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Semi-Analytical Option Pricing Under Double Heston Jump-Diffusion Hybrid Model

Rehez Ahlip[a*], Laurence A. F. Park[a] and Ante Prodan[a]

[a]*School of Computing, Engineering and Mathematics , Western Sydney University, Australia*
[*]*Corresponding author*

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Abstract

We examine European call options in the jump-diffusion version of the Double Heston stochastic volatility model for the underlying price process to provide a more flexible model for the term structure of volatility. We assume, in addition, that the stochastic interest rate is governed by the Cox– Ross – Ingersoll (CIR) dynamics. The instantaneous volatilities are correlated with the dynamics of the stock price process, whereas the short-term rate is assumed to be independent of the dynamics of the price process and its volatility. The main result furnishes a semi-analytical formula for the price of the European call option in the hybrid call option/interest rates model. Numerical results show that the model implied volatilities are comparable for in-sample but outperform out-of-sample implied volatilities compared to the benchmark Heston model [1], and Double Heston volatility model put forward by Christoffersen *et al.* [2] for calls on the S&P 500 index.

1. Introduction

In this paper we derive a semi-analytical pricing formula for European options in a model where the volatility of the stock price process is specified by a jump diffusion version of double Heston volatility model considered by Christoffersen et al.[2], whereas, the interest rate is governed by CIR dynamics postulated in Cox *et al.* [3]. In particular, the model put forward in the present work allows for a non-zero correlation between the stock price process and its instantaneous volatilities. According to the model given by (2.1), the CIR interest rate processes are independent of one another, and they are also independent of the stock price process and its volatility, which in turn is jointly governed by a jump process an extension of Heston's model. It is well established that the Heston model is not always able to fit the implied volatility smile very well, particularly at short maturities Gatheral [4]. Further, these models are particularly restrictive in their modeling of the relationship between the volatility level and the slope of the smirk, crucially the Heston one factor model can generate steep smirks at a given volatility level but cannot generate both for a given parametrization. Christofferson *et al.* [2], considered a two-factor structure for the volatility and demonstrate that the two-factor model gives much more flexibility in controlling the level and slope of the smirk. In their empirical estimates, one of the factors has a high mean reversion and determines the correlation between the short-term returns and variance. The other factor has lower mean reversion and determines the correlation between long-term returns and variance. Recchioni *et al.* [5] consider a two factor model, specifically, the dynamics of the asset price is described through two stochastic factors, one related to the stochastic volatility and the second to the stochastic interest rate.

In papers by Bakshi *et al.* [6], Bates [7] and Duffie *et al.* [8], the authors showed that stochastic volatility models do not offer reliable prices for close to expiration derivatives. This motivated Bates [7] and Bakshi *et al.* [6] to introduce jumps to the dynamics of the underlying. However, as observed by Andersen and Andreasen [9] and Alizadeh *et al.* (2002), the addition of jumps to the dynamics of the underlying is not sufficient to capture the sudden increase in volatility due to market turbulence. Since the overall volatility in financial markets consists of a highly persistent slow moving and a rapid moving components, Eraker *et al.* [10] proposed to introduce jump process to the dynamics of the volatility process in order to enhance the cross-sectional impact on option prices(see also Lewis [11]). A distinct advantage of an affine specification using Lévy processes as building block leads to analytically tractable pricing formulas for volatility derivatives, such as VIX options, as well as efficient numerical methods for pricing European options on the underlying asset, Cont *et al.* [12]. As observed by Gatheral [4] a more significant aspect as to why we consider jumps, though jumps have very little effect on the shape of the volatility surface

***Email addresses and ORCID numbers*: r.ahlip@westernsydney.edu.au, 0000-0003-2426-4265 (R. Ahlip),l.park@westernsydney.edu.au, 0000-0003-0201-4409 (L. A. F. Park),a.prodan@westernsydney.edu.au, 0000-0001-9684-7967 (A. Prodan)**

