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Trees of The Normalizer of the Modular Group in The Picard Group

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Abstract. In this study, we investigate trees arising from the imprimitive action of the normalizer of Modular group in the Picard group on extended rational numbers. We determine the farthest vertex from a given vertex in hyperbolic paths of minimal lengths. We also include some results of the suborbital graph $\bar{F}_{u,N}$ related to a continued fraction representation of a rational number.

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1. Introduction

It is well known, since Cantor's first works on the theory of cardinality, that the rationals are countable. However, it is not so simple to give an explicit enumeration of all of them. There are some methods that illustrate the countability of the positive rational numbers and related sets. Techniques include radix representations, Gödel numbering, the fundamental theorem of arithmetic, continued fractions, Egyptian fractions, and the sequence of ratios of successive hyperbinary representation numbers [20]. Among these methods, we mention about Farey fractions and Calkin-Wilf tree.

Farey fractions which give an useful classification of the rational numbers is one of the oldest, while the Calkin-Wilf tree is newest. In [10], Jones, Singerman, Wicks studied the the suborbital graphs of the modular group and showed that the most basic one turns out to be well-known Farey graph related to Farey fractions. The Calkin-Wilf tree is an infinite binary tree whose vertex set is the set of positive reduced rational numbers. In this tree, every positive reduced rational number a/b is the tail of the two edges. The heads of these edges are the positive rational numbers $a/(a+b)$ and $(a+b)/b$. Then Nathanson describes a generalizaton of the Calkin-Wilf tree to a forest of rooted infinite binary trees of rational functions of the form $(az+b)/(cz+d)$ [19]. Hence, the close relationship between enumeration of the rational numbers and the modular group has been presented in both of studies [10, 19]. Inspiring this relation, we examine the trees of the normalizer of the modular group in the present study.

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We start to remind the groups we will work and some basic properties. Let $\mathbb{H}$ denote the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The Picard group is denoted by $\mathbb{P}$ and contains all linear fractional transformations

$$T : z \to \frac{az+b}{cz+d}\text{ , where } a, b, c \text{ and } d \in \mathbb{Z}[i] \text{ and } ad - bc = 1.$$

$\mathbb{P} = PSL(2, \mathbb{Z}[i])$ is an important subgroup of $PSL(2, \mathbb{C})$. Now $PSL(2, \mathbb{R})$ is the group of all conformal homeomorphisms of $\mathbb{H}$. A Fuchsian group is a discrete subgroup of $PSL(2, \mathbb{R})$.The modular group $\Gamma = PSL(2, \mathbb{Z})$ is perhaps the most important and certainly best-known Fuchsian group. It is known that every finitely generated Fuchsian groups has a unique presentation with generators and relations [9]. The normalizer of the modular group Γ in the Picard group $\mathbb{P}$ will be denoted by $N_{\mathbb{P}}(\Gamma)$ and in Lemma 3.2 we give a characterization of $N_{\mathbb{P}}(\Gamma)$. The presentation of $N_{\mathbb{P}}(\Gamma)$ is [23]

$$N_{\mathbb{P}}(\Gamma) = \langle u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, y = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, r = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; u^2 = y^3 = r^2 = (ry)^2 = (ru)^2 = 1\rangle.$$

2. Continued Fractions

We recall definitions, notations, and some preliminary results of continued fractions for the sake of completeness. In general, a (simple) continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

where the letters $a_0, a_1, a_2, \ldots$ denote independent variables, and may be interpreted as one wants (e.g. real or complex numbers, functions, etc.). In this study, the letters $a_1, a_2, \ldots$ denote positive integers. The letter a_0 denotes an integer.

Notation 2.1. We write

$$[a_0; a_1, a_2, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{}{+\cfrac{1}{a_n}}}}$$

if the number of terms is finite, and

$$[a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

for an infinite number of terms.

We give some well-known facts from [6] as following theorems without proofs.

Theorem 2.2. *Any rational number can be represented as a finite continued fraction.*

3. Suborbital Graphs of the Normalizer

The reader is refereed to [1–3, 8, 11–17] for some relevant previous work on suborbital graphs. The general descriptions can be found in these papers. Let (G, Δ) be a transitive permutation group, consisting of a group G acting on a set Δ transitively. An equivalence relation $\approx$ on Δ is called *G-invariant* if, whenever $\alpha, \beta \in \Delta$ satisfy $\alpha \approx \beta$, then $g(\alpha) \approx g(\beta)$ for all $g \in G$.

