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On Domination Polynomials of Caterpillar Graphs

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Abstract. The domination polynomial of a simple graph G is calculated with $D(G,x) = \sum_{i=1}^{n} d(G,i)x^i$ such that $d(G,i)$ is the number of the dominating sets of G of size i. In this paper we study the domination polynomials of caterpillar graphs.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph whose vertex set V and the edge set E. For the open neighborhood of a vertex v in a graph G, the notation $N_G(v)$ is used as $N_G(v) = \{u|(u, v) \in E(G)\}$ and the closed neighborhood of v is used as $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set, if every vertex in G either is element of S or is adjacent to at least one vertex in S. The domination number of a graph G is denoted with $\gamma(G)$ and it is equal to the minimum cardinality of a dominating set in G. Fundamental notions of domination theory are outlined in the book [5].

Domination polynomials of cycles, cubic graphs and some other graphs are calculated by Alikhani and Peng [1–3]. The numbers of domination sets of cactus chains [7] and benzenoid chains [8] are determined. Similar to the domination polynomials, Hosoya polynomial [6], independence polynomial and matching polynomial [4] are well studied.

In this paper we calculate the domination polynomials of caterpillar graphs and for a special case we obtain the domination polynomials of comb graphs. A caterpillar graph is a tree consisted a path and vertices directly connected to this path. Comb graphs are a special form of caterpillar graphs which are obtained by adding a vertex to every vertex of a path with an edge.

Definition 1.1 ([3]). Let $\mathcal{D}(G, i)$ be the family of dominating sets of G with size i and let $d(G, i) = |\mathcal{D}(G, i)|$. Then the domination polynomial $D(G, x)$ of G is defined as

$$D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$$

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such that the $\gamma(G)$ is the domination number [3].

Example 1.2. We find the domination polynomial of P_4 path with 4 vertices respectively $\{v_1, v_2, v_3, v_4\}$. The domination number $\gamma(P_4)$ is equal to 2.

Dominating sets of cardinality 2 are $\mathcal{D}(G, 2) = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}\}$,

Dominating sets of cardinality 3 are $\mathcal{D}(G, 3) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}$, and the domination sets of cardinality 4 is $\mathcal{D}(G, 4) = \{v_1, v_2, v_3, v_4\}$. So that the domination polynomial of P_4 is $D(P_4, x) = x^4 + 4x^3 + 4x^2$.

Theorem 1.3 ([3]). *If a graph G consists of m components $G_1, G_2, \ldots\ldots, G_m$ then*

$$D(G, x) = D(G_1, x) \cdots D(G_m, x).$$

Lemma 1.4 ([3]). *Domination polynomial the star graph $K_{1,n}$ is*

$$D(K_{1,n}, x) = x^n + (1 + x)^n.$$

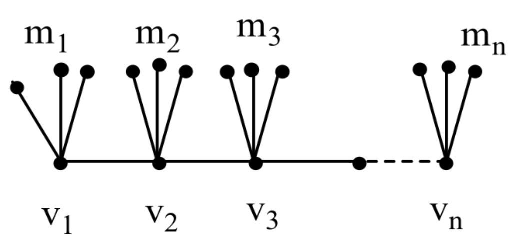

Figure 1. Caterpillar graph $C_n(m_1 + 1, m_2 + 1, \ldots, m_n + 1)$

Definition 1.5. Let G a caterpillar graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ of a path and the number of the pendant vertices are denoted with $m_1, m_2, \ldots, m_n$ to the $v_1, v_2, \ldots, v_n$ respectively. G is denoted with $G = C_n(m_1 + 1, m_2 + 1, \ldots, m_n + 1)$ as in Figure 1.

Lemma 1.6.

$$\gamma((C_n(m_1 + 1, m_2 + 1, \ldots, m_n + 1)) = n.$$

Proof. v_1 dominates m_1 pendant vertices which are connected to the own. Like this minimum cardinality dominating set is $\{v_1, v_2, \ldots, v_n\}$. Therefore $\gamma((C_n(m_1 + 1, m_2 + 1, \ldots, m_n + 1)) = n$ □

2. Results

In first we find the domination polynomial of bistar graph $C_2(m_1 + 1, m_2 + 1)$ depicted in Figure 2.

Theorem 2.1. *The domination polynomial of bistar graph $G = C_2(m_1 + 1, m_2 + 1)$ is*

$$D(C_2, x) = x^2(1 + x)^{m_1+m_2} + x(x^{m_1}(1 + x)^{m_2} + x^{m_2}(1 + x)^{m_1}) + x^{m_1+m_2}.$$

Proof. We investigate three cases. First case v_1 and v_2 are contained by domination sets, second case one of v_1 and v_2 is contained by domination sets and third case non of them is is contained by dominating set.