for long-dated options; the impact on the shape of the volatility surface is all at the short-expiration end, and further might explain why the skew is so steep for very short expirations and why the very short-dated term structure of skew is inconsistent with any stochastic volatility model. In this paper we have demonstrated implied volatilities based Double Heston Jump-Diffusion Hybrid Model for the underlying asset and volatility dynamics clearly outperform implied volatilities based on single and Double Heston volatility models when compared with market implied volatilities compatible with observations of Carr *et al.* [13] and Christofferson *et al.* [2] with regard to out-of-sample implied volatilities. Van Haastrecht *et al.* [14] have extended the stochastic volatility model of Schöbel and Zhu [15] to equity/currency derivatives by including stochastic interest rates and assuming all driving model factors to be instantaneously correlated. Since their model is based on the Gaussian processes, it enjoys analytical tractability even in the most general case of a full correlation structure. On the other hand,, when the squared volatility is driven by the CIR process and the interest rate is driven either by the Vasicek [16] or the Cox *et al.* [3] process, a full correlation structure leads to intractability of equity options even under a partial correlation of the driving factors, as have been documented by, among others, Van Haastrecht and Pelsser [17] and Grzelak and Oosterlee [18], [19] who examined, in particular, the Heston/Vasicek and Heston/CIR hybrid models (see also Grzelak *et al.* [20], where the Schöbel–Zhu/Hull–White and Heston/Hull–White models for equity derivatives are studied). Andrei Cozma *et al.* [21] consider the Heston-CIR stochastic-local volatility model in the context of foreign exchange markets under a full correlation structure. They derive a full truncation scheme for simulating the stochastic volatility component and the stochastic domestic and foreign interest rates. More recently Andrei Cozma *et al.* [21] propose a calibration technique for four-factor foreign-exchange hybrid local-stochastic volatility models (LSV) with stochastic short rates. However, their model specification do not include jumps . In this paper we do not follow this line of research here and we focus instead on finding a semi-analytical solution, since this goal can be achieved under Assumptions (A.1)–(A.6).
In this paper we extend the results put forward in Ahlip-Rutkowski [22] by considering the double Heston Volatility model, further we provide a complete pricing formula which speeds numerical calibration substantially (refer to Lemma 4.3) Our goal is to derive a semi-analytical solution for prices of plain-vanilla options in a model in which the volatility components are specified by the extended double Heston model with log-normal and exponential jumps, whereas the short-term interest rate is governed by the independent CIR processes. The model thus incorporates important empirical characteristics of stock price return variability: (a) the correlation between the stock price dynamics and its stochastic volatility, (b) the presence of jumps in the stock price process and in one of the stochastic factors and a second stochastic factor the usual Heston volatility and (c) the random character of interest rate. The practical importance of this feature of newly developed equity models is rather clear in view of the existence of complex equity products that have a short lifetime and are sensitive to smiles or skews in the market.
The paper is organised as follows. In Section 2, we set the option pricing model examined in this work. The options pricing problem is introduced in Section 3. The main result, Theorem 4.1 of Section 4, furnishes the pricing formula for European call options. And in particular the result in Lemma 4.3 is crucial in the derivation of the semi analytical pricing formula Section 4,which in turn significantly speeds up calibration of the model parameters to market and most important the model implied volatility surface . It is worth stressing that the independence of volatility and interest rates appears to be a crucial assumption from the point of view of analytical tractability and thus it cannot be relaxed. Numerical illustrations of our method are provided in Section 5 where the Single Heston, Double Heston and Double Heston jump-diffusion models are compared applied to S&P 500 index data and further our model can fit market implied volatilities across strikes and maturities particularly well for out-of-sample options.

2. The double Heston-Jump diffusion/CIR model

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be an underlying probability space. Let the stock price process $S=(S_t)_{t\in[0,T]}$, its instantaneous squared volatility $v=(v_t)_{t\in[0,T]}$, the short-term interest rate $r=(r_t)_{t\in[0,T]}$ be governed by the following system of SDEs:

$$\begin{cases} dS_t = S_t\,(r_t-\lambda_S\mu_S)\,dt + S_t\,\sqrt{v_t}\,dW_t^S + S_t\,\sqrt{\widehat{v}_t}\,d\widehat{W}_t^S + S_t dZ_t^S, \\ dv_t = \big(\theta-\kappa v_t\big)dt + \sigma_v\sqrt{v_t}\,dW_t^v + dZ_t^v, \\ d\widehat{v}_t = \big(\widehat{\theta}-\widehat{\kappa}\widehat{v}_t\big)dt + \sigma_{\widehat{v}}\sqrt{\widehat{v}_t}\,d\widehat{W}_t^v, \\ dr_t = \big(a-br_t\big)dt + \sigma_r\sqrt{r_t}\,dW_t^r. \end{cases} \tag{2.1}$$

We work under the following standing assumptions:

