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Research Article

Morita-like equivalence for fair semigroups

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Abstract

In this paper, we mainly investigate Morita-like equivalence and Morita context for right fair semigroups. If two right fair semigroups S and T are Morita-like equivalent, that is, there is a category equivalence $F : US-\text{Act} \rightleftharpoons: UT\text{-Act} : G$, we characterize the two functors F and G using Hom functor and tense product functor. Also, we investigate Morita context for right fair semigroups and obtain equivalence between two right unitary act categories.

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Keywords. fair semigroup, Morita equivalence, Morita context, equivalence, functor

1. Introduction

Morita equivalence theory characterizes the relationship of module categories of two rings with 1. Morita theory is very useful in studying ring theory. Later, many authors tried to extend Morita equivalence theory to rings without 1. Fuller [4] obtained equivalence between a completely additive subcategory of the module category over a ring (not necessarily with 1) and a full unitary module category over a ring with 1. Abrams [1], Ánh and Márki [2] obtained Morita equivalences by replacing the usual module categories with the categories of unital modules. Xu, Shum and Turner-Smith [17] further extended Morita theory and introduced Morita-like equivalence of module categories. García, Marín and Ouyang etc. [5, 14, 15] studied xst-rings, which is named after Xu, Shum and Turner-Smith.

Morita theory has also been extended to semigroup theory [3, 7–13, 16]. Banaschewski [3] and Knauer [8] independently studied Morita theory of monoids and got the same conclusions. Since then, many results on Morita theory of semigroups have been obtained. Talwar [16] generalized this theory to semigroups with local units. Lawson [12] reformulated Morita theory of semigroups with local units in [16] and characterized equivalence of two semigroups with local units by joint enlargement. Recently, Liu [13] got equivalence between two subcategories of unital act categories over two arbitrary semigroups based on Morita context. Laan and Márki [11] introduced the notion of fair semigroups, which corresponds to xst-rings. They investigated Morita equivalence of these semigroups. It is worth to further consider Morita-like equivalence for fair semigroups. In this paper, we will use the tools in [15] to study Morita-like equivalence and Morita context for these semigroups.

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The paper is constructed as follows. In Section 2, we give some basic notions and results on fair semigroups which is used in this paper. In Section 3, the Morita-like equivalence is investigated and some properties of Morita-like equivalence are obtained. We study Morita context for right fair semigroups and obtain equivalence between two unitary act categories in Section 4.

2. Preliminaries

Let S be a semigroup. A set M is a right S-act if there is a map from $M \times S$ to M, denoted $(m, s) \to ms$, such that $m(st) = (ms)t$ $(\forall s, t \in S, m \in M)$. If M is a right S-act, we write M_S. If $M = MS$, we call M to be unitary. If for all $m \in M$, there exists $s \in S$ such that $ms = m$, we call M to be s-unital.

One can similarly define left S-acts.

For two right S-acts M and N, $f : M \to N$ is said to be S-morphism if f satisfies $f(ms) = f(m)s$, $(\forall m \in M, s \in S)$. The set of all S-morphisms from M_S to N_S is denoted by $\mathrm{Hom}_S(M, N)$. The set of all S-morphisms from M to itself is denoted by $\mathrm{End}_S(M)$. Let S-Act be the category of right S-acts on a semigroup S.

The unitary right S-acts together with the S-morphisms form a full subcategory of S-Act, which we shall denote by US-Act.

Let S and T be two semigroups. An S-T-biact is a set M which is both left S-act and right T-act and $(sm)t = s(mt)$ for all $s \in S, t \in T$ and all $m \in M$. A biact is said to be unital if it is left and right unital. If M and N are S-T-biact, a map $f : M \to N$ is called a biact morphism if f satisfies $f(sm) = sf(m)$ and $f(mt) = f(m)t$ for all $m \in M, s \in S, t \in T$.

An equivalence ρ on M_S is a congruence if for all $s \in S, m, n \in M$,

$$(m, n) \in \rho \Rightarrow (ms, ns) \in \rho.$$

If ρ is a congruence on M, then M/ρ is also a right S-act. The act M/ρ is called a quotient act.

For a right S-act A_S and a left S-act ${}_SB$, the tensor product $A \otimes_S B$ exists. In fact, $A \otimes_S B = (A \times B)/\sigma$, where σ is the congruence on $A \times B$ generated by

$$\mathcal{R} = \{((xs, y), (x, sy))|x \in A, y \in B, s \in S\}.$$

We denote the element $(x, y)\sigma$ of $A \otimes_S B$ by $x \otimes y$.

By Proposition 1.4.10 of [6], we have that $a \otimes b = c \otimes d \iff (a, b) = (c, d)$ or there is a sequence

$$(a, b) = (x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_n, y_n) = (c, d)$$

such that either $((x_i, y_i), (x_{i+1}, y_{i+1})) \in T$ or $((x_{i+1}, y_{i+1}), (x_i, y_i)) \in T$, where $1 \leq i \leq n - 1$.

If A is a right S-act and B is an S-T-biact, then $A \otimes_S B$ is a right T-act with

$$(a \otimes b)t = a \otimes bt;$$

similarly, if A is a T-S-biact and B is a left S-act, then $A \otimes_S B$ is a left T-act with

$$t(a \otimes b) = ta \otimes b$$

(Proposition 3.1, [16]).

Let S, T be two semigroups. For an S-T-biact ${}_SM_T$ and a left S-act ${}_SN$, we have that $\mathrm{Hom}_S(M, N)$ is a left T-act with action $(t \cdot f)(m) = f(m \cdot t)$; for an S-T-biact ${}_SM_T$ and a right S-act N_T, we have that $\mathrm{Hom}_T(M, N)$ is a right S-act with action $(f \cdot s)(m) = f(s \cdot m)$; for a right S-act M_S and a T-S-biact ${}_TN_S$, we have that $\mathrm{Hom}_S(M, N)$ is a left T-act with action $(t \cdot f)(m) = t \cdot f(m)$; for a left T-act ${}_TM$ and a T-S-biact ${}_TN_S$, we have that $\mathrm{Hom}_T(M, N)$ is a right S-act with action $(f \cdot s)(m) = f(m) \cdot s$.