The equivalence classes are called blocks, and the block containing α is denoted by $[\alpha]$.

We call (G, Δ) *imprimitive* if Δ admits some G-invariant equivalence relation different from

i. the identity relation, $\alpha \approx \beta$ if and only if $\alpha = \beta$;
ii. the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Delta$.

Otherwise (G, Δ) is called *primitive*. These two relations are supposed to be trivial relations. Clearly, a primitive group must be transitive, for if not the orbits would form a system of blocks. The converse is false, but we have the following useful result.

Lemma 3.1 ([4]). *Let (G, Δ) be a transitive permutation group. (G, Δ) is primitive if and only if G_α, the stabilizer of $\alpha \in \Delta$, is a maximal subgroup of G for each $\alpha \in \Delta$.*

From the above lemma we see that whenever, for some α, $G_\alpha \not\leq H \lneq G$, then Ω admits some G-invariant equivalence relation other than the trivial cases. Because of the transitivity, every element of Ω has the form $g(\alpha)$ for some $g \in G$. Thus one of the non-trivial G-invariant equivalence relation on Ω is given as follows:

$$g(\alpha) \approx g'(\alpha) \text{ if and only if } g' \in gH.$$

Lemma 3.2 ([23]). *The elements of $N_{\mathbb{P}}(\Gamma)$ consist of the mappings of the form :*

$$T(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{Z} \text{ with } ad-bc = \pm 1.$$

If we set $G = N_{\mathbb{P}}(\Gamma)$, $\Delta = \hat{\mathbb{Q}}$, $H = \bar{\Gamma}_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N_{\mathbb{P}}(\Gamma) | \, c \equiv 0 \, mod(N) \right\}$, and $G_\alpha = N_{\mathbb{P}}(\Gamma)_\infty$, then we clearly see that $N_{\mathbb{P}}(\Gamma)_\infty \lneq \bar{\Gamma}_0(N) \lneq N_{\mathbb{P}}(\Gamma)$.

We define the following $N_{\mathbb{P}}(\Gamma)$ invariant equivalence relation " $\underset{N}{\approx}$ " on $\hat{\mathbb{Q}}$. Since $N_{\mathbb{P}}(\Gamma)$ acts transitively on $\hat{\mathbb{Q}}$, every element of $\hat{\mathbb{Q}}$ has the form $g(\infty)$ for some $g \in N_{\mathbb{P}}(\Gamma)$. So, it is easily seen that,

$$g(\infty) \underset{N}{\approx} g'(\infty) \Longleftrightarrow g' \in gN_{\mathbb{P}}(\Gamma)$$

gives a $N_{\mathbb{P}}(\Gamma)-$invariant imprimitive equivalence relation.

Theorem 3.3 ([23] Block condition). *Let $v = \frac{r}{s}$, $w = \frac{x}{y} \in \hat{\mathbb{Q}}$. Then $v \underset{N}{\approx} w$ if and only if $ry - sx \equiv 0(modN)$ or $sx - ry \equiv 0(modN)$.*

Let (G, Δ) be a transitive permutation group. Then G acts on $\Delta \times \Delta$ by $g(\alpha, \beta) = (g(\alpha), g(\beta))$, $(g \in G, \alpha, \beta \in \Delta)$. The orbits of this action are called *suborbitals* of G .

In this study, G is $N_{\mathbb{P}}(\Gamma)$ and Δ is $\hat{\mathbb{Q}}$. We now consider the suborbital graphs for the action $N_{\mathbb{P}}(\Gamma)$ on $\hat{\mathbb{Q}}$. Since $N_{\mathbb{P}}(\Gamma)$ acts transitively on $\hat{\mathbb{Q}}$, each suborbital contains a pair $(\infty, u/N)$ for some $u/N \in \hat{\mathbb{Q}}$ such that $(u, N) = 1$. We denote this suborbital by $\bar{O}(u, N)$ and corresponding suborbital graph $\bar{G}(u, N)$ by $\bar{G}_{u,N}$.