(A.1) Processes $W^S=(W_t^S)_{t\in[0,T]}$, $W^v=(W_t^v)_{t\in[0,T]}$ are correlated Brownian motions with a constant correlation coefficient, so that the quadratic covariation between the processes W^S and W^v satisfies $d[W^S,W^v]_t=\rho\,dt$ for some constant $\rho\in[-1,1]$.
(A.2) Processes $\widehat{W}^S=(\widehat{W}_t^S)_{t\in[0,T]}$, $\widehat{W}^v=(\widehat{W}_t^v)_{t\in[0,T]}$ are correlated Brownian motions with a constant correlation coefficient, so that the quadratic covariation between the processes $\widehat{W}^S$ and $\widehat{W}^v$ satisfies $d[\widehat{W}^S,\widehat{W}^v]_t=\widehat{\rho}\,dt$ for some constant $\widehat{\rho}\in[-1,1]$. Further the processes $W^v=(W_t^v)_{t\in[0,T]}$ and $\widehat{W}^v=(\widehat{W}_t^v)_{t\in[0,T]}$ are independent.
(A.3) Processes $W^r=(W_t^r)_{t\in[0,T]}$ is independent of the Brownian motions W^S, $\widehat{W}^S$ and W^v, $\widehat{W}^v$.
(A.4) The process $Z_t^S=\sum_{k=1}^{N_t^S}J_k^S$ is the compound Poisson process; specifically, the Poisson process N^S has the intensity $\lambda_S>0$ and the random variables $\ln(1+J_k^S)$, $k=1,2,\dots$ have the probability distribution $N(\ln[1+\mu_S]-\frac{1}{2}\sigma_S^2,\sigma_S^2)$; hence the jump sizes $(J_k^S)_{k=1}^{\infty}$ are lognormally distributed on $(-1,\infty)$ with mean $\mu_S>-1$.
(A.5) The process $Z_t^v=\sum_{k=1}^{N_t^v}J_k^v$ is the compound Poisson process; specifically, the Poisson process N^v has the intensity $\lambda_v>0$ and the jump sizes J_k^v are exponentially distributed with mean μ_v.
(A.6) The Poisson process N^v and sequence of random variables $(J_k^v)_{k=1}^{\infty}$ are independent of the Brownian motions $W^S,W^v,\widehat{W}^S,\widehat{W}^v,W^r$.
(A.7) The model's parameters satisfy the stability conditions: $2\theta>\sigma_v^2>0$, $2\widehat{\theta}>\sigma_{\widehat{v}}^2>0$ and $2a>\sigma_r^2>0$ (see, for instance, Wong and Heyde [23]).

Note that we postulate that the instantaneous squared volatility processes v, $\widehat{v}$ the short-term interest rate r are independent stochastic processes. We will argue in what follows that this assumption is indeed crucial for analytical tractability. For brevity, we refer to the model given by SDEs (2.1) under Assumptions (A.1)–(A.6) as the Double Heston/CIR jump-diffusion hybrid model(DHJDH).

3. Call option

We will first establish the general representation for the value European call option with maturity $T>0$ and a constant strike level $K>0$. The probability measure $\mathbb{P}$ is interpreted as the spot martingale measure (i.e., the risk-neutral probability). We denote by $\mathbb{F}=(\mathscr{F}_t)_{t\in[0,T]}$ the filtration generated by the Brownian motions $W^S, W^v, \widehat{W}^v, W^r$ and the compound Poisson processes Z^S and Z^v. We write $\mathbb{E}_t^{\mathbb{P}}(\cdot)$ and $\mathbb{P}_t(\cdot)$ to denote the conditional expectation and the conditional probability under $\mathbb{P}$ with respect to the σ-field $\mathscr{F}_t$, respectively. Hence the arbitrage price $C_t(T,K)$ of the call option at time $t\in[0,T]$ is given as the conditional expectation with respect to the σ-field $\mathscr{F}_t$ of the option's payoff at expiration discounted by the money market account, that is,

$$C_t(T,K)=\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)C_T(T,K)\right\}=\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)(S_T-K)^+\right\}$$

or, equivalently,

$$C_t(T,K)=\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)S_T\mathbb{1}_{\{S_T>K\}}\right\}-K\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)\mathbb{1}_{\{S_T>K\}}\right\}.$$

Similarly, the arbitrage price of the discount bond maturing at time T equals, for every $t\in[0,T]$,

$$B(t,T)=\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\int_t^T r_u\,du\right)\right\}$$

(see Musiela and Rutkowski ([24], Chapter 14)).
As a preliminary step towards the general valuation result presented in Section 4, we state the following well-known proposition (see, e.g, Cox *et al.* [3] or Chapter 10 in Musiela and Rutkowski [24]).