A functor F is said to be faithful if it is injective. A semigroup S is called factorisable if every element of S is a product of two elements.

Corresponding to the notion of xst-rings investigated by García and Marín in [5], Laan and Márki [11] introduced the notion of fair semigroups.

Definition 2.1. [11] Let S be a semigroup. If every subact of an unitary right S-act is unital, S is said to be a right fair semigroup.

Similarly, we can define left fair semigroups. A fair semigroup is a semigroup S that is both a left fair semigroup and a right fair semigroup.

By Proposition 2.3 in [11], S is a fair semigroup if and only if every right unital s-act is s-unital.

An ideal I of S is said to be unitary if I is unitary as S-act. Let $U(S_S)$ be the disjoint union of all right unitary ideals of S. It is obvious that $U(S_S)$ is the largest right unitary ideal of S. Similarly, we can define the left ideal $U(_SS)$. If S is factorisable, then $U(S_S) = U(_SS) = S$.

Let S be a right fair semigroup. Laan and Márki [11] proved that $U(S_S)$ is a two-sided ideal and $U(_SS) = U(S_S)$. We denote $U(_SS) = U(S_S)$ by $U(S)$. By Corollary 2.13 in [11], we have

$$U(S) = \{s \in S \mid s = su = vs \text{ for some } u, v \in S\}.$$

The set $U(S)$ is said to be the unitary part of S.

A right S-act M_S is nonsingular, if $m = m'$ $(m, m' \in M)$ whenever $ms = m's$ for all $s \in S$ ([11]). A semigroup S has common weak right local units, if for all $s, t \in S$, there is an element $u \in S$ such that $s = su$ and $t = tu$ ([11]). Let S be a fair semigroup such that $U(S)$ has common weak right local units. By Proposition 3.2 in [11], every act in US-Act is nonsingular. Define $\mu_M : M \otimes_S S \to M$ by $m \otimes s \mapsto ms$. Let

$$FS\text{-Act} = \{M \in S\text{-Act} \mid M \otimes S \cong M\}.$$

If S has common weak right local units, then FS-Act $= US$-Act ([11]).

We generalize the definition of Morita-like equivalence to semigroup theory.

Definition 2.2. Let S and T be two fair semigroups. S and T are Morita-like equivalent if US-Act is equivalent to UT-Act.

3. Morita-like equivalence for fair semigroups

Let M_S be a right S-act. Denote by $U(M_S)$ the union of all unitary subacts of M_S. Then $U(M)$ is the largest subact of M_S which is unitary. The following proposition gives a characterization of $U(M_S)$, which is analogous to Proposition 2.2 in [15].

Proposition 3.1. *Let S be a fair semigroup. For $M \in US$-Act, we have $U(M_S) = MU(S_S)$. In particular, if S is a factorisable right fair semigroup, then $U(M_S) = MS$.*

Proof. Since $MU(S_S)$ is a unitary subact of M_S, we have that $MU(S_S) \subseteq U(M_S)$.

If N is a unitary subact of M_S, we get $NU(S_S) \subseteq NS = N$. Assume $NU(S_S) \neq N$. There exitsts an element $x \in N = NS$ but $x \notin NU(S)$. Hence, we have $x = n_1s_1$, where $n_1 \in N$ and $s_1 \in S$. Since $U(S)$ is a right ideal and $n_1s_1 \notin NU(S)$, we have $n_1 \in N\backslash NU(S)$ and $s_1 \notin U(S)$. For $n_1 \in NS\backslash NU(S)$, we have $n_1 = n_2s_2$, where $n_2 \in N$ and $s_2 \in S\backslash U(S)$.

Continuing in this way, we can get two sequences $n_1, n_2, \ldots, n_k, \ldots \in N$ and $s_1, s_2, \ldots, s_k, \ldots \in S$ such that $n_ks_ks_{k-1}\cdots s_1 \notin NU(S_S)$, for any positive integral k. By Theorem 2.6 of [11], we can find some $u \in S$ such that $s_k \cdots s_1 = s_k \cdots s_1u$. By Lemma 2.12 (2) in [11], we get $s_ks_{k-1}\cdots s_1 \in U(S)$. This is a contradiction. Hence, $NU(S_S) = N$.

For any unitary subact N of M_S, we have

$$N = NU(S_S) \subseteq MU(S_S) \subseteq U(M_S).$$

Particularly, let $N = U(M_S)$, we have $U(M_S) = MU(S_S)$. $\square$

If R is a right xst-ring and M is an R-module, there is a natural equivalence $M \cong \mathrm{Hom}_R(U(R), M)U(R)$ ([15]). The following statement gives a similar result for right fair semigroup S, where $U(S)$ has common weak right local units.

Proposition 3.2. *Let S be a right fair semigroup such that $U(S)$ has common weak right local units. For $M \in US$-Act, define*

$$\theta_M : M_S \longrightarrow Hom_S(U(S_S), M_S)U(S_S)$$

by putting $\theta_M(x)(s) = xs$, where $x \in M$ and $s \in U(S_S)$, then θ_M is a natural S-isomorphism. Moreover,

$$\theta : I_{US-Act} \longrightarrow Hom_S(U(S_S), -)U(S_S)$$

is a natural equivalence.

Particularly, if S is a factorisable right fair semigroup, we have

$$\theta : I_{US-Act} \longrightarrow Hom_S(S, -)S$$

is a natural equivalence.

Proof. Firstly, we show that θ_M is injective. For all $x, y \in M$, if $\theta_M(x) = \theta_M(y)$, we have $\theta_M(x)(s) = \theta_M(y)(s)$, for all $s \in U(S_S)$. That is, $xs = ys$. Hence, we get $x = y$ by Proposition 3.2 in [11].

