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Hosoya Polynomial of Graphs Belonging to Twist Knots

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Abstract. There is a deep relationship between knot theory and graph theory. Graphs are effective tools for studies into knot theory. It is possible to switch between the two theories by forming the graphs of knots. Thus, the study areas of the two theories meet on a common ground. In this study, we firstly introduce Hosoya polynomial and we work out the Hosoya polynomials for graphs of twist knots which are one of interesting families of knots and try to get a generalization for them.

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1. Introduction

In graph theory, there are a lot of invariants for a graph like polynomials. These polynomials are related by structure of a graph, which is invariant under graph automorphisms [3]. Polynomials have been quite handy when dealing with knots and links. What is a graph? It takes place a set of points called vertices and a set of edges which connect vertices.

Distance is an important concept in graph theory and it has many appliacations to other sciences. The Hosoya polynomial of a graph was defined by H. Hosoya in 1988. The main profit of the Hosoya polynomial is that it includes a opulence of information for distance-based graph invariants. Such as, acknowledging the Hosoya polynomial of a graph, it is simple to state the well-known Weiner index of a graph as the first derivative of the polynomial at point 1 [2]. The Hosoya polynomial has been by now investigated on trees, composite graphs, benzenoid graphs, tori, zig-zag open-ended nanotubes, certain graph decorations, armchair open-ended nanotubes, zigzag polyhex nanotorus, TUC4C8(S) nanotubes, pentachains, polyphenyl chains, as well as on Fibonacci and Lucas cubes and Hanoi graphs [1].

Definition 1.1 ([2]). Let $G = (V, E)$ be connected and distance-based graph. The distance $d(u, v)$ between any two vertices u and v is the minimum of the lenghts of paths between u and v. The topological diameter $d(G)$ of a graph G (i.e. the longest topological distance in G) is defined as

$$d(G) = \max_{u,v \in V(G)} \{d(u, v)\}.$$

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Let $d(G,k)$, $k \geq 0$, be the number of vertex pairs at distance k. The Hosoya polynomial of G is defined as follows [2]:

$$H(G,y) = \sum_{k=0}^{d(G)} d(G,k)y^k$$

where $d(G,0) = n$ such that n is the number of vertices in G.

Twist knots are one of interesting families of knots. They have various remarkable combinatorial properties of knots. In this study, we determine the Hosoya polynomials for graphs of twist knots and get two generalizations for these calculations.

Definition 1.2 ([4]). A twist knot which is denoted T_n is gotten by twisting two parallel strands n times and subsequently hooking the ends together to be alternating knot, as seen in Figure 1.

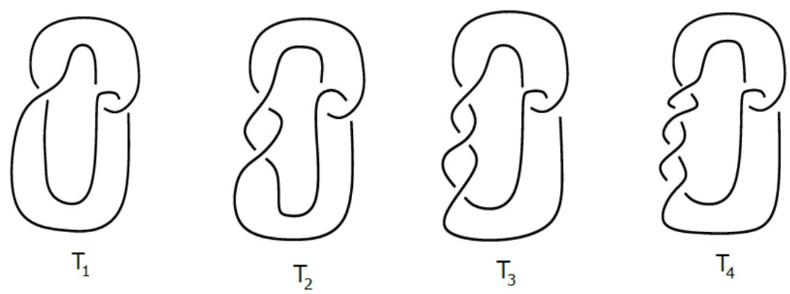

Figure 1. Some twist knots

Now we proceed that: At first, we will obtain regular projections of twist knots from their regular diagrams (see Figure 2). Then, we will shadow these projections in a checkered pattern such that the sides of an edge get different colors (see Figure 3). And then, we will get a point in the centers of each dark region. We obtain the graphs of twist knots by combining these points with the paths passing through the crossing points of the dark regions (see Figure 4 and Figure 5).