Theorem 3.4 ([23] Edge condition). *$r/s \longrightarrow x/y$ is an edge in $\bar{G}_{u,N}$ if and only if*

(i) *$x \equiv ur\,(modN)$, $y \equiv us\,(modN)$, $ry - sx = N$ or*
(ii) *$x \equiv -ur\,(modN)$, $y \equiv -us\,(modN)$, $ry - sx = -N$ or*
(iii) *$x \equiv ur\,(modN)$, $y \equiv us\,(modN)$, $ry - sx = -N$ or*
(iv) *$x \equiv -ur\,(modN)$, $y \equiv -us\,(modN)$, $ry - sx = N$.*

Since the action $N_{\mathbb{P}}(\Gamma)$ on $\hat{\mathbb{Q}}$ is transitive, $N_{\mathbb{P}}(\Gamma)$ permutes the blocks transitively; so the subgraphs are all isomorphic. Hence it is sufficient to study only one block. On the other hand, it is clear that each non-trivial suborbital graph contains a pair $(\infty, u/n)$ for some $u/n \in \hat{\mathbb{Q}}$ where $(u, n) = 1$. Therefore, we study on the following case: We denote by $\bar{F}_{u,N}$ the subgraph of $\bar{G}_{u,N}$ such that its vertices are in the block $[\infty] = \left\{ \frac{x}{y} \in \hat{\mathbb{Q}} \,|\, y \equiv 0\,(modN) \right\}$.

Theorem 3.5 ([23]). *$r/s \longrightarrow x/y$ is an edge in $\bar{F}_{u,N}$ if and only if*

(i) *$x \equiv ur\,(modN)$, $ry - sx = N$ or*
(ii) *$x \equiv -ur\,(modN)$, $ry - sx = -N$ or*
(iii) *$x \equiv ur\,(modN)$, $ry - sx = -N$ or*
(iv) *$x \equiv -ur\,(modN)$, $ry - sx = N$.*

Lemma 3.6. *If $(u, N) = 1$ then there exists an integer k such that $u^2 + ku + 1 \equiv 0 \mod N$.*

Proof. Let $(u, N) = 1$, then there exists integers x, y such that $ux - yN = 1$ giving $ux \equiv 1 \mod N$. Thus $ux(-u^2 - 1) \equiv -u^2 - 1 \mod N$.Putting $k = x(-u^2 - 1)$ gives $u^2 + ku + 1 \equiv 0 \mod N$. □

4. Main Calculations

Theorem 4.1. *Let u, n be relatively prime positive integers. For $\bar{F}_{u,N}$, the following statements hold:*

(i) *The farthest vertex which can be joined with $\frac{u}{N}$ is $\dfrac{u + \frac{1}{k}}{N}$, where k is the unique integer such that $1 \leq k \leq N$ and $u^2 + ku + 1 \equiv 0 \pmod N$. The nearest vertex does not exist.*

(ii) *The farthest vertex which can be joined with* $\frac{u+\frac{1}{k}}{N}$ *is* $\dfrac{u+\frac{1}{k-\frac{1}{k}}}{N}$, *where* k *is the unique integer such that* $1 \le k \le N$ *and* $u^2+ku+1 \equiv 0 \pmod{N}$. *The nearest vertex does not exist.*

Proof. We first prove (i). The existence of an integer k such that $u^2+ku+1 \equiv 0 \pmod{N}$ is due to Lemma 3.6. Now we can assume that $1 \le k \le N$. To see this, if $k > N$ we choose k_1 such that $k \equiv k_1 \pmod{N}$. We have $ku+1 \equiv k_1u+1 \pmod{N}$, this gives

$$u^2+k_1u+1 \equiv u^2+ku+1 \equiv 0 \pmod{N}.$$

Now we show the uniqueness of k. Let m be another integer such that $1 \le m \le N$ and $u^2+mu+1 \equiv 0 \pmod{N}$. Hence $mu \equiv -u^2-1 \equiv ku \pmod{N}$, so we have $(k-m)u \equiv 0 \pmod{N}$. Because $(u,N)=1$, $k-m \equiv 0 \pmod{N}$. Thus $k=m$ since $|k-m| < N$. Now suppose there exists an edge $\frac{u}{N} \to \frac{x}{y}$ in $\bar{F}_{u,N}$ and $\frac{u}{N} < \frac{x}{y}$. We can write $\frac{x}{y}$ in the form

$$\frac{x}{y} = \frac{u}{N} + \frac{Nx}{Ny} - \frac{uy}{Ny} = \frac{u+\frac{Nx-uy}{y}}{N}.$$

With this and the fact that $uy < Nx$, we can replace $\frac{x}{y}$ with $\frac{u+\frac{t}{s}}{N}$, where $\frac{t}{s}$ is in $\mathbb{Q}^+$. So we have

$$\frac{u}{N} \to \frac{u+\frac{t}{s}}{N} = \frac{su+t}{sN}.$$

Theorem 3.5 gives numerical informations when this edge exists. From this point onward, we aim to analyze each of the cases of this theorem.

