Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (6) (2019), 1815 – 1823
DOI : 10.15672/HJMS.2018.646

Research article

On J-rigid rings

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Abstract

Let R be a ring with an endomorphism σ. We introduce the notion of σ-J-rigid rings as a generalization of σ-rigid rings, and investigate its properties. It is proved that a ring R is σ-J-rigid if and only if $R[[x;\sigma]]$ is $\bar{\sigma}$-J-rigid, while the σ-J-rigid property is not Morita invariant. Moreover, we prove that every ring isomorphism preserves J-rigid structure, and several known results are extended.

Mathematics Subject Classification (2010). 16S36, 16U99

Keywords. rigid rings, reduced rings, Jacobson radical, σ-J-rigid rings, over-rings

1. Introduction

Throughout this paper, R denotes an associative ring with identity and σ is an endomorphism of R. We denote the set of invertible elements of R, the Jacobson radical, the upper nil radical (i.e., the sum of all nil ideals), the set of all nilpotent elements of R and the ring of n-by-n matrices over R, by $U(R)$, $J(R)$, $nil^*(R)$, $nil(R)$ and $M_n(R)$, respectively. In what follows, $\mathbb{Z}$ denotes the ring of integer numbers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n.

According to Krempa [10], an endomorphism σ of a ring R is said to be *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. Later a ring R is called *σ-rigid* if there exists a σ-rigid endomorphism of R in Hong et al.'s article [7]. We recall that a ring is said to be *reduced* if it has no non-zero nilpotent element. Note that any rigid endomorphism of a ring is monomorphism and σ-rigid rings are reduced by Hong et al. [7]. In this work, we introduce and study σ-J-rigid rings as a generalization of rigid rings. A ring R with an endomorphism σ is called *σ-J-rigid* if for each $a \in R$, $a\sigma(a) = 0$ implies $a \in J(R)$. Among of the results, we show that local rings are σ-J-rigid for an endomorphism σ. We also study some famous extensions of σ-J-rigid rings. Suppose that the endomorphism σ is monomorphism. We say that an over-ring A of R is a Jordan extension of R if σ can be extended to an automorphism of A and $A = \cup_{k=0}^{\infty}\sigma^k(R)$. Jordan showed with the technique of left localization to the Ore extension $R[x;\sigma]$ with respect to the set of powers of x, that for any pair $(R;\sigma)$, such an extension A always exists. In this paper, we prove

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Received: 20.07.2017; Accepted: 15.08.2018

that for a ring R with a monomorphism σ and some additional conditions, σ-J-rigidity and some related properties transfer from R to A and viceversa.

2. Some properties of J-rigid rings

Definition 2.1. A ring R with an endomorphism σ is called *σ-J-rigid* if for each $a \in R$, $a\sigma(a) = 0$ implies that $a \in J(R)$, and a subring S of R is called *σ-J-rigid* if S satisfies the same condition as R and $\sigma(S) \subseteq S$.

For J-semisimple rings, the concepts of σ-rigid and σ-J-rigid are equivalent. Also, σ-rigid rings are σ-J-rigid, but the following example shows that the converse is not true, in general.

Example 2.2. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field and $\sigma : R \to R$ be an endomorphism defined by $\sigma(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that R is a σ-J-rigid ring. But, since:

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\sigma(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then R is not σ-rigid.

Now, let R is a σ-J rigid ring and I a σ-ideal (i.e., $\sigma(I) \subseteq I$), then I is also σ-J-rigid. In fact, for any $a \in I$ with $a\sigma(a) = 0$, then we have $a \in I \cap J(R) = J(I)$. Using this fact to $R = \prod_{i \in I} R_i$, if R is σ-J rigid, then so is every R_i as an ideal of R. Conversely, if every R_i is σ-J rigid, then clearly so is $R = \prod_{i \in I} R_i$. In particular we have:

Corollary 2.3. *Let e be a non-zero central idempotent of a ring R. Then eR and $(1-e)R$ are σ-J-rigid rings if and only if so is R.*

Proposition 2.4. *Let σ be an endomorphism of R such that $\sigma(eRe) \subseteq eRe$ and R be a σ-J-rigid ring. Then eRe is σ-J-rigid for any $e^2 = e$ of R.*

Proof. If $(ere)\sigma(ere) = 0$, then $ere \in J(R)$ by σ-J-rigidity of R. So $ere = e(ere)e \in eJ(R)e = J(eRe)$, as desired. □

Although σ-rigid rings are reduced by Hong [7], the above example shows that σ-J-rigid rings are not necessarily reduced. Also, reduced rings are not necessarily σ-J-rigid, by the following examples.