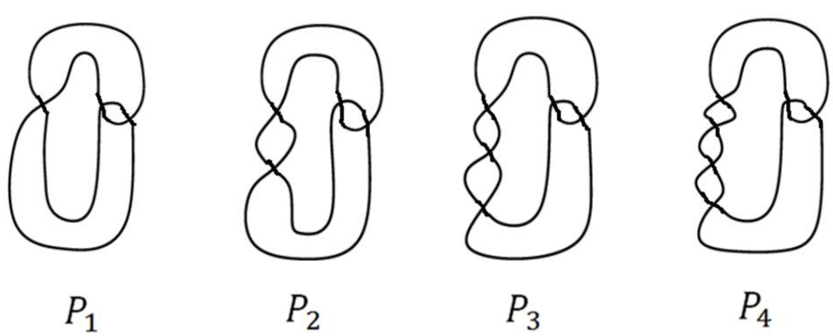

Figure 2. Some projections of twist knots

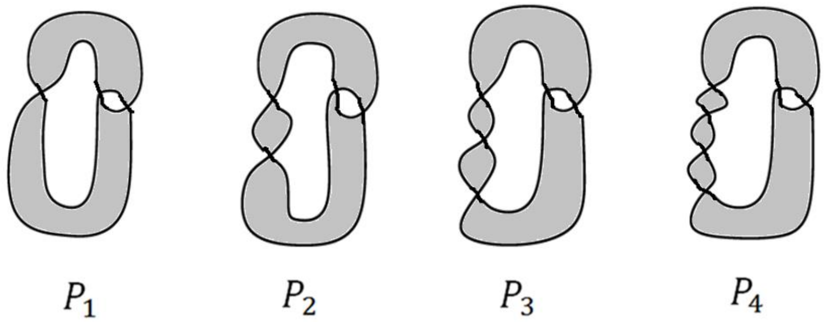

FIGURE 3. Some projections in a checkered pattern of twist knots

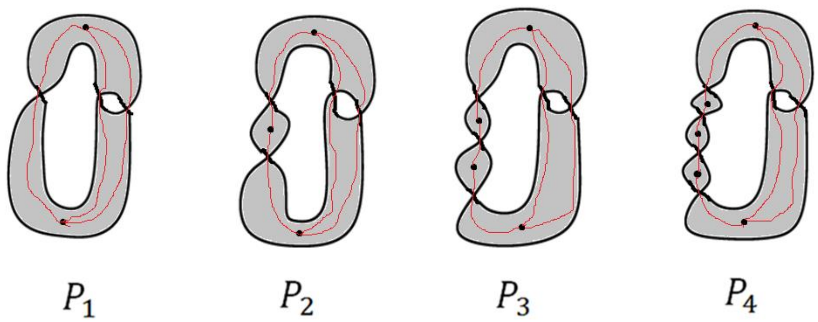

FIGURE 4. Obtaining some graphs of twist knots

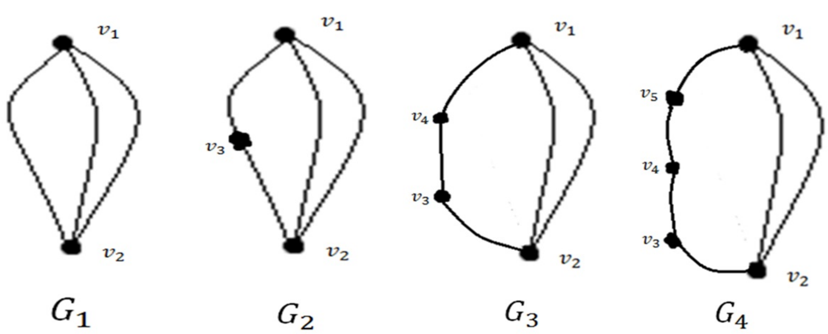

FIGURE 5. Some graphs of twist knots

As seen in Figure 5, we yield the following results:

$d(G_1, 0) = 2$	$d(G_2, 0) = 3$	$d(G_3, 0) = 4$	$d(G_4, 0) = 5$
$d(G_1, 1) = 1$	$d(G_2, 1) = 3$	$d(G_3, 1) = 4$	$d(G_4, 1) = 5$
$d(G_1) = 1$	$d(G_2) = 1$	$d(G_3) = 2$	$d(G_4) = 2$

2. RESULTS

Definition 2.1. Let $D_k = \{(u, v)|\ u, v \in V(G)$ and $d(u, v) = k\}$ be a set and we denote the number of elements of D_k by $|D_k|$ i.e. $d(G, k) = |D_k|$, $k \geq 0$.