Case 1: In this case we have $su+t \equiv u^2 \pmod{N}$ and $u(sN)-N(su+t)=N$, which implies $t=-1$. Therefore $su-1 \equiv u^2 \pmod{N}$. Since $u^2+ku+1 \equiv 0 \pmod{N}$, we have $su-1 \equiv -ku-1 \pmod{N}$, that is, $su \equiv -ku \pmod{N}$. Since $(u,N)=1$, we have $s \equiv -k \pmod{N}$. In other words, $s=-k-Nz$ for some $z \in \mathbb{N} \cup \{0\}$. Thus $\frac{t}{s} = \frac{1}{Nz+k}$. Next, we find the largest value of $\frac{t}{s}$ by defining a function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$,

$$f(z) = \frac{u+\frac{1}{Nz+k}}{N}.$$

The derivative of f is $f'(z) = \frac{-1}{(Nz+k)^2} < 0$, which is negative for every non-negative z. This implies that the maximum occurs at $z=0$, that is,

$$\frac{u+\frac{1}{k}}{N} = \frac{uk+1}{kN}. \tag{4.1}$$

It remains to show that $\frac{u+\frac{1}{k}}{N} = \frac{uk+1}{kN}$ is a vertex in $\bar{F}_{u,N}$. To see this, we show that it is an irreducible fraction. It is true that $(ku+1,k) = (ku+1-ku,k) = (1,k) = 1$ and since $u^2+ku+1 = Ny$ for some $y \in \mathbb{Z}$, we have $(ku+1,N) = (ku+1-Ny,N) = (-u^2,N) = 1$. Thus, $(ku+1,kN)=1$. We conclude that $\dfrac{u+\frac{1}{k}}{N}$ is vertex in $\bar{F}_{u,N}$ and is the farthest one being joined with $\frac{u}{N}$. We also see that

$$\lim_{z\to\infty} \frac{u+\frac{1}{Nz+k}}{N} = \frac{u}{N}. \tag{4.2}$$

This implies that there is no such nearest point being joined with the vertex $\frac{u}{N}$.

Case 2: In this case, we obtain that $su+t \equiv -u^2 \pmod{N}$ and $u(sN)-N(us+t)=-N$, which implies $t=1$. Thus, $su+1 \equiv -u^2 \pmod{N}$ and we know $u^2+ku+1 \equiv 0 \pmod{N}$. This implies that $su+1 \equiv ku+1 \pmod{N}$, that is, $su \equiv ku \pmod{N}$. The fact $(u,N)=1$ implies that $s \equiv k \pmod{N}$. Therefore, $s = Nz+k$ for some $z \in \mathbb{N} \cup \{0\}$ and

$$\frac{t}{s} = \frac{1}{Nz+k}.$$

This case is done by using a similar argument to that of the first case.

Case 3: We have $su+t \equiv u^2 \pmod{N}$ and $u(sN)-N(su+t)=-N$, which implies $t=1$. Then $su+1 \equiv u^2 \pmod{N}$. Since $u^2+ku+1 \equiv 0 \pmod{N}$, $su+1 \equiv -ku-1 \pmod{N}$. Hence $su+1+ku+1 = Nz$ for some $z\mathbb{N} \cup \{0\}$, that is, $s = \frac{Nz-ku-2}{u}$. Thus

$$\frac{t}{s} = \frac{u}{Nz-ku-2}.$$

We again find the greatest value of $\frac{t}{s}$ by defining a function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$,

$$f(z) = \frac{u + \frac{u}{Nz-ku-2}}{N}.$$

And the derivative of f is $f'(z) = \dfrac{-u}{(Nz-ku-2)^2} < 0$, so the greatest value of the function f is taken at $z = 0$ and this value is

$$\frac{u - \frac{1}{k+\frac{2}{u}}}{N}.$$

But $\dfrac{u - \frac{1}{k+\frac{2}{u}}}{N}$ is nearer to $\frac{u}{N}$ than $\dfrac{u + \frac{1}{k}}{N}$. Therefore the farthest vertex which can be joined with $\frac{u}{N}$ is $\dfrac{u + \frac{1}{k}}{N}$. Now we know that

$$\lim_{z\to\infty} \frac{u + \frac{u}{Nz-ku-2}}{N} = \frac{u}{N}.$$

Hence the nearest vertex does not exist.