Example 2.5. Let S be any ring and $R = S \times S$. Define $\sigma(a, b) = (b, a)$ for all $(a, b) \in R$, each S as subring of R is not σ-subring (i.e., $\sigma(S) \nsubseteq S$) and so is not σ-J-rigid. So we get the desired conclusion by taking R be any reduced ring.

Example 2.6. Let $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \pmod 2\}$ be a ring with additive and multiplicative pairwise. Then R is a commutative reduced ring. Suppose $\sigma : R \to R$ is an endomorphism defined by $\sigma((a, b)) = (b, a)$. We have $(2, 0)\sigma((2, 0)) = (0, 0)$ and $(2, 0) \notin J(R)$, since R is J-semisimple.

For an ideal I of a ring R with an endomorphism σ, if I is a σ-ideal (i.e. $\sigma(I) \subseteq I$), then $\bar{\sigma} : R/I \to R/I$ defined by $\bar{\sigma}(r + I) = \sigma(r) + I$ is an endomorphism of R/I.

Proposition 2.7. *Let R be a ring with an endomorphism σ and I be a σ-ideal of R such that $I \subseteq J(R)$. If R/I is $\bar{\sigma}$-J-rigid, then R is σ-J-rigid.*

Proof. Suppose $r\sigma(r) = 0$. Therefore, $\bar{r}\bar{\sigma}(\bar{r}) = \bar{0}$. Since R/I is $\bar{\sigma}$-J-rigid, then $\bar{r} \in J(R/I)$ and so $r \in J(R)$. □

In the following, we state an example of rings which satisfies the condition of Proposition 2.7.

Example 2.8. Let $T(R)$ be the ring of countably infinite upper triangular matrices over a σ-J-rigid ring R with $\bar{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ for each $A = (a_{ij}) \in T(R)$ and I be the ideal of $T(R)$ with all diagonal elements zero. It is easy to see that $I \subseteq J(T(R))$. Also, $T(R)/I \cong \prod_{i\in\mathbb{N}} R$ is $\bar{\sigma}$-J-rigid. So $T(R)$ is $\bar{\sigma}$-J-rigid, by above proposition.

The converse of Proposition 2.7 is not true with help of the next example.

Example 2.9. Let R denote the localization of $\mathbb{Z}$ at $3\mathbb{Z}$. Consider the ring of quaternions Q over the ring R, that is, a free R-module with basis $1, i, j, k$. Then Q is a non-commutative domain and $J(Q) = 3Q$. So Q is σ-J-rigid for any monomorphism σ of Q. On the other hand, $Q/J(Q)$ is isomorphic to 2×2 full matrix ring over $\mathbb{Z}_3$ via an isomorphism f defined by

$$f(\frac{a_0}{b_0} + \frac{a_1}{b_1}i + \frac{a_2}{b_2}j + \frac{a_3}{b_3}k + 3Q) = \begin{pmatrix} a_0b_0^{-1}+a_1b_1^{-1}-a_2b_2^{-1} & a_1b_1^{-1}+a_2b_2^{-1}-a_3b_3^{-1} \\ a_1b_1^{-1}+a_2b_2^{-1}+a_3b_3^{-1} & a_0b_0^{-1}-a_1b_1^{-1}+a_2b_2^{-1} \end{pmatrix},$$

where the entries of the matrix are read modulo the ideal $< 3 >$ of $\mathbb{Z}$. Define $\bar{\sigma} : M_2(\mathbb{Z}_3) \to M_2(\mathbb{Z}_3)$ such that

$$\bar{\sigma}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}.$$

Then $M_2(\mathbb{Z}_3)$ is not $\bar{\sigma}$-J-rigid and so is not $Q/J(Q)$.

According to Hang et al. [7], for a ring R with an endomorphism σ, a σ-ideal I is called *σ-rigid* if for each $a \in R$, $a\sigma(a) \in I$ implies that $a \in I$.

Proposition 2.10. *Let I be a σ-rigid ideal of a σ-J-rigid ring R. Then R/I is $\bar{\sigma}$-J-rigid.*

Proof. Suppose $\bar{r}\bar{\sigma}(\bar{r}) = \bar{0}$. Therefore, $r\sigma(r) \in I$. So $r \in I$, since I is σ-rigid ideal and hence, $\bar{r} \in J(R/I)$. □

Now, we prove that the class of σ-J rigid rings contains local rings as a proper subclass.