Next, we show that θ_M is surjective. We know that $\mathrm{Hom}_S(U(S_S), M_S)U(S_S)$ is the largest unitary subact of $\mathrm{Hom}_S(U(S_S), M_S)$ by Proposition 3.1. For all $g \in \mathrm{Hom}_S(U(S_S), M_S)U(S_S)$, suppose that $g = g^{'}s^{'}$, where $g^{'} \in \mathrm{Hom}_S(U(S), M_S)$, $s^{'} \in U(S_S)$, there exits $u \in S$ satisfying $s^{'} = s^{'}u$ by Lemma 2.12 in [11]. Hence,

$$g = g^{'}s^{'} = g^{'}s^{'}u = gu.$$

Since S is a right fair semigroup, there exists a positive integer n such that $u^n \in U(S_S)$ by Corollary 2.10 in [11]. Also, we have $g = gu^n$. Therefore,

$$g(x^{'}) = (gu^n)(x^{'}) = g(u^n x^{'}) = g(u^n)x^{'} = \theta_M(g(u^n))(x^{'}),$$

where $x^{'} \in U(S_S)$. This concludes that $g = \theta_M(g(u^n))$ and so $\mathrm{Hom}_S(U(S_S), M_S)U(S) \subseteq \mathrm{Im}\theta_M$.

Since $\mathrm{Im}\theta_M$ is a subact of $\mathrm{Hom}_S(U(S_S), M_S)U(S_S)$, we have

$$\mathrm{Im}\theta_M = \mathrm{Hom}_S(U(S_S), M_S)U(S_S).$$

This implies that θ_M is surjective.

Clearly, θ_M is a natural isomorphism. $\square$

Remark 3.3. (i) If S is a factorisable right fair semigroup, then

$$S_S \cong \mathrm{Hom}_S(S, S_S)S = \mathrm{End}_S(S_S)S.$$

(ii) Let S be a right fair semigroup. For $M \in US$-Act, if ${}_TM_S$ is a T-S-biact, then

$$\theta_M : M_S \longrightarrow \mathrm{Hom}_S(U(S_S), M_S)U(S_S)$$

is a T-S-isomorphism.

Let $\{M_i \mid i \in I\}$ be a family of S-Act, the coproduct $\coprod_{i \in I} M_i$ is the disjoint union of M_i.

Let S be a semigroup. For all $M \in US$-Act, if M is an epimorphic image of $\coprod P$, we call the unitary S-act P to be a generator of US-Act.

Proposition 3.4. *Let S be a right fair semigroup, and P be an object in US-Act. Then the following are equivalent:*

(1) P is a generator of US-Act.

(2) The functor $Hom_S(P_S, -)$ is faithful.

Proof. (1) $\Longrightarrow$ (2) For $M, N \in US$-Act, suppose that f and g are two S-morphisms from M to N such that $\mathrm{Hom}(P, f) = \mathrm{Hom}(P, g)$. For all $m \in M$, since P is a generator, there is an S-morphism $\mu : \coprod P \longrightarrow M$ such that $m = \mu(p)$. Hence

$$f(m) = f\mu(m) = g\mu(m) = g(m).$$

This implies that $f = g$. Therefore, $\mathrm{Hom}(P, -)$ is faithful.

(2) $\Longrightarrow$ (1) Let $I = \mathrm{Hom}_S(P_S, S_S)$ and $\mu : P^{(I)} \longrightarrow S$ be a morphism indexed by the elements of I. If μ is not epic, we have

$$\mathrm{Hom}(P, S) \longrightarrow \mathrm{Hom}(P, S/\mathrm{Im}\mu)$$

For all $f_1, f_2 \in \mathrm{Hom}(P, S)$, we have $\mu f_1 = \mu f_2$. This is contrary to that $\mathrm{Hom}(P, -)$ is faithful. $\square$

By Proposition 3.5 in [11], we can now establish the following result:

Proposition 3.5. *Let S be a right fair semigroup such that $S' = U(S_S)$ has common weak right local units.*

(1) Every right unitary S-act is a right unitary S'-act, and vice versa.

(2) $Hom_S(M, N) = Hom_{S'}(M, N)$, for any M,$N \in US$-Act.

(3) $S^{'}$ is a generator in US-Act.

Proof. We can obtain (1) and (2) by the proof of Proposition 3.5 in [11].

(3) For all $M \in US$-Act, we have that M is also a unitary right S'-act by (1). So M is s-unital by Proposition 2.3 in [11]. Let $g : S'^M \longrightarrow M$ defined by $s \mapsto ms$. This proves that S' generates M. $\square$

Let S and T be two semigroups. If ${}_SP_T$ and ${}_TQ_S$ are biacts, $\tau : P \otimes_T Q \longrightarrow S$ and $\mu : Q \otimes_S P \longrightarrow T$ are biact morphisms correspondingly as written

$$\tau(p \otimes q) =< p, q >, \quad \mu(q \otimes p) = [q, p]$$

such that

$$< p_1, q > \cdot p_2 = p_1 \cdot [q, p_2], \quad [q_1, p] \cdot q_2 = q_1 \cdot < p, q_2 >,$$

for all $p, p_1, p_2 \in P$ and $q, q_1, q_2 \in Q$. The set $(S, T, {}_SP_T, {}_TQ_S, \tau, \mu)$ is said to be a Morita context.

Note that $U(T)$ is a unitary $U(T)$-act, it follows that $U(T)$ is also s-unital as $U(T)$-act. That is, for all $t \in U(T)$, there is $t^{'} \in U(T)$ such that $t = tt^{'}$. Hence, for all $f \in \mathrm{Hom}_T(Q, M)U(T)$, there is $t \in U(T)$ such that $ft = f$.

Corresponding to Theorem 3.1 in [15], we give the following theorem. We use a different way to prove this result.