Case 4: We obtain that $su + t \equiv -u^2 \pmod N$ and $u(sN) - N(su + t) = N$, which implies $t = -1$. Thus $su - 1 \equiv -u^2 \pmod N$. Because $u^2 + ku + 1 \equiv 0 \pmod N$, $su - 1 \equiv ku + 1 \pmod N$, that is, $-su + 1 \equiv -ku - 1 \pmod N$. So $-su + 1 + ku + 1 = Nz$ for some $z \in \mathbb{N} \cup \{0\}$, then $s = \frac{-(Nz-ku-2)}{u}$. Therefore

$$\frac{t}{s} = \frac{u}{Nz - ku - 2}.$$

The remaining proof is similar to the third case.

Now we prove the (*ii*). By the above proof of existence, let k be an integer such that $1 \le k \le N$ and $u^2 + ku + 1 \equiv 0 \pmod N$. We have shown that k is unique. From the proof of (*i*), we can suppose that

$$\frac{u + \frac{1}{k}}{N} < \frac{u + \frac{t}{s}}{N}$$

and

$$\frac{ku+1}{kN} = \frac{u + \frac{1}{k}}{N} \to \frac{u + \frac{t}{s}}{N} = \frac{us+t}{sN},$$

where $\frac{t}{s}$ is in $\mathbb{Q}^+$. We start working on each case as in the proof of (*i*).

Case 1: In this case, we have $us + t \equiv u^2k + u \pmod N$ and $Ns(ku + 1) - kN(su + t) = N$, which implies $s = kt + 1$. Thus $u + kut + t \equiv u^2k + u \pmod N$ and we get $t(ku + 1) \equiv u^2k \pmod N$. We observe that $-u^2t \equiv u^2k \pmod N$. Moreover, $-t \equiv k \pmod N$ since $(u, N) = 1$. So $t = -Nz - k$ for some $t \in \mathbb{N} \cup \{0\}$, that is, $s = 1 - k(Nz + k)$. Therefore

$$\frac{t}{s} = \frac{Nz + k}{k(Nz + k) - 1}.$$

Next, we find the largest value of $\frac{t}{s}$ by defining a function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ by

$$f(z) = \frac{u + \frac{Nz+k}{k(Nz+k)-1}}{N}.$$

Since the derivative of f is $f'(z) = \dfrac{-1}{(k(Nz+k)-1)^2} < 0$, then f has a maximum at $z = 0$ and this value is

$$\frac{u + \frac{k}{k^2-1}}{N} = \frac{(k^2-1)u + k}{(k^2-1)N}.$$

Now we will show $((k^2 - 1)u + k, (k^2 - 1)N) = 1$. Suppose that $((k^2 - 1)u + k, k^2 - 1) = a$, then a divides $k^2 - 1$, which means a divides $(k^2 - 1)u$. Since a divides $(k^2 - 1)u + k$, a divides k. Because a divides $k^2 - 1$, a divides -1 and hence $a = \pm 1$.

Next we assume that $((k^2 - 1)u + k, N) = b$, then b divides $(k^2 - 1)u + k$. So we have $k(ku + 1) - u = (k^2 - 1)u + k \equiv 0 \pmod b$. Since $u^2 + ku + 1 \equiv 0 \pmod N$ and b divides N, $u^2 + ku + 1 \equiv 0 \pmod b$. As we have shown $k(ku + 1) - u \equiv 0 \pmod b$ and $-u^2 \equiv ku + 1 \pmod b$, then $k(-u^2) - u \equiv 0 \pmod b$.