Proposition 2.11. *Let R be a local ring. Then R is σ-J rigid for any endomorphism σ of R.*

Proof. Let R be a local ring. Then $J(R) = m$ in which m is the only maximal ideal of R. Suppose that $\bar{\sigma}$ is an endomorphism of R/m such that defined by $\bar{\sigma}(\bar{r}) = \sigma(r) + m$. Let $r\sigma(r) = 0$ for $r \in R$. Then $\bar{r}\bar{\sigma}(\bar{r}) = \bar{0}$. Since R/m is division ring, then $r \in m$ or $\sigma(r) \in m$. If $\sigma(r) \in m$, then $\sigma(r)$ and consequently r are not invertible. Hence $r \in m$, as desired. □

The converse of the above proposition is not true by the following example.

Example 2.12. Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. The only non-zero proper ideals of R are $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Hence, R is not a local ring, but R is a σ-J- rigid ring by the Example 2.2.

Proposition 2.13. *Let R be a ring with an endomorphism σ, S be a ring and $\alpha : R \to S$ be a ring isomorphism. Then R is σ-J-rigid if and only if S is $\alpha\sigma\alpha^{-1}$-J-rigid.*

Proof. Suppose that R is σ-J-rigid. Let $s(\alpha\sigma\alpha^{-1})(s) = 0$ for some $s \in S$. So $\alpha^{-1}(s)\sigma(\alpha^{-1}(s)) = 0$ and thus $\alpha^{-1}(s) \in J(R)$, since R is σ-J-rigid. Therefore, $s = \alpha(\alpha^{-1}(s)) \in \alpha(J(R)) = J(S)$. Conversely, suppose S is $\alpha\sigma\alpha^{-1}$-J-rigid. Let $r\sigma(r) = 0$. Then $\alpha(r)\alpha(\sigma(r)) = 0$ and so $\alpha(r)\alpha(\sigma(\alpha^{-1}(\alpha(r))) = 0$. Therefore, $\alpha(r) \in J(S)$. Thus $r \in \alpha^{-1}(J(S)) \subseteq J(R)$ and we are done. □

3. Extensions of J-rigid rings

Let R and S be two rings with endomorphisms α and β, respectively and M be an (R,S)-bimodule. Then

$$T(R,M,S) = R \oplus M \oplus S = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\}$$

is a ring by usual addition and the following multiplication:

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rm'+ms' \\ 0 & ss' \end{pmatrix}.$$

Also,

$$\sigma(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}) = \begin{pmatrix} \alpha(r) & m \\ 0 & \beta(s) \end{pmatrix}.$$

is an endomorphism of T.

Theorem 3.1. *Let R and S be two rings with endomorphisms α and β, respectively and M be an (R,S)-bimodule. Then $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is σ-J-rigid if and only if R and S are α-J-rigid and β-J-rigid, respectively.*

Proof. Let R and S be α-J-rigid and β-J-rigid, respectively and $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in T$ with $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\sigma(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}) = 0$. Then we have $r\alpha(r) = 0$ and $s\beta(s) = 0$. This implies that $r \in J(R)$ and $s \in J(S)$. Therefore, $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in \begin{pmatrix} J(R) & M \\ 0 & J(S) \end{pmatrix} = J(T)$, as desired. Conversely, let $r\alpha(r) = s\beta(s) = 0$ for some $r \in R$ and $s \in S$. Thus $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}\sigma(\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}) = 0$ and so $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in J(T)$, by σ-J-rigidity of T. Hence, $r \in J(R)$ and $s \in J(S)$ and the result follows. □

Corollary 3.2. *Let R be a ring with an endomorphism σ. Then*

(i) *$T(R,M)$ is a $\bar{\sigma}$-J-rigid ring if and only if R is σ-J-rigid.*

(ii) *The trivial extension $T(R,R)$ is $\bar{\sigma}$-J-rigid if and only if R is σ-J-rigid.*

In Proposition 2.4, we proved that if R is a σ-J-rigid ring, then so is eRe. In the following, we give an example which shows that σ-J-rigidity of R does not transfer to the full matrix ring $M_n(R)$ and so J-rigid property is not Morita invariant.

Example 3.3. Let k be a field with monomorphism σ. Then k is σ-J-rigid. Now, let $R = M_2(k)$ and define $\bar{\sigma} : R \to R$ such that:

$$\bar{\sigma}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}.$$

Then

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\bar{\sigma}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin J(R) = 0$$

and consequently R is not $\bar{\sigma}$-J-rigid.