Theorem 3.6. *Let S and T be two semigroups. Suppose $(S, T, {}_SP_T, {}_TQ_S, \tau, \mu)$ is a Morita context, where $\tau : P \otimes_T Q \longrightarrow S$ and $\mu : Q \otimes_S P \longrightarrow T$ are biact morphisms such that $Im\tau = U(S_S)$ and $Im\mu = U(T_T)$. Then we have the following natural functor isomorphisms:*

$$- \otimes_S P \cong Hom_S(Q_S, -)U(T_T), \quad - \otimes_T Q \cong Hom_T(P_T, -)U(S_S).$$

Proof. Define a map $\tau_M : M \otimes_S P \longrightarrow \mathrm{Hom}_S(Q_S, M_S)U(T_T)$ by $m \otimes p \longmapsto m\tau(p \otimes -)t$, that is

$$\tau_M(m \otimes p) = m\tau(p \otimes -)t,$$

where $t \in U(T)$ such that $m\tau(p \otimes -)t = m\tau(p \otimes -)$.

We claim that τ_M is well-defined.

Suppose that $m \otimes p = m^{'} \otimes p^{'}$, we have $(m, p) = (m^{'}, p^{'})$ in which case

$$m\tau(p \otimes -)t = m^{'}\tau(p^{'} \otimes -)t$$

or

$$\begin{array}{ll} m = m_1 s_1, & s_1 p = t_1 p_1, \\ m_1 t_1 = m_2 s_2, & s_2 p_1 = t_2 p_2, \\ \cdots & \cdots \\ m_{n-1} t_{n-1} = m_n s_n, & s_n p_{n-1} = t_n p^{'}, \\ m_n t_n = m^{'}, & \end{array}$$

where $m_1, m_2, \cdots, m_n \in M, p_1, p_2, \cdots, p_n \in P, s_1, s_2, \cdots, s_n, t_1, t_2, \cdots, t_n \in S$. Then

$$\begin{aligned} m\tau(p \otimes -) &= m_1 s_1 \tau(p \otimes -) = m_1 \tau(s_1 p_1 \otimes -) = m_1 \tau(t_1 p_1 \otimes -) \\ &= m_1 t_1 \tau(p_1 \otimes -) = m_2 s_2 \tau(p_1 \otimes -) \\ &= \cdots\cdots \\ &= m_n s_n \tau(p_{n-1} \otimes -) = m_n \tau(s_n p_{n-1} \otimes -) \\ &= m_n \tau(t_n p^{'} \otimes -) = m_n t_n \tau(p^{'} \otimes -) = m^{'} \tau(p^{'} \otimes -). \end{aligned}$$

Define $\overline{\mu} : \mathrm{Hom}_S(Q_S, M_S)U(T_T) \longrightarrow M \otimes_S P$ by $f \longmapsto f(q) \otimes p$, where $f \in Hom_S(Q_S, M_S)$, $t \in U(T)$ such that $ft = f$ and $t = \mu(q \otimes p)$. Suppose $ft_1 = f = ft_2$, where $t_1 = \mu(q \otimes p), t_2 = \mu(q^{'} \otimes p^{'})$. We have

$$\begin{aligned} f(q) \otimes p &= (f \cdot t)(q) \otimes p = f(t \cdot q) \otimes p = f(\mu(q^{'} \otimes p^{'})q) \otimes p \\ &= f(q^{'} \tau(p^{'} \otimes q)) \otimes p = f(q^{'}) \tau(p^{'} \otimes q) \otimes p \\ &= f(q^{'}) \otimes p^{'} \mu(q \otimes p) = f(q^{'}) \otimes p^{'} \mu(q^{'} \otimes p^{'}) \\ &= f(q^{'}) \tau(p^{'} \otimes q^{'}) \otimes p^{'} = f(q^{'} \tau(p^{'} \otimes q^{'})) \otimes p^{'} \\ &= f(tq^{'}) \otimes p^{'} = ft(q^{'}) \otimes p^{'} = f(q^{'}) \otimes p^{'}. \end{aligned}$$

So $\overline{\mu}$ is well-defined.

For all $m \otimes p \in M \otimes_S P$, we have

$$\begin{aligned} \overline{\mu\tau}(m \otimes p) &= \overline{\mu}(m\tau(p \otimes -)t) = \overline{\mu}(m\tau(p \otimes -)\mu(q^{'} \otimes p^{'})) \\ &= m\tau(p \otimes q^{'}) \otimes p^{'} = m \otimes \tau(p \otimes q^{'})p^{'} \\ &= m \otimes p\mu(q^{'} \otimes p^{'}) = (m \otimes p)t, \end{aligned}$$

where $t = \mu(q^{'} \otimes p^{'})$ with $m\tau(p \otimes -)t = m\tau(p \otimes -)$. Hence, we have $\overline{\mu\tau} = I_{m \otimes p}$.

For all $f \in \mathrm{Hom}_T(Q, M)U(T)$, suppose $ft = f$ and $t = \mu(q \otimes p)$. We have

$$\overline{\tau\mu}(f) = \overline{\tau\mu}(ft) = \overline{\tau}(f(q) \otimes p) = f(q)\tau(p \otimes -)t.$$

For all $x \in Q$, we have

$$f(q)\tau(p \otimes x) = f(q\tau(p \otimes x)) = f(\mu(q \otimes p)x) = f(tx) = ft(x).$$

Thus, $f(q)\tau(p \otimes -) = ft(-) = f(-)$. It concludes $\overline{\tau\mu} = I_{\mathrm{Hom}_S(Q_S, M_S)U(T_T)}$.

The naturality is obvious. Hence, we get $- \otimes_S P \cong \mathrm{Hom}_S(Q_S, -)U(T_T)$.

Similarly, we have $- \otimes_T Q \cong \mathrm{Hom}_T(P_T, -)U(S_S)$. □

We can obtain the following corollary by the above theorem.

