F_{2k-1}$ and then we choose $a_k = (-1)^k$ and $b_k = F_{2k+1} = F_{2k+2} - F_{2k}$ in formula (2.1), we obtain the following equations

$$\begin{aligned} I - 2J &= (-1)^{n+1} F_{2n+1} + 1, \\ J + 2I &= (-1)^{n+2} F_{2n+2} - 1. \end{aligned}$$

Now, solving the above of equations yields

$$\sum_{k=1}^{n}(-1)^k F_{2k} = \frac{1}{5}\left[(-1)^n \left(F_{2n+2} + F_{2n} - 1\right)\right], \tag{4.5}$$

$$\sum_{k=1}^{n}(-1)^k F_{2k+1} = \frac{1}{5}\left[(-1)^n \left(F_{2n+3} + F_{2n+1} - 3\right)\right]. \tag{4.6}$$

One of the most interesting sequences of numbers which appears particularly in the area of mathematical analysis of algorithms is the sequence of Harmonic numbers. Indeed, it is the sum of the reciprocal of the first n positive integers; that is, $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$.
Recall that the generalized Ibn al-Haytham formula (2.1) is the following identity:

$$a_{n+1}\left(\sum_{k=1}^{n} b_k\right) = \sum_{k=1}^{n} b_k a_k + \sum_{k=1}^{n}\left(\sum_{j=1}^{k} b_j\right)(a_{k+1} - a_k).$$

Now, by choosing $a_k = k$ and $b_k = \frac{1}{k}$ in the above formula, we get the following well known identity [3] for harmonic numbers

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n.$$

Next, we choose $a_k = \binom{k}{2}$ and $b_k = \frac{1}{k}$. Hence, we obtain $a_{k+1} - a_k = \binom{k+1}{2} - \binom{k}{2} = \binom{k}{1}$. Based on the two well-known binomial identities $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ and $\sum_{k=1}^{n}\binom{k+m}{m} = \binom{n+m+1}{m}$ [3], we obtain another beautiful harmonic numbers identity, namely

$$\sum_{k=1}^{n}\binom{k}{1}H_k = \binom{n+1}{2}H_n - \frac{1}{2}\binom{n}{2}.$$

More generally, it is not hard to see that by putting $a_k = \binom{k+r}{r+1}$ $(r \geq 0)$ and $b_k = \frac{1}{k}$ and applying similar argument one can obtain the following general identity for harmonic numbers

$$\sum_{k=1}^{n}\binom{k+r}{r}H_k = \binom{n+r+1}{r+1}H_n - \frac{1}{r+1}\left[\binom{n+r+1}{r+1} - 1\right].$$

As another interesting harmonic numbers identity, we will find an identity related to the following sum:

$$\sum_{k=1}^{n}\frac{H_k}{k(k+1)}.$$

Recall that for the complex number s, the zeta function is defined as $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$. Hence for the natural number m, we observe that $\zeta(m) = lim_{n\to\infty} H_n^{(m)}$.
Now, by choosing $a_k = -\frac{1}{k}$ and $b_k = \frac{1}{k}$ in the generalized Ibn al-Haytham formula (2.1), we get

$$-\frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{k} = -\sum_{k=1}^{n}\frac{1}{k}.\frac{1}{k} + \sum_{k=1}^{n} H_k \frac{1}{k(k+1)},$$

or equivalently the following identity:

$$\sum_{k=1}^{n}\frac{H_k}{k(k+1)} = H_n^{(2)} - \frac{1}{n+1}H_n.$$

As an immediate consequence of the above identity, one can obtain the following infinite sum connection between zeta function and harmonic numbers

$$\sum_{k=1}^{\infty}\frac{H_k}{k(k+1)} = \zeta(2).$$

In a similar way, one can prove the following identities

$$\sum_{k=1}^{\infty} H_k\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) = \zeta(3). \tag{4.7}$$

$$\sum_{k=1}^{\infty} H_k\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) = \zeta(4).$$

Using the following two simple tricks,

$$H_n = \sum_{k=1}^{\infty}\frac{1}{k} - \sum_{k=1}^{\infty}\frac{1}{k+n}, \qquad 2H_{n-1} = \sum_{k=1}^{n-1}\left(\frac{1}{k} + \frac{1}{n-k}\right),$$

one can get the following identity

$$\sum_{k=1}^{\infty}\frac{1}{k^2}H_k = 2\zeta(3). \tag{4.8}$$

Indeed, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2}H_n &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{k} - \frac{1}{k+n}\right) \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{nk(n+k)} \\
&= \sum_{n=k+1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{nk(n-k)} \\
&= \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{1}{nk(n-k)} \\
&= \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{1}{n^2}\left(\frac{1}{k} + \frac{1}{n-k}\right) \\
&= 2\sum_{n=1}^{\infty}\frac{1}{n^2}H_{n-1} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}H_n - 2\zeta(2).
\end{aligned}$$

Therefore, Considering the identities (4.7) and (4.8), we also obtain

$$\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}H_k = \zeta(3).$$

As our last example if we set $a_k = (-1)^k \binom{n-1}{k-1}$ and $b_k = \frac{1}{k}$, then by the convention $a_{n+1} = 0$ and $a_{k+1} - a_k = (-1)^{k+1}\binom{n}{k}$. Now considering the absorption identity $\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$ [3], we get

$$\begin{aligned} 0 &= \sum_{k=1}^{n}(-1)^k \frac{1}{k}\binom{n-1}{k-1} + \sum_{k=1}^{n} H_k(-1)^{k+1}\binom{n}{k} \\ &= \frac{1}{n}\sum_{k=1}^{n}(-1)^k \frac{n}{k}\binom{n-1}{k-1} + \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}H_k \\ &= \frac{1}{n}\sum_{k=1}^{n}(-1)^k\binom{n}{k} + \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}H_k \\ &= \frac{1}{n}\{(1+(-1))^n - 1\} + \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}H_k \end{aligned}$$

or equivalently

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}H_k = \frac{1}{n}.$$

For more *systematic* treatment of *combinatorial* summation using the generalized Ibn al-Haytahm formulas, see [7].

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