Let $F \cup \{0\}$ be the free monoid generated by $U = \{u_1, \ldots, u_t\}$ with 0 added, and M be a factor of F by setting certain monomial in U to 0. In fact for some positive integer $n \geq 2$, $M'^n = 0$, where $M' = M \backslash \{e\}$ and e is the identity of M. In [5] the authors defined and studied the skew monoid ring $R[M;\alpha]$, by taking its elements to be finite formal combinations $\sum_{g\in M} r_g g$ with usual addition and multiplication subject to the relation $u_i r = \alpha(r)u_i$ for each $1 \leq i \leq t$. Clearly for any endomorphism σ of R, if $\alpha\sigma = \sigma\alpha$, then $\bar{\sigma} : R[M;\alpha] \to R[M;\alpha]$ with $\bar{\sigma}(\sum_{g\in M} r_g g) = \sum_{g\in M} \sigma(r_g)g$ is an endomorphism of $R[M;\alpha]$.

Theorem 3.4. *The ring R is σ-J-rigid if and only if $R[M;\alpha]$ is $\bar{\sigma}$-J-rigid.*

Proof. Let R be σ-J-rigid and $(\sum_{g\in M} r_g g)\bar{\sigma}(\sum_{g\in M} r_g g) = 0$. Then $r_e\sigma(r_e) = 0$ and so $r_e \in J(R)$, by σ-J-rigidity of R. Thus $(\sum_{g\in M} r_g g) \in J(R[M;\alpha])$, by [5, Theorem 2.9]. Conversely, let $R[M;\alpha]$ be $\bar{\sigma}$-J-rigid and for $r \in R$ we have $r\sigma(r) = 0$. So $(re)(\sigma(r)e) = 0$. Therefore $(re)\bar{\sigma}(re) = 0$ and so $re \in J(R[M;\alpha])$. Then $r \in J(R)$, by [5, Theorem 2.9] and the proof is complete. □

Let R be a ring and α denotes an endomorphism of R with $\alpha(1) = 1$. In [2] Chen et al. introduced *skew triangular matrix* ring as a set of all triangular matrices with addition point-wise and a new multiplication subject to the condition $E_{ij}r = \alpha^{j-i}(r)E_{ij}$ and denoted it by $T_n(R,\alpha)$. The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R,n,\alpha)$; and the subring of the skew triangular matrices with constant diagonals is denoted by $T(R,n,\alpha)$. It is well-known that $T(R,n,\alpha) \cong R[x;\alpha]/(x^n)$, where $R[x;\alpha]$ is the skew polynomial ring with multiplication subject to the condition $xr = \alpha(r)x$ for each $r \in R$, and (x^n) is the ideal generated by x^n. The rings $S(R,n,\alpha)$ and $T(R,n,\alpha)$ fit into the structure introduced above with $U = \{E_{12}, E_{23}, \ldots, E_{n-1,n}\}$ and $U = \{E_{12} + E_{23} + \cdots + E_{n-1,n}\}$, respectively.

We consider the following two subrings of $S(R,n,\alpha)$, as follow (see [6, Page 13]).

$$A(R,n,\alpha) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{n-j+1} a_j E_{i,i+j-1} + \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} \sum_{i=1}^{n-j+1} a_{i,i+j-1} E_{i,i+j-1};$$

$$B(R,n,\alpha) = \{A + rE_{1k} \mid A \in A(R,n,\alpha) \text{ and } r \in R\} \qquad n = 2k \geq 4.$$

In [12] showed that $A(R,n,\alpha)$ and $B(R,n,\alpha)$ are also fit into the structure $R[M;\alpha]$. If σ is an endomorphism of R such that $\alpha\sigma = \sigma\alpha$, then $\bar{\sigma} : S(R,n,\alpha) \to S(R,n,\alpha)$, given by $\bar{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ is an endomorphism of $S(R,n,\sigma)$. Now, as a corollary of Theorem 3.4, we have the following result.