Corollary 3.7. *Let S and T be two fair semigroups. Assume $(S, T, {}_SP_T, {}_TQ_S, \tau, \mu)$ is a Morita context, where ${}_SP_T$ and ${}_TQ_S$ are unitary biacts, $\mu : Q \otimes_S P \to T$ and $\tau : P \otimes_T Q \to S$ are biact morphisms such that $Im\tau = U(S_S)$ and $Im\mu = U(T_T)$. We have the following natural functor isomorphisms:*

$$\begin{array}{ll} - \otimes_S P \cong Hom_S(Q_S, -)U(T), & - \otimes_T Q \cong Hom_T(P_T, -)U(S), \\ Q \otimes_S - \cong U(T)Hom_S({}_SP, -), & P \otimes_T - \cong U(S)Hom_T({}_TQ, -). \end{array}$$

Theorem 3.3 in [15] gives some properties of Morita-like equivalence of right-xst rings. We shall give the corresponding properties for right fair semigroups.

Theorem 3.8. *Let S and T be two right fair semigroups such that $U(S_S)$ and $U(T_T)$ have common weak right local units. Assume S and T are Morita-like equivalence via $F : US\text{-}Act \rightleftharpoons UT\text{-}Act : G$. Let $P = F(U(S_S))$ and $Q = G(U(T_T))$. Then*

(1) *P_T and Q_S are respectively generators in UT-Act and US-Act.*
(2) *$End(P_T)U(S_S) \cong U(S_S)$ and $End(Q_S)U(T_T) \cong U(T_T)$ as semigroups.*
(3) *${}_SP_T$ and ${}_TQ_S$ are unitary biacts.*
(4) *${}_SP_T \cong Hom_S(Q_S, U(S_S))U(T_T)$ and ${}_TQ_S \cong Hom_T(P_T, U(T_T))U(S_S)$.*
(5) *$F \approx Hom_S(Q_S, -)U(T_T)$ and $G \approx Hom_T(P_T, -)U(S_S)$.*
(6) *${}_SP_T$ and ${}_TQ_S$ induce a Morita context $(S, T, {}_SP_T, {}_TQ_S, \tau, \mu)$ such that τ and μ are biact homorphisms such that $Im\tau = U(S_S)$ and $Im\mu = U(T_T)$, respectively.*

Moreover, if we define multiplications in $Q \otimes_S P$ and $P \otimes_T Q$, respectively:

$$(y \otimes x)(y' \otimes x') = y \otimes \tau(x \otimes y')x',\ (x \otimes y)(x' \otimes y') = x \otimes \mu(y \otimes x')y',$$

where $x, x' \in P$ and $y, y' \in Q$, then $Q \otimes_S P$ and $P \otimes_T Q$ are semigroups, and μ and τ are semigroup homomorphisms.

(7) $F \approx - \otimes_S P$ and $G \approx - \otimes_T Q$.

Proof. We denote $U(S_S)=S'$ and $U(T_T)=T'$.

(1) For $M, N \in UT$-Act. Suppose $f, g \in \mathrm{Hom}_T(M, N)$ and $f \neq g$. Since

$$G : Hom_T(M, N) \longmapsto Hom_S(G(M), G(N))$$

is an abelian group isomorphism, we have $G(f) \neq G(g)$. By Proposition 3.5, we have that S' is a generator in US-Act. It follows that $Hom_S(S'_S, -)$ is faithful by Proposition 3.4. Hence, $\mathrm{Hom}_S(S', -)G(f) \neq \mathrm{Hom}_S(S', -)G(g)$. Therefore, $F(G(f))F(h) \neq F(G(g))F(h)$, where h is a morphism from $S^{'}$ to $G(M)$. That is, $fF(h) \neq gF(h)$. This concludes that $\mathrm{Hom}(F(S^{'}), -)$ is faithful and $P = F(S^{'})$ is a generator.

Similarly, Q_S is a generator in US-Act.

(2) Since $F : US\text{-Act} \rightleftharpoons UT\text{-Act} : G$ is category equivalence and $Q = G(T^{'})$, we have $\mathrm{End}_T(T^{'}) \cong \mathrm{End}_T(Q)$. By Proposition 3.2, we obtain $\mathrm{End}_T(T'_T)T' \cong T'$. Therefore, $\mathrm{End}_S(Q_S)T' \cong T'$ as semigroups.

Similarly, $\mathrm{End}_T(P_T)S' \cong S'$.

(3) In virtue of (1) and (2), we know that ${}_SP_T$ and ${}_TQ_S$ are unitary biacts.

(4) Since

$$\mathrm{Hom}_S(Q_S, S'_S) \cong \mathrm{Hom}_T(F(Q_S), F(S'_S)) \cong \mathrm{Hom}_T(T'_T, P_T),$$

we obtain

$$\mathrm{Hom}_S(Q_S, S'_S)T' \cong \mathrm{Hom}_T(F(Q_S), F(S'_S))T' \cong \mathrm{Hom}_T(T'_T, P_T)T' \cong {}_SP_T,$$

in virtue of Proposition 3.2.

Similarly, we have

$$\mathrm{Hom}_S(Q_S, S^{'}_S)T^{'} \cong \mathrm{Hom}_T(F(Q_S), F(S^{'}_S))T^{'} \cong \mathrm{Hom}_T(T^{'}_T, P_T)T^{'} \cong {}_SP_T.$$

(5) By Proposition 3.2, for all $M \in US$-Act, we can get the natural isomorphisms

$$F(M_S) \cong \mathrm{Hom}_T(T^{'}_T, F(M_S))T^{'} \cong \mathrm{Hom}_T(G(T^{'}_T), GF(M_S))T^{'} \cong \mathrm{Hom}_S(Q_S, M_S)T^{'}$$

This implies that $F \approx \mathrm{Hom}_S(Q_S, -)T^{'}$ and $G \approx \mathrm{Hom}_T(P_T, -)S^{'}$.

(6) By (2) and (4), we can identify ${}_TQ_S$ with $\mathrm{Hom}_T(P_T, T^{'}_T)S^{'}$ and $S^{'}$ with $\mathrm{End}(P_T)S^{'}$. Set

$$< \, , \, >: Q \times P \longrightarrow T^{'}$$

by $< y, x >= yx$, and

$$[\ ,\ ] : P \times Q \longrightarrow S'$$

by $[x, y]x' = x < y, x' >= x(yx')$ for $x, x' \in P$ and $y \in Q$.