First case: Let v_1 and v_2 are contained in domination set.

$\mathcal{D}(G, 2) = \{v_1, v_2\}$, $d(G, 2) = 1$ and the domination polynomial term x^2,

$\mathcal{D}(G, 3) = \{v_1, v_2, v\}$ where v is chosen from $m_1 + m_2$ pendant vertices and $d(G, 3) = \binom{m_1+m_2}{1}$ and the domination polynomial term $\binom{m_1+m_2}{1}x^3$. If we continue like this

$\mathcal{D}(G, m_1 + m_2 + 2) = \{v_1, v_2, m_1 + m_2 \textit{ pendant vertices}\}$, $d(G, m_1 + m_2 + 2) = \binom{m_1+m_2}{m_1+m_2}$, and the domination polynomial term $x^{m_1+m_2+2}$. Thus for the first case,

$$D(C_2, x) = x^2 + \binom{m_1 + m_2}{1}x^3 + \cdots + \binom{m_1 + m_2}{m_1 + m_2}x^{m_1+m_2+2}$$

$$D(C_2, x) = x^2[1 + \binom{m_1 + m_2}{1}x + \cdots + \binom{m_1 + m_2}{m_1 + m_2}x^{m_1+m_2}]$$

$$D(C_2, x) = x^2(1 + x)^{m_1+m_2}$$

Second case: One of v_1 and v_2 is contained in the dominating sets. In the beginning we see v_1. v_1 and all of the m_2 pendant vertices dominate the graph. For this situation $d(G, m_2 + 1) = 1$ and the domination polynomial term x^{m_2+1}. The next dominating set family $\mathcal{D}(G, m_2 + 2) = \{v_1, m_2 \textit{ pendant vertices}, v\}$ such that v is chosen from m_1 pendant vertices. Thus $d(G, m_2 + 2) = \binom{m_1}{1}$ and the domination polynomial term $\binom{m_1}{1}x^{m_2+2}$. In this way domination polynomial is

$$\begin{aligned} & x^{m_2+1} + \binom{m_1}{1}x^{m_2+2} + \cdots + \binom{m_1}{m_1}x^{m_1+m_2+1} \\ &= x^{m_2+1}\left[1 + \binom{m_1}{1}x^1 + \cdots + \binom{m_1}{m_1}x^{m_1}\right] \\ &= x^{m_2+1}(1 + x)^{m_1}. \end{aligned}$$

Now it is supposed only v_2 is contained in the dominating set. Similar to the previous situation minimum cardinality dominating set is obtained by combining v_2 and all of the m_1 pendant vertices. Domination polynomial of this situation is

$$= x^{m_1+1}(1 + x)^{m_2}.$$

Main domination polynomial of second case is

$$\begin{aligned} & x^{m_1+1}(1 + x)^{m_2} + x^{m_2+1}(1 + x)^{m_1} \\ &= x(x^{m_1}(1 + x)^{m_2} + x^{m_2}(1 + x)^{m_1}) \end{aligned}$$

Third case: Non of v_1 and v_2 is contained by dominating set. For this case there is only one dominating set consisted from $m_1 + m_2$ pendant vertices. The domination polynomial of this case is $x^{m_1+m_2}$. This complete the proof. □

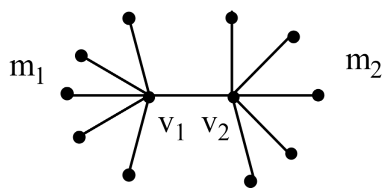

FIGURE 2. Bistar graph

We prove the domination polynomial of caterpillar graph $C_n(m_1 + 1, m_2 + 1, \ldots, m_n + 1)$ by a similar way to the obtaining domination polynomial of bistar graph. In the next theorem we denoted the summation of pendant vertices with m such that $m = m_1 + m_2 + \cdots + m_n$.

Theorem 2.2. *The domination polynomial of caterpillar graph $G = C_n(m_1 + 1, m_2 + 1, \ldots, m_n + 1)$ is*

$$\begin{aligned} D(G, x) &= x^n(1 + x)^m + x^{n-1}\Big(\sum_{i=1}^{n} x^{m_i}(1 + x)^{m-m_i}\Big) \\ &\quad + x^{n-2}\Big(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} x^{m_i+m_j}(1 + x)^{m-(m_i+m_j)}\Big) + \cdots + x^m \end{aligned}$$

Proof. The minimum cardinality dominating set is $\{v_1, v_2, \ldots, v_n\}$ of the $C_n(m_1+1, m_2+1, \ldots, m_n+1)$ caterpillar graph (1.6 Lemma). There are $n + 1$ cases change to the whether a vertex is contained in a dominating set or not.