Corollary 3.5. *Let R be a ring with endomorphisms α and σ such that $\alpha\sigma = \sigma\alpha$. Then the following statements are equivalent.*

(i) *R is σ-J-rigid.*
(ii) *$S(R,n,\alpha)$ is $\bar{\sigma}$-J-rigid.*
(iii) *$A(R,n,\alpha)$ is $\bar{\sigma}$-J-rigid.*
(iv) *$B(R,n,\alpha)$ is $\bar{\sigma}$-J-rigid.*
(v) *$T(R,n,\alpha)$ is $\bar{\sigma}$-J-rigid.*
(vi) *$R[x;\alpha]/(x^n)$ is $\bar{\sigma}$-J-rigid.*

Let R be a ring with endomorphism α and σ such that $\alpha\sigma = \sigma\alpha$. Recall that $\bar{\sigma} : R[[x;\alpha]] \to R[[x;\alpha]]$ given by $\bar{\sigma}(\sum_{i=0}^{\infty} a_i x^i) = \sum_{i=0}^{\infty} \sigma(a_i)x^i$ is an endomorphism.

Proposition 3.6. *R is a σ-J-rigid ring if and only if $R[[x;\alpha]]$ is a $\bar{\sigma}$-J-rigid ring.*

Proof. First, suppose that R is σ-J-rigid and $f(x)\bar{\sigma}(f(x)) = 0$, where $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Therefore $a_0\sigma(a_0) = 0$ and hence $a_0 \in J(R)$. Next, let $g(x) = \sum_{i=0}^{\infty} b_i x^i$ be an arbitrary element of $R[[x;\alpha]]$. Thus $1 - a_0 b_0$ is invertible and so $1 - f(x)g(x)$ is an invertible series of $R[[x;\alpha]]$. So $f(x) \in J(R[[x;\alpha]])$. The converse is proved by the similar method. □

Corollary 3.7. *A ring R is σ-J-rigid if and only if $R[[x]]$ is $\bar{\sigma}$-J-rigid.*

Let R be a ring with endomorphism σ. A subring S of R is called σ-subring if $\sigma(S) \subseteq S$. In the following we give two examples which show that σ-subrings of a σ-J-rigid ring need not be σ-J-rigid.

Example 3.8. Let F be a field. We note that $F[x]$ is a subring of $F[[x]]$. Define an endomorphism $\sigma : F[[x]] \to F[[x]]$ by $\sigma(f(x)) = f(0)$ for $f(x) \in F[[x]]$. We consider $f(x) = ax$ for $a \neq 0$. We have $f(x)\sigma(f(x)) = 0$, but $f(x) \notin J(F[x])$. Then $F[x]$ is not σ-J-rigid. Now we show that $F[[x]]$ is σ-J-rigid. Let $f(x)\sigma(f(x)) = 0$ where $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Then $a_i a_0 = 0$ for $i \geq 0$. It is clear that $1 - f(x)g(x)$ is an invertible series of $F[[x]]$ for each $g(x) \in F[[x]]$. So $f(x) \in J(F[[x]])$ and we are done.

Example 3.9. For any countable field K, there exists a nil algebra S over K such that $S[x]$ is Jacobson radical (i.e. $J(S[x]) = S[x]$) but $nil^*(S[x]) = 0$ by [3, Lemma 2.5]. Let $R = K + S$. Then R is a local ring, and so are $R[[x]]$ and $R[[x]][[y]]$. This means that $R[[x]][[y]]$ is σ-J-rigid for any endomorphism σ. We claim that subring $R[x][y]$ of $R[[x]][[y]]$ is not σ-J-rigid. In fact, $J(R[x][y]) \subseteq nil^*(R[x])[y] = nil^*(S[x])[y] = 0$ holds by [3, Lemma 2.4]. Indeed, this result is duo to Amitsur [1]. If $R[x][y]$ is σ-J-rigid, then it is σ-rigid, and so is reduced. This is an obvious contradiction.

Recall that an algebra over a commutative ring S is just a ring R equipped with a specified ring homomorphism ϕ from S to the center of R. Then ϕ is used to define products of elements of S with elements of R. In fact for $s \in S$ and $r \in R$, we set sr equal to $\phi(s)r$. Using this product, we can view R as an S-module.

Dorroh [4] introduced the *Dorroh extension* of R by S in which R is an algebra over a non-zero commutative ring S. In fact $D = R \times S$ is the ring with operators

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2) \quad , \quad (r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2),$$

where $r_i \in R$, $s_i \in S$.

For an S-endomorphism σ of R and the Dorroh extension D of R by S, the non-zero map $\bar{\sigma} : D \to D$ defined by $\bar{\sigma}((r, s)) = (\sigma(r), s)$ is an S-algebra homomorphism.