Let $d(G,k)$, $k \geq 0$, be the number of vertex pairs at distance k. The Hosoya polynomial of G is defined as follows:

$$H(G,y) = \sum_{k=0}^{d(G)} d(G,k)y^k$$

where $d(G,0) = n$ such that n is the number of vertices in G.

Now we compute the Hosoya polynomials for graphs of twist knots:

For G_1:

$$D_0 = \{v_1, v_2\} \Rightarrow |D_0| = d(G_1, 0) = 2,$$
$$D_1 = \{(v_1, v_2)\} \Rightarrow |D_1| = d(G_1, 1) = 1,$$
$$\Rightarrow H(G_1, y) = 2y^0 + 1y^1$$
$$H(G_1, y) = 2 + y$$

For G_2:

$$D_0 = \{v_1, v_2, v_3\} \Rightarrow |D_0| = d(G_2, 0) = 3,$$
$$D_1 = \{(v_1, v_2), (v_1, v_3), (v_2, v_3)\} \Rightarrow |D_1| = d(G_2, 1) = 3,$$
$$\Rightarrow H(G_2, y) = 3y^0 + 3y^1$$
$$H(G_2, y) = 3 + 3y$$

For G_3:

$$D_0 = \{v_1, v_2, v_3, v_4\} \Rightarrow |D_0| = d(G_3, 0) = 4,$$
$$D_1 = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_3, v_4)\} \Rightarrow |D_1| = d(G_3, 1) = 4,$$
$$D_2 = \{(v_1, v_3), (v_2, v_4)\} \Rightarrow |D_2| = d(G_3, 2) = 2,$$
$$\Rightarrow H(G_3, y) = 4y^0 + 4y^1 + 2y^2$$
$$H(G_3, y) = 4 + 4y + 2y^2$$

For G_4:

$$D_0 = \{v_1, v_2, v_3, v_4, v_5\} \Rightarrow |D_0| = d(G_4, 0) = 5,$$
$$D_1 = \{(v_1, v_2), (v_1, v_5), (v_2, v_3), (v_3, v_4), (v_4, v_5)\} \Rightarrow |D_1| = d(G_4, 1) = 5,$$
$$D_2 = \{(v_1, v_3), (v_1, v_4), (v_2, v_4), (v_2, v_5), (v_3, v_5)\} \Rightarrow |D_2| = d(G_4, 2) = 5,$$
$$\Rightarrow H(G_4, y) = 5y^0 + 5y^1 + 5y^2$$
$$H(G_4, y) = 5 + 5y + 5y^2$$

If the above operations are continued, the following equations are obtained for some graphs:

$$H(G_5, y) = 6 + 6y + 6y^2 + 3y^3$$
$$H(G_6, y) = 7 + 7y + 7y^2 + 7y^3$$
$$H(G_7, y) = 8 + 8y + 8y^2 + 8y^3 + 4y^4$$
$$H(G_8, y) = 9 + 9y + 9y^2 + 9y^3 + 9y^4$$
$$H(G_9, y) = 10 + 10y + 10y^2 + 10y^3 + 10y^4 + 5y^5$$

Finally, we can indicate the following two generalization conclusions for Hosoya polynomials for graphs of twist knots.

Theorem 2.2. *For $k = 2n$ (i.e. $G_{k=2n}$, $n = 1, 2, 3, \ldots$)*

$$H(G_{2n}, y) = (2n+1)y^0 + (2n+1)y^1 + (2n+1)y^2 + \cdots + (2n+1)y^{n-1} + (2n+1)y^n$$

Proof. It isn't difficult to be seen the proof of the theorem by induction method. □

Theorem 2.3. *For $k = 2n - 1$ (i.e. $G_{k=2n-1}$, $n = 1, 2, 3, \ldots$)*

$$H(G_{2n-1}, y) = 2ny^0 + 2ny^1 + 2ny^2 + \cdots + 2ny^{n-1} + ny^n$$

Proof. It isn't difficult to be seen the proof of the theorem by induction method. □

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