Case 1: The set $\{v_1, v_2, \ldots, v_n\}$ dominate the caterpillar so $d(G, n) = 1$ and the domination polynomial term x^n. The next dominating set family is $\mathcal{D}(G, n + 1) = \{v_1, v_2, \ldots, v_n, v\}$ such that v is chosen from m pendant vertices and $d(G, n + 1) = \binom{m}{1}$. Thus domination polynomial term $\binom{m}{1}x^{n+1}$. By continuing this way the domination polynomial of the first case is obtained as,

$$x^n + \binom{m}{1}x^{n+1} + \cdots + \binom{m}{m}x^{m+n}$$

$$= x^n[1 + \binom{m}{1}x^1 + \cdots + \binom{m}{m}x^m]$$
$$= x^n(1+x)^m.$$

Case 2: One of the $\{v_1, v_2, \ldots, v_n\}$ is not contained in the dominating set. Therefore, there are n distinct situations. It is supposed v_1 is not contained in dominating sets. This is provided by combining $\{v_2, \ldots, v_n\}$ with m_1 pendant vertices which has minimum cardinality $d(G, m_1 + n - 1) = 1$. Thus domination polynomial term x^{m_1+n-1}.

The next domination set family $\mathcal{D}(G, m_1 + n) = \{v_2, \ldots, v_n, m_1 \textit{ pendant vertices}, v\}$ where v is chosen from pendant vertices except m_1. Thus the domination polynomial term $\binom{m-m_1}{1}x^{m_1+n}$. By this way domination polynomial for v_1 is

$$x^{m_1+n-1} + \binom{m-m_1}{1}x^{m_1+n} + \cdots + \binom{m-m_1}{m-m_1}x^{m+n-1}$$
$$= x^{m_1+n-1}[1 + \binom{m-m_1}{1}x^1 + \cdots + \binom{m-m_1}{m-m_1}x^{m-m_1}]$$
$$= x^{m_1+n-1}(1+x)^{m-m_1}.$$

Main domination polynomial of the second case is obtained by investigating situations of each n vertices $\{v_1, v_2, \ldots, v_n\}$. Therefore main domination polynomial of the second case is

$$= x^{n-1}\Big(\sum_{i=1}^{n} x^{m_i}(1+x)^{m-m_i}\Big).$$

Last case: It is supposed non of the $\{v_1, v_2, \ldots, v_n\}$ is contained in the dominating set. This is provided by combining the m pendant vertices. $d(G, m) = 1$ and the domination polynomial term of this case is x^m. So that the domination polynomial of caterpillar is

$$\begin{aligned} D(G,x) \quad = \quad & x^n(1+x)^m + x^{n-1}\Big(\sum_{i=1}^{n} x^{m_i}(1+x)^{m-m_i}\Big) \\ & + x^{n-2}\Big(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} x^{m_i+m_j}(1+x)^{m-(m_i+m_j)}\Big) + \cdots + x^m. \quad \square \end{aligned}$$

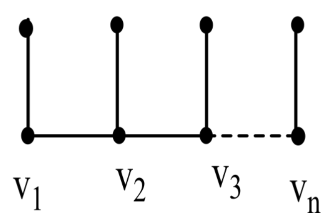

FIGURE 3. Comb graph $C_n(2, 2, \ldots, 2)$

On the comb graph $m_1 = m_2 = \cdots = m_n = 1$ and $m = \sum_{i=1}^{n} m_i = n$. We obtain domination polynomial of comb graphs in the next theorem.

Theorem 2.3. *The domination polynomial of comb graphs which is depicted in Figure 3 is*

$$D(G,x) = x^n(x+2)^n.$$

Proof. We use Theorem 2.2 for $m_1 = m_2 = \cdots = m_n = 1$ and $m = n$.

$$D(G,x) = x^n(1+x)^n + x^{n-1}\Big(\binom{n}{1}x^1(1+x)^{n-1}\Big) + x^{n-2}\Big(\binom{n}{2}x^2(1+x)^{n-2}\Big) + \cdots + x^n$$
$$D(G,x) = x^n\Big((1+x)^n + \binom{n}{1}(1+x)^{n-1} + \binom{n}{2}(1+x)^{n-2} + \cdots + 1\Big)$$

$$D(G,x) = x^n((1+x+1)^n)$$
$$D(G,x) = x^n(x+2)^n. \qquad \square$$

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