Theorem 3.10. *Let $D = R \times S$ be the Dorroh extension of R by S such that S is a reduced ring. Then R is σ-J-rigid if and only if D is $\bar{\sigma}$-J-rigid.*

Proof. Let R be σ-J-rigid and $(r, s)\bar{\sigma}((r, s)) = 0$. Then $s^2 = r\sigma(r) + sr + s\sigma(r) = 0$. Thus, $s = 0$ and consequently $r\sigma(r) = 0$. Therefore, $r \in J(R)$. We claim that $(r, s) = (r, 0) \in J(D)$. Proving this, we need to show that if $r_1 \in R$ and $s \in S$, then $(0, 1) - (r, 0)(r_1, s) = (-rr_1 - sr, 1) \in U(D)$; equivalently, we need to prove that there exists $r_2 \in R$ such that $(-rr_1 - sr, 1)(r_2, 1) = (0, 1)$. Since $r \in J(R)$, then $(1 - rr_1) \in U(R)$ and $(-sr) \in J(R)$. Therefore, $(1 - rr_1 - sr) \in U(R)$. Put $r_2 = (1 - rr_1 - sr)^{-1} - 1$. So $(1 - rr_1 - sr)(1 + r_2) = 1$ and consequently $-rr_1 r_2 - srr_2 - rr_1 - sr + r_2 = 0$. This implies that $(-rr_1 - sr, 1)(r_2, 1) = (0, 1)$ and the claim is proved. Hence D is $\bar{\alpha}$-J-rigid. Conversely, let D be $\bar{\sigma}$-J-rigid and $r \in R$ with $r\sigma(r) = 0$. Then $(r, 0)(\sigma(r), 0) = 0$ and so $(r, 0)\bar{\sigma}((r, 0)) = 0$. Since D is $\bar{\sigma}$-J-rigid, then we have $(r, 0) \in J(D)$. Now, we claim that $r \in J(R)$. Let r_1 be an arbitrary element of R. Thus there exist $r_2 \in R$ and $s \in S$ such that $((0, 1) - (r, 0)(r_1, 0))(r_2, s) = (0, 1)$. Therefore $s = 1$ and hence $-rr_1 r_2 - rr_1 + r_2 = 0$. So $(1 - rr_1)r_2 = rr_1$ and consequently $(1 - rr_1)(1 + r_2) = 1$. Thus $(1 - rr_1) \in U(R)$ and hence $r \in J(R)$. This implies R is σ-J-rigid and the proof is complete. □

Now, we consider Jordan's construction of the ring $A(R, \sigma)$. Let $A = A(R, \sigma)$ be the subset

$$\{x^{-i} r x^i \mid r \in R, i \geq 0\}$$

of the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$. For each $j \geq 0$,

$$x^{-i} r x^i = x^{-(i+j)} \sigma^j(r) x^{(i+j)}.$$

It follows that the set of all such elements forms a subring of $R[x, x^{-1}; \alpha]$ with

$$x^{-i} r x^i + x^{-j} s x^j = x^{-(i+j)}(\sigma^j(r) + \sigma^i(s)) x^{(i+j)}$$

and

$$(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}(\sigma^j(r)\sigma^i(s))x^{(i+j)}$$

for $r, s \in R$ and $i, j \geq 0$. Note that, $\sigma(x^{-i}rx^i) = x^{-i}\sigma(r)x^i$ is actually an automorphism of $A(R, \sigma)$.

Lemma 3.11. *Let σ be an endomorphism of R with $\sigma^k = id_R$ for some $k \geqslant 2$. Then $J(R) = J(A) \cap R$.*

Proof. Let $r \in J(R)$. We show that $1 - rb \in U(A)$ for each $b \in A$. Let $b = x^{-i}sx^i$. So, $1 - rb = x^{-i}(1 - \sigma^i(r)s)x^i$. Since $\sigma^k = id_R$ for some $k \geqslant 2$ then σ is an epimorphism of R. Therefore, $\sigma(J(R)) \subseteq J(R)$. So $1 - \sigma^i(r)s \in U(R)$. Therefore, by [9, Proposition 3.1] $1 - rb \in U(A)$, as desired. Now, let $a \in J(A) \cap R$. We show that $1 - ab \in U(R)$ for all $b \in R$. Since $R \subseteq A$ and $a \in J(A)$, we have $1 - ab \in U(A)$. So there exists $n \geq 0$ such that $\sigma^n(1 - ab) \in U(R)$, by [9, Proposition 3.1]. Since $\sigma^k = id_R$ for some $k \geqslant 2$, then $(1 - ab) \in U(R)$ and the result follows. □

Now, we state an example of rings which satisfies the condition of the above lemma.