Suppose b does not divide u. As $(u,N)=1$ and b divides N, so $(u,b)=1$. Moreover, $-ku-1\equiv 0 \pmod{b}$. Since we have $u^2\equiv -ku-1 \pmod{b}$, then b divides u^2, which gives a contradiction. Hence b divides u. Thus $b=1$ since $(u,N)=1$. And b divides N. Therefore $((k^2-1)u+k,(k^2-1)N)=1$, that is,

$$\frac{u+\frac{k}{k^2-1}}{N}=\frac{u+\frac{1}{k-\frac{1}{k}}}{N}$$

is a vertex in $\bar{F}_{u,N}$ and is also the farthest vertex which can be joined with $\dfrac{u+\frac{1}{k}}{N}$. Since

$$\lim_{z\to\infty}\frac{u+\frac{Nz+k}{k(Nz+k)-1}}{N}=\frac{u+\frac{1}{k}}{N},$$

there is no such a nearest vertex.

Case 2: We obtain that $su+t\equiv -u^2k-u \pmod{N}$ and $Ns(ku+1)-kN(su+t)=-N$, which implies $s=kt-1$. Thus $kut-u+t\equiv -u^2k-u \pmod{N}$. We observe that $t(ku+1)\equiv -u^k \pmod{N}$. Since $u^2+ku+1\equiv 0 \pmod{N}$, $u^2t\equiv u^2k \pmod{N}$. As $(u,N)=1$, $t\equiv k \pmod{N}$, that is, $t=Nz+k$ for some $z\in\mathbb{N}\cup\{0\}$. So $s=k(Nz+k)-1$. Hence

$$\frac{t}{s}=\frac{Nz+k}{k(Nz+k)-1}$$

and the remaining proof is the same as the above proof.

Case 3: We have $su+t\equiv u^2k+u \pmod{N}$ and $Ns(ku+1)-kN(su+t)=-N$, which implies $s=kt-1$. So we have $kut-u+t\equiv u^2k+u \pmod{N}$, that is, $t(ku+1)-ku^2\equiv 2u \pmod{N}$. Since $u^2+ku+1\equiv 0 \pmod{N}$, $-tu^2-ku^2\equiv 2u \pmod{N}$. Moreover, $-tu-ku\equiv 2 \pmod{N}$ by $(u,N)=1$. Then $tu+ku\equiv -2 \pmod{N}$, that is, $t=\frac{Nz-ku-2}{u}$ for some $z\in\mathbb{N}\cup\{0\}$. Thus $s=\frac{k(Nz-ku-2)-u}{u}$, so

$$\frac{t}{s}=\frac{Nz-ku-2}{k(Nz-ku-2)-u}.$$

Now we find the greatest value of $\frac{t}{s}$ by defining a function $f:\mathbb{R}^+\cup\{0\}\to\mathbb{R}$,

$$f(z)=\frac{u+\frac{Nz-ku-2}{k(Nz-ku-2)-u}}{N}.$$

Since $f'(z)=\dfrac{-u}{(k(Nz-ku-2)-u)^2}<0$, the greatest value of f is taken at $z=0$ and this value is

$$\frac{u+\cfrac{1}{k+\cfrac{1}{k+\frac{2}{u}}}}{N}.$$

But, $\dfrac{u+\cfrac{1}{k+\cfrac{1}{k+\frac{2}{u}}}}{N}$ is nearer to $\dfrac{u+\frac{1}{k}}{N}$ than $\dfrac{u+\frac{1}{k-\frac{1}{k}}}{N}$. So the farthest one being joined with $\dfrac{u+\frac{1}{k}}{N}$ is $\dfrac{u+\frac{1}{k-\frac{1}{k}}}{N}$. As we have

$$\lim_{z\to\infty}\frac{u+\frac{Nz-ku-2}{k(Nz-ku-2)-u}}{N}=\frac{u+\frac{1}{k}}{N},$$

the nearest vertex does not exist.

Case 4: We obtain that $su+t\equiv -u^2k-u \pmod{N}$ and $Ns(ku+1)-kN(su+t)=N$, which implies $s=kt+1$. Therefore $u+kut+t\equiv -u^2k-u \pmod{N}$ and we have $t(ku+1)\equiv -ku^2-2u \pmod{N}$. As $u^2+ku+1\equiv 0 \pmod{N}$, $-tu^2\equiv -ku^2-2u \pmod{N}$. Since $(u,N)=1$, $-tu\equiv -uk-2 \pmod{N}$, that is, $t=\frac{-(Nz-ku-2)}{u}$ for some $z\in\mathbb{N}\cup\{0\}$. Moreover, $s=\frac{-(k(Nz-ku-2)-u)}{u}$. Thus

$$\frac{t}{s}=\frac{Nz-ku-2}{k(Nz-ku-2)-N}.$$

The proof is similar to the proof for the third case. □

Corollary 4.2. *If* $(u, N) = 1$ *and* $u^2 + ku + 1 \equiv 0 \pmod N$*, then*

$$\varphi(\frac{u}{N}) = \frac{u + \frac{1}{k}}{N}, \quad \varphi(\frac{u + \frac{1}{k}}{N}) = \frac{u + \frac{1}{k - \frac{1}{k}}}{N}.$$