Clearly, $<\ ,\ >$ and $[\ ,\ ]$ are both balanced mappings.

Define

$$\tau : P \otimes_T Q \longrightarrow S', \mu : Q \otimes_S P \longrightarrow T'$$

by

$$\tau(x \otimes y) = [x, y],\ \ \mu(y \otimes x) =< y, x >= yx.$$

Since ${}_SP_T$ and ${}_TQ_S$ are unitary biacts, τ and μ are biact morphisms. We can easily check that $(S, T, {}_SP_T, {}_TQ_S, \tau, \mu)$ is a Morita context.

Since P_T is a generator in UT-Act, we have

$$T'_T = \bigcup_{g \in \mathrm{Hom}_T(P_T, T'_T)} gP.$$

For all $t \in T'$, there exist $x \in P$ and $g \in \mathrm{Hom}_T(P_T, T'_T)$ such that $t = gx$. Since ${}_SP$ is a unitary left S-act, we can write $x = sx'$, for some $s \in S, x' \in P$. Hence,

$$t = g(sx') =< g, sx' >=< gs, x' >= \mu(gs \otimes x').$$

So μ is surjective.

Since Q_S is a generator in US-Act, for all $s \in S'$, we have $s = fy$, where $f \in \mathrm{Hom}_S(Q_S, S'_S)$ and $y \in Q$. As Q is a unitary left T'-act, there exist $t \in T'$ and $y' \in Q$ such that $y = ty'$. Therefore, we can write $s = fty'$. Since T' has common weak right local units, there exists $e \in T'$ such that $te = t$ for all $t \in T'$.

Let $e = \mu(q \otimes p)$, where $q \in Q$ and $p \in P$. Since $ft \in \mathrm{Hom}_S(Q_S, S_S)T'$ and $ftq \in S$, we have

$$s = (fte)y' = (ft < q, p > y' = ftq[p, y'] = [ftqp, y'] = \tau((ftq)p \otimes y').$$

So τ is surjective.

In addition, we can verify that $Q \otimes_S P$ and $P \otimes_T Q$ are semigroups about the multiplications defined above. So μ and τ are semigroup isomorphisms.

(6) From the proof process of Theorem 3.6, (5) and (6), we know that

$$F \cong \mathrm{Hom}_S(Q_S, -)T' \cong - \otimes_S P$$

and

$$G \cong \mathrm{Hom}_T(P_T, -)S' \cong - \otimes_T Q$$

are naturally isomorphisms. □

4. Morita context for fair semigroups

Definition 4.1. A right S-act M is said to be strong s-unital, if for all $m_1, m_2 \in M$, there exists $s \in S$ such that $m_1 s = m_2 s$.

Definition 4.2. A Morita context $(S, T, {}_SP_T, {}_TQ_S, \tau, \mu)$ is unital, if ${}_SP_T$ and ${}_TQ_S$ are unitary biacts.

Ouyang etc. [15] obtained Morita-like equivalence based on Morita context for right-xst rings (Theorem 3.4). We also get similar result for right fair semigroups, but we need the assumption that $U(S)$ has common weak right local units.

Theorem 4.3. *Let S and T be two right fair semigroups such that $U(S)$ and $U(T)$ have common weak right local units. Assume $(S, T, {}_SP_T, {}_TQ_S, \tau, \mu)$ is a Morita context, where ${}_SP_T$ and ${}_TQ_S$ are strong s-unital and $Im\tau = U(S_S)$ and $Im\mu = U(T_T)$. We have the following conditions:*

(1) Q_S and P_T are respectively generators of US-Act and UT-Act.

(2) $P \otimes_T Q \overset{\tau}{\cong} {}_SU(S_S)_S$ and $Q \otimes_S P \overset{\mu}{\cong} {}_TU(T_T)_T$ as biacts.

Furthermore, if we define multiplications in $P \otimes_T Q$ and $Q \otimes_S P$ respectively by:

$$(x \otimes y)(x' \otimes y') = x \otimes \mu(y \otimes x')y',\ (y \otimes x)(y' \otimes x') = y \otimes \tau(x \otimes y')x',$$

where $x, x' \in P$ and $y, y' \in Q$, then $P \otimes_T Q$ and $Q \otimes_S P$ are semigroups, and τ and μ are semigroup isomorphisms.

(3) ${}_SP_T \cong Hom_S(Q_S, U(S_S))U(T_T)$ and ${}_TQ_S \cong Hom_T(P_T, U(T_T))U(S_S)$ as biacts.

(4) $U(T_T) \cong End(Q_S)U(T_T)$ and $U(S_S) \cong End(P_T)U(S_S)$ as semigroups.

(5) The functor pair $(- \otimes_S P, - \otimes_T Q)$ defines an equivalence $- \otimes_S P : US\text{-}Act \rightleftharpoons UT\text{-}Act : - \otimes_T Q$. That is, S and T are Morita-like equivalence semigroups.

(6) The functor pair $(Hom_S(Q_S, -)U(T_T), Hom_T(P_T, -)U(S_S))$ defines an equivalence $Hom_S(Q_S, -)U(T_T) : US\text{-}Act \rightleftharpoons UT\text{-}Act : Hom_T(P_T, -)U(S_S)$. That is, S and T are Morita-like equivalence semigroups.

(7) The lattice of right ideals of $U(S_S)$ (resp., $U(T_T)$) is isomorphic to the lattice of subacts of P_T (resp., Q_R).

Furthermore, these induce lattice isomorphisms between the lattices of two-sided ideals of $U(S_S)$ (resp., $U(T_T)$) and the lattice of subacts of ${}_SP_T$ (resp., ${}_TQ_S$).

Proof. For convenience, put $S' = U(S_S)$ and $T' = U(T_T)$.