Example 3.12. Let R be a ring and n be a positive integer number. Suppose that $S = \oplus_{i=1}^n R_i$, where $R_i = R$ for each $1 \leq i \leq n$. Define $\sigma : S \to S$, given by $\sigma(a_1, a_2, \ldots, a_n) = (a_n, a_1, a_2, \ldots, a_{n-1})$. Then σ is a monomorphism and $\sigma^n = id_S$.

Note that R is id_R-J-rigid if and only if $a^2 = 0$ implies that $a \in J(R)$ for each $a \in R$. According to [3] a ring R is called *J-reduced* if $nil(R) \subseteq J(R)$. Clearly, J-reduced rings are id_R-J-rigid.

Theorem 3.13. *Let σ be an endomorphism of R and with $\sigma^k = id_R$ for some $k \geqslant 2$. Then R is an id-J-rigid ring if and only if so is A.*

Proof. Let R be an id_R-J-rigid ring and $p^2 = 0$ for $p \in A$. We have $(x^{-i}rx^i)^2 = x^{-2i}\sigma^i(r^2)x^{2i} = 0$ for some $i \geq 0$ and $r \in R$ (as designed in [9]). Since σ is monomorphism, then $r^2 = 0$. Hence $r \in J(R)$. Since $\sigma^j(r) \in J(R)$ for each $j \geq 0$, hence $(1 - \sigma^j(r)\sigma^i(s)) \in U(R)$ for each $s \in R$. Therefore $x^{-(i+j)}(1 - \sigma^j(r)\sigma^i(s))x^{(i+j)} \in U(R)$ for each $j \geq 0$ and $s \in R$, as desired. Conversely, let A be identity-J-reduced and $r^2 = 0$ for $r \in R$. So $r \in J(A)$. By Lemma 3.11, $r \in J(R)$. The proof is complete. □

Theorem 3.14. *Let σ be an endomorphism of R with $\sigma^k = id_R$ for some $k \geqslant 2$. Then R is a σ-J-rigid ring if and only if so is A.*

Proof. Suppose R is σ-J-rigid and $p\sigma(p) = 0$ for $p \in A$. We claim that $p \in J(A)$. For, we prove $1 - pq \in U(A)$ for each $q \in A$. Let $p = x^{-i}rx^i$ and $q = x^{-j}sx^j$ such that $r, s \in R$. We have $1 - pq = x^{-(i+j)}(1 - \sigma^j(r)\sigma^i(s))x^{i+j}$. From $(x^{-i}rx^i)\sigma(x^{-i}rx^i) = 0$ and by extension σ to a mapping from A (as designed in [9]), $x^{-i}r\sigma(r)x^i = 0$. Hence, $r\sigma(r) = 0$. Therefore, $r \in J(R)$. Since $J(R)$ is σ-ideal, hence $\sigma^i(r) \in J(R)$. Thus $(1 - \sigma^j(r)\sigma^i(s)) \in U(R)$. So, by [9, Proposition 3.1], we have $1 - pq \in U(A)$, as desired. Conversely, by Lemma 3.11 is trivial. □

According to [13], a ring R is said to be *σ-J-skew Armendariz* if whenever $f(x)g(x) = 0$, where $f(x) = \sum_{i=0}^m a_i x^i$ and $g(x) = \sum_{j=0}^n b_j x^j$ in $R[x; \sigma]$, then $a_i\alpha^i(b_j) \in J(R)$, for each i and j.

Theorem 3.15. *Let σ be an endomorphism of R with $\sigma^k = id_R$ for some $k \geqslant 2$. Then R is σ-J-skew Armendariz if and only if so is A.*

Proof. Let R be a σ-J-skew Armendariz ring. Suppose $f = \sum_{i=0}^m a_i x^i$ and $g = \sum_{i=0}^n b_j x^j$ are elements of $A[x; \sigma]$ with $fg = 0$. We prove that $a_i\sigma^i(b_j) \in J(A)$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$. So, we show that $1 - a_i\sigma^i(b_j)u \in U(A)$ for each $u \in A$. Since $A = \cup_{k=0}^\infty \sigma^{-k}(R)$, hence $\sigma^k(1 - a_i\sigma^i(b_j)u) \in R$ for some $k \geq 0$. From $fg = 0$, we have