Definition 4.3. Let $v_1, v_2, ..., v_m$ be vertices of $\bar{F}_{u,N}$, then configurations $v_1 \to v_2 \to ... \to v_m$ and $v_1 \to v_2 \to ...$ are called a path and an infinite path in $\bar{F}_{u,N}$, respectively.If $v_m = v_1$, then the above path is said to be a circuit (or closed path) in $\bar{F}_{u,N}$.

Let $\{t_m\}$ be a sequence of Möbius transformation

$$t_m(z) = \frac{a_m}{b_m + z}, \quad a_m \neq 0,$$

and let $T_m(z) = t_1 t_2 ... t_m(z)$, $m \geq 1$ such that T_0 is the identity map.

Note that $T_m(\infty) = T_{m-1}(0)$. If one computes $t_1(0), t_2(0), t_1 t_2 t_3(0)$ and so on, form a continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{4.3}$$

and the value of the continued fraction (4.3) when it exists is equal to the limit of the sequence $\{T_m(0)\}$. In this study we work with special Möbius transformation

$$t_m(z) := t(z) = \frac{1}{k - z} = \frac{-1}{-k + z}.$$

To see some relations between continued fractions and hyperbolic paths of suborbital graphs, by using Theorem 4.1 we can give the following infinite path

$$\frac{1}{0} \to \frac{u}{N} \to \frac{u + \frac{1}{k}}{N} \to \frac{u + \cfrac{1}{k - \frac{1}{k}}}{N} \to \frac{u + \cfrac{1}{k - \cfrac{1}{k - \frac{1}{k}}}}{N} \to ....$$

The above path gives rise to a continued fraction, if $k \leq 2$,

$$\cfrac{1}{k - \cfrac{1}{k - \cfrac{1}{k - \ddots}}} \tag{4.4}$$

which is a special case of the continued fractions from the following theorem.

Theorem 4.4 ([22])**.** *Let* $|b_m| \geq 1 + |a_m|$ *for all* $m \in \mathbb{N}$*. Then the continued fraction* (4.3) *converges to some value* v *with* $|v| \leq 1$*.*

Corollary 4.5. *The continued fraction* (4.4) *converges to* $\frac{k - \sqrt{k^2 - 4}}{2}$.

Proof. Since we have $a_m = -1$ and $b_m = -k$ where $k \geq 2$, then $|b_m| \geq 1 + |a_m|$. By Theorem 4.4, the continued fraction (4.4) converges to v with $|v| \leq 1$, that is, $\lim_{m \to \infty} T_m(0) = v$. As we know

$$T_m(0) = \frac{1}{k - T_{m-1}(0)},$$

$T_m(0)(k - T_{m-1}(0)) = 1$ and since $\lim_{m \to \infty} T_m(0) = \lim_{m \to \infty} T_{m-1}(0)$, we have $v(k - v) = 1$. Moreover, $v^2 - kv + 1 = 0$ and $v = \dfrac{k \pm \sqrt{k^2 - 4}}{2}$. We observe that if $k = 2$, then $v = 1$. If $k > 2$, then $v = \dfrac{k - \sqrt{k^2 - 4}}{2}$ since $v \leq 1$. □

5. Conclusions

First time, Jones et al. shows that a shortest path in suborbital graphs of the modular group can be expressed as a continued fraction [10]. Then, Değer et al. prove that a shortest path in trees of these suborbital graphs is a special case of Pringsheim continued fraction [7]. The main important development is revealed by Sarma et al [21]. The authors shows that the subgraph $F_{1,2}$ can be defined as a new kind of continued fraction and any irrational numbers has a unique $F_{1,2}$ expansion. Similar one is done for the subgraph $F_{1,3}$ in [18]. We can deduce from all these studies that suborbital graphs of modular groups can be considered as a method in the classification of rational numbers. Present study is an extension of some results in [7]. Their proofs are straightforward adjustment of those in [5, 7].

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