(1) Define $f_p : Q \longrightarrow S'$ by $q \mapsto \tau(p \otimes q)$. We have

$$f_p(qs) = \tau(p \otimes qs) = \tau(p \otimes q)s.$$

Hence, $f_p \in \mathrm{Hom}_S(Q, S^{'})$. For all $s \in S^{'} = Im\tau$, we have $s = \tau(p \otimes q) = f_p(q)$. Define $\theta : Q^{\mathrm{Hom}(Q,S^{'})} \longrightarrow S^{'}$ by $q \longmapsto f_p(q)$. Therefore, Q_S generates S'.

Since $S^{'}$ is a generaor in US-Act by Proposition 3.5, we have that Q_S generates M and Q_S is a generator in US-Act.

Similarly, P_T is a generator in UT-Act.

(2) Suppose $x = q \otimes p$, $y = q^{'} \otimes p^{'}$ and $\mu(x) = \mu(y)$. As P is strong s-unital, there exists $t \in T^{'}$ such that $pt = p^{'}t$. Set $t = \mu(v \otimes u)$. Then

$$\begin{aligned} x &= q \otimes p = q \otimes pt = q \otimes p\mu(v \otimes u) = q \otimes \tau(p \otimes v)u = q\tau(p \otimes v) \otimes u \\ &= \mu(q \otimes p)v \otimes u = \mu(q^{'} \otimes p^{'})v \otimes u = q^{'} \otimes p^{'}\tau(v \otimes u) = y. \end{aligned}$$

So μ is injective. This implies that μ is a biact isomorphism and so $Q \otimes_S P \cong T'$ as biact.

Next, to prove that $Q \otimes_S P$ is a semigroup, for any $x, x', x'' \in P$ and $y, y', y'' \in Q$, we have

$$\begin{aligned} ((y \otimes x)(y' \otimes x'))(y'' \otimes x'') &= (y \otimes \tau(x \otimes y')x')(y'' \otimes x'') \\ &= y \otimes \tau(x \otimes y')\tau(x' \otimes y'')x'' \\ &= (y \otimes x)(y' \otimes \tau(x' \otimes y'')x'') \\ &= (y \otimes x)((y' \otimes x')(y'' \otimes x'')). \end{aligned}$$

Hence, the multiplication satisfies the associative law. This proves that $Q \otimes_S P$ is a semigroup.

Similarly, $P \otimes_T Q$ is also a semigroup.

Clearly, μ and τ preserve the multiplication operations, respectively. So μ and τ are semigroup isomorphisms.

(3) Define $\theta : P \to \mathrm{Hom}_S(Q_S, S^{'}_S)$ by $p \mapsto \theta(p)$ for $p \in P$, where $\theta(p)(q) = \tau(p \otimes q)$ for $q \in Q$.

Clearly, θ is a morphism.

Suppose $\theta(p) = \theta(p^{'})$, we have $\tau(p \otimes q) = \tau(p^{'} \otimes q^{'})$, for all $q \in Q$. Since P is strong s-unital, there exists $t \in T^{'}$ such that $pt = p^{'}t$, where $t = \mu(v \otimes u)$. Hence

$$p = pt = p\mu(v \otimes u) = \tau(p \otimes v)u = \tau(p^{'} \otimes v)u = p^{'}\mu(v \otimes u) = p^{'}t = p^{'}.$$

It follows that θ is injective. Thus $\mathrm{Im}\theta \cong P$ is a unitary S-T-subbiact of $\mathrm{Hom}_S(Q_S, S^{'}_S)$. From Proposition 3.1, we know that $\mathrm{Hom}_S(Q_S, S^{'}_S)T^{'}$ is the largest unitary right T-subact of $\mathrm{Hom}_S(Q_S, S^{'}_S)$. Then $\mathrm{Im}\theta \subseteq \mathrm{Hom}_S(Q_S, S^{'}_S)T^{'}$. For $f \in \mathrm{Hom}_S(Q_S, S^{'}_S)T^{'}$, since $T^{'}$ is s-unital, there exists $e \in T^{'}$ such that $fe = f$. Suppose $e = \mu(y \otimes x)$, where $y \in Q$ and $x \in P$. Let $x^{'} = f(y)x$. Then $x' \in P$ and

$$\begin{aligned} \theta(x^{'})(y) &= \theta(f(y)x)(y) = \tau(f(y)x \otimes y) \\ &= f(y)\tau(x \otimes y) = f(y\tau(x \otimes y)) \\ &= f(\mu(y \otimes x)y) = f(ey) \\ &= (fe)(y) = f(y). \end{aligned}$$

Therefore, $f = \theta(x^{'})$ and $\mathrm{Im}\theta = \mathrm{Hom}_S(Q_S, S^{'}_S)T^{'}$. Hence, ${}_SP_T \cong \mathrm{Hom}_S(Q_S, S^{'}_S)T^{'}$.

Similarly, we can prove ${}_TQ_S \cong \mathrm{Hom}_T(P_T, T^{'}_T)S'$.

(4) Define $\lambda : T' \to \mathrm{End}(Q_S)$ by $\lambda(t)(y) = ty$ for $y \in Q$. For $t, u \in T'$, suppose $\lambda(t) = \lambda(u)$. Then $ty = \lambda(t)(y) = \lambda(u)(y) = uy$ for $y \in Q$. By assumption, $\exists\ e \in T'$ such that $te = t$ and $ue = u$. Let $e = \mu(y \otimes x)$, where $y \in Q$, $x \in P$. Then

$$t = te = t\mu(y \otimes x) = \mu(ty \otimes x) = \mu(uy \otimes x) = u\mu(y \otimes x) = ue = u.$$

This proves that λ is injective and $\mathrm{Im}\lambda \cong T^{'}$. Hence, $\mathrm{Im}\lambda$ is a subsemigroup of $\mathrm{End}(Q_S)$. Since $\mathrm{Im}\lambda$ is a right unitary $T^{'}$-act, $Im\lambda$ is a subact of $\mathrm{End}(Q_S)T'$ by Proposition 3.2. For all $f \in \mathrm{End}(Q_S)T^{'}$, there exists $e \in T^{'}$ such that $fe = f$. Suppose $e = \mu(y \otimes x)$, where $y \in Q$ and $x \in P$. Set $t^{'} = \mu(f(y) \otimes x)$. For all $q \in Q$, we have

$$\begin{aligned} \lambda(t')(q) &= t'q = \mu(f(y) \otimes x)q = f(y)\tau(x \otimes q) \\ &= f(y\tau(x \otimes q)) = f(\mu(y \otimes x)q) = f(eq) = (fe)(q) = f(q). \end{aligned}$$

Hence, $f = \lambda(t^{'})$ and $\mathrm{Im}\lambda = \mathrm{End}(Q_S)T^{'}$. Then $T^{'} \cong \mathrm{End}(Q_S)T^{'}$ as semigroup.