$\sigma^k(f)\sigma^k(g) = 0_{R[x;\sigma]}$. Therefore, $\sigma^k(a_i)\sigma^{i+k}(b_j) \in J(R)$ for each $0 \le i \le m$ and $0 \le j \le n$. So $1 - \sigma^k(a_i)\sigma^{i+k}(b_j)r \in U(R)$ for all $r \in R$. Specially, $1 - \sigma^k(a_i)\sigma^{i+k}(b_j)\sigma^k(u) \in U(R)$. So, there exists $w \in R$ such that $(1 - \sigma^k(a_i)\sigma^{i+k}(b_j)\sigma^k(u))w = 1$. By construction of A, there exists $c \in A$ such that $\sigma^k(c) = w$. Therefore, $\sigma^k((1-a_i\sigma^i(b_j)u)c) = 1 = \sigma^k(1)$. Thus $1 - a_i\sigma^i(b_j)u \in U(A)$, as desired. Conversely, by Lemma 3.11 is trivial. □

Next, we show that every σ-J-skew Armendariz is not σ-J-rigid, by the following example.

Example 3.16. Consider $R = \mathbb{Z}_2[x]$, a commutative polynomial ring over the ring of integers modulo 2. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. We show that R is σ-J-skew Armendariz. To see this, let $p = \sum_{i=0}^{m} f_i y^i$ and $q = \sum_{j=0}^{n} g_j y^j \in R[y;\sigma]$. Assume that $pq = 0$. Therefore, $\sum_{l=0}^{m+n}\sum_{i+j=l} f_i\sigma^i(g_j)x^l = 0$. Suppose that $f_s \ne 0$ and $f_0 = \cdots = f_{s-1} = 0$, where $0 \le s \le m$. So $\sum_{i=0}^{s} f_i\sigma^i(g_{s-i}) = 0$, implies that $f_s\sigma^s(g_0) = 0$ and consequently $f_s g_0(0) = 0$. Thus, $g_0(0) = 0$. Also, by considering the equation $\sum_{i=0}^{s+1} f_i\sigma^i(g_{s+1-i}) = 0$, we obtain $f_s\sigma^s(g_1) + f_{s+1}\sigma^{s+1}(g_0) = 0$ and so $f_s g_1(0) = 0$. This implies that $g_1(0) = 0$. Continuing this process, we have

$$g_0(0) = g_1(0) = \cdots = g_n(0) = 0.$$

Thus, $f_i\sigma^i(g_j) = 0$ for each $0 \le i \le m$ and $0 \le j \le n$. Therefore R is σ-J-skew Armendariz. But R is not σ-J-rigid, because $x\sigma(x) = 0$, but $x \notin J(R)$.

The next example shows that there exists an id-J-rigid ring R such that $R[x;id]$ is not id-J-rigid.

Example 3.17. Let R be the ring as in Example 3.9. Clearly, R is a local ring with $J(R) = S$, where S is nil algebra. So R is an id_R-J-rigid ring. If $R[x]$ is id_R-J-rigid, then we are done. Otherwise, we choose the ring $R[x][y]$. Since $J(R[x][y]) = I[y]$ for some nil ideal of $R[x]$ and $nil^*(R[x]) = nil^*(S[x])$ and we have $nil^*(S[x]) = 0$ by [3, Lemma 2.5]. So $J(R[x][y]) = 0$. This implies that $R[x][y]$ is not an id_R-J-rigid ring. Assume on the contrary, since $J(R[x][y]) = 0$ then it is an id_R-rigid ring. So it is reduced by Hong et al. [7], an obvious contradiction.

Matczuk investigated a characterization of σ-rigid rings in [11] and by using the over-ring A, gave positive answer to the question posed in Hong et al. [8]. That is, he proved that the following conditions are equivalent:
(1) σ is monomorphism, R is reduced and σ-skew Armendariz.
(2) R is σ-rigid.
(3) $R[x;\sigma]$ is reduced.

We finish this article by a question on σ-J-rigid rings. Under which conditions or properties, can we say σ-J-rigid rings and σ-J-skew Armendariz rings are equivalent? Are the following conditions equivalent?
(1) σ is monomorphism, R is id-J-rigid and σ-J-skew Armendariz.
(2) R is σ-J-rigid.
(3) $R[x;\sigma]$ is id-J-rigid.

Acknowledgment. This paper is supported by Islamic Azad University Central Tehran Branch (IAUCTB). The authors want to thank the authority of IAUCTB for their support to complete this research. Also, we are grateful to Professor Weixing Chen for many useful suggestions during this work. The authors would like to thank referee for carefully reading the paper and for his/her suggestions.

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