Similarly, $U(S_S) \cong \mathrm{End}(P_T)U(S_S)$ as semigroup.

(5) Firstly, we prove $- \otimes_S P \otimes_T Q \cong I_{US\text{-Act}}$. Since $P \otimes Q \cong S^{'}$, we can prove

$$M \otimes_S S^{'} \cong M, \text{ for all } M_S \in US\text{-Act}.$$

Define $\sigma : M \otimes_S S^{'} \to M$ by

$$\sigma(m \otimes s) = ms, \text{ for } m \otimes s \in M \otimes_S S^{'}.$$

Clealy, σ is a morphism. By Proposition 3.5, M is a right unitary S-act. Hence, σ is surjective.

Assume $\sigma(m \otimes s) = ms = m^{'}s^{'} = \sigma(m^{'} \otimes s^{'})$. Since $S^{'}$ has common weak right local units, there exists $e \in S^{'}$ such that $se = s^{'}e$. Hence,

$$m \otimes s = m \otimes se = ms \otimes e = m^{'}s^{'} \otimes e = m^{'} \otimes s^{'}e = m^{'} \otimes s^{'}.$$

So σ is injective. Therefore,

$$(M \otimes_S P) \otimes_T Q \cong M \otimes_S (P \otimes_T Q) \overset{I \otimes \tau}{\cong} M \otimes_S S' \overset{\sigma}{\cong} M.$$

The naturality is obvious. So we get the desired result.

Similarly, we can get $-\otimes_T Q \otimes_S P \cong I_{UT\text{-Act}}$.

(6) By (5) and Theorem 3.6, we can prove (6).

(7) Set

$$\mathcal{I} = \{I \mid I \text{ is a right ideal of } S^{'}\};$$
$$\mathcal{N} = \{N \mid N \text{ is a subact of } P\}.$$

Define $f : \mathcal{I} \longrightarrow \mathcal{N}$ by $I \longmapsto IP$, where $IP = \{ip | i \in I, p \in P\}$ and $g : \mathcal{N} \longrightarrow \mathcal{I}$ by $N \longmapsto \tau(N \otimes_T Q)$, where $\tau(N \otimes_T Q) = \{\tau(n \otimes q) | n \in N, q \in Q\}$.

Clearly, f and g are inverse lattice isomorphisms between $\mathcal{I}$ and $\mathcal{N}$.

Similarly, the lattice of right ideals of T' is isomorphic to the lattice of subacts of Q_S.

Obviously, these isomorphisms induce lattice isomorphisms between the lattice of ideals of S' $(resp., T')$ and the lattice of subacts of ${}_SP_T$ $(resp., {}_TQ_S)$. □

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References

[1] G. Abrams, *Morita equivalence for rings with local units,* Commun. Algebra, **11**, 801–837, 1983.

[2] P.N. Ánh and L.Márki, *Morita equivalence for rings without identity,* Tsukuba J. Math. **11**, 1–16, 1987.

[3] B. Banaschewski, *Functors into categories of M-sets,* Abh. Math. Sem. Univ. Hamburg, **38**, 49–64, 1972.

[4] K.R. Fuller, *Density and equivalence,* J. Algebra **29**, 528–550, 1974.

[5] J.L. García and L.Marín, *Rings having a Morita-like equivalence,* Commun. Algebra, **27**, 665–680, 1999.

[6] J.M. Howie, *Fundamentals of semigroup theory*, Clarendon press, Oxford, 1995.

[7] M. Kilp, U. Knauer and A.V. Mikhalev, *Monoids Acts and Categories,* Degruyter Expositions in Mathematics, 2000.

[8] U. Knauer, *Projectivity of acts and Morita equivalence of monoids,* Semigroup Forum **3**, 359–370, 1972.

[9] V. Laan and L. Márki, *Strong Morita equivalence of semigroups with local units,* J. Pure Appl. Algebra, **215**, 2538–2546, 2011.

[10] V. Laan and L. Márki, *Morita invariants for semigroups with local units,* Monatsh. Math. **166**, 441–451, 2012.

[11] V. Laan and L. Márki, *Fair semigroups and Morita equivalence,* Semigroup Forum, **92**, 633–644, 2016.

[12] M.V. Lawson, *Morita equivalence of semigroups with local units,* J. Pure Appl. Algebra **215**, 455–470, 2011.

[13] H. Liu, *Morita equivalence based on Morita context for arbitrary semigroups,* Hacet. J. Math. Stat. **45** (4), 1083–1090, 2016.

[14] B.Y. Ouyang and W.T. Tong, *Morita context and Morita-like equivalence for the xst-rings,* Acta Math. Sinica (English Series), **19**, 371–380, 2003.

[15] B.Y. Ouyang, L.R. Zhou and W.T. Tong, *Characterizations of Morita-like Equivalences for right xst-rings,* Algebra Colloquium, **14**, 85–95, 2007.

[16] S. Talwar, *Morita equivalence for semigroups,* J. Aust. Math. Soc. (Series A), **59**, 81–111, 1995.

[17] Y.H. Xu, K.P. Shum and R.F. Turner-Smith, *Morita-like equivalence of infinite matrix subrings,* J. Algebra, **159**, 423–435, 1993.