Proposition 3.1. *The price at date t of the discount bond maturing at time $T>t$ in the CIR model are given by the following expressions*

$$B(t,T)=\exp\big(m(t,T)-n(t,T)r_t\big),$$

$$m(t,T)=\frac{2a}{\sigma_r^2}\log\left[\frac{\widetilde{\gamma}e^{\frac{1}{2}b(T-t)}}{\widetilde{\gamma}\cosh(\widetilde{\gamma}(T-t))+\frac{1}{2}b\sinh(\widetilde{\gamma}(T-t))}\right],$$
$$n(t,T)=\frac{\sinh(\widetilde{\gamma}(T-t))}{\widetilde{\gamma}\cosh(\widetilde{\gamma}(T-t))+\frac{1}{2}b\sinh(\widetilde{\gamma}(T-t))}.$$

and

$$\widetilde{\gamma}=\frac{1}{2}\sqrt{b^2+2\sigma_r^2}.$$

The dynamics of the bond price under the spot martingale measure $\mathbb{P}$ is given by

$$dB(t,T)=B(t,T)\big(r_t\,dt-\sigma_r n(t,T)\sqrt{r_t}\,dW_t^r\big),$$

The following result is also well known (see, for instance, Section 11.3.1 in Musiela and Rutkowski [24]).

Lemma 3.2. *The forward rate $F(t,T)$ at time t for settlement date T equals*

$$F(t,T)=\frac{S_t}{B(t,T)}. \tag{3.1}$$

Since manifestly $S_T=F(T,T)$, the option's payoff at expiration can also be expressed as follows

$$C_T(T,K)=F(T,T)\mathbb{1}_{\{F(T,T)>K\}}-K\mathbb{1}_{\{F(T,T)>K\}}.$$

Consequently, the option's value at time $t\in[0,T]$ admits the following representation

$$\begin{aligned}C_t(T,K)\quad&=\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\textstyle\int_t^T r_u\,du\right)F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\right\}\\&\quad-K\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\textstyle\int_t^T r_u\,du\right)\mathbb{1}_{\{F(T,T)>K\}}\right\}.\end{aligned}$$

In what follows, we will frequently use the notation $x_t=\ln F(t,T)$ where $t\in[0,T]$.

4. Pricing formula for the European call option

In this section we present the main result of the paper, which furnishes a semi-analytical formula for the arbitrage price of the call option of European style under the Double Heston Jump- Diffusion Hybrid model for the stock price process combined with the independent CIR model for short-term rate.

Theorem 4.1. *Let the model be given by SDEs (2.1) under Assumptions (A.1)–(A.6). Then the price of the European call option equals, for every $t \in [0,T]$,*

$$C_t(T,K) = S_t P_1\left(t,S_t,v_t,\widehat{v}_t,r_t,K\right) - KB(t,T)P_2\left(t,S_t,v_t,\widehat{v}_t,r_t,K\right)$$

where the bond price $B(t,T)$ is given in Proposition 3.1, and the functions P_1 and P_2 are given by

$$P_1\left(t,S_t,v_t,\widehat{v}_t,r_t,K\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(f_1(\phi)\,\frac{\exp(-i\phi\ln K)}{i\phi}\right)d\phi. \tag{4.1}$$

and

$$P_2\left(t,S_t,v_t,\widehat{v}_t,r_t,K\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(f_2(\phi)\,\frac{\exp(-i\phi\ln K)}{i\phi}\right)d\phi.$$

where the $\mathscr{F}_t$-conditional characteristic functions $f_j(\phi) = f_j(\phi,t,S_t,v_t,\widehat{v}_t,r_t)$, $j=1,2$ of the random variable $x_T = \ln(S_T)$ under the probability measure $\widehat{\mathbb{P}}_T$ (see Definition 4.6) and $\mathbb{P}_T$ (see Definition 4.4), respectively, are given by

$$\begin{aligned}
f_1(\phi) \;=\; & c_t \exp\left[\lambda_S\tau\left((1+\mu_S)^{i\phi}e^{-\frac{1}{2}(\phi^2+i\phi)\sigma_S^2}-1\right)\right] \\
& \times\exp\left[-\left(i\phi\lambda_S\mu_S\tau+\lambda_v\tau\left(\frac{\rho(1+i\phi)\mu_v}{\sigma_v+\rho(1+i\phi)\mu_v}\right)\right)\right] \\
& \times\exp\left[-\left(\frac{(1+i\phi)\rho}{\sigma_v}(v_t+\theta\tau)+\frac{(1+i\phi)\widehat{\rho}}{\sigma_{\widehat{v}}}(\widehat{v}_t+\widehat{\theta}\tau)\right)\right] \\
& \times\exp\left[-i\phi\left(n(t,T)r_t+a\int_t^T n(u,T)\,du\right)\right] \\
& \times\exp\left[-G_1(\tau,s_1,s_2)v_t-G_2(\tau,s_3,s_4)\widehat{v}_t-G_3(\tau,s_5,s_6)r_t\right] \\
& \times\exp\left[-\theta H_1(\tau,s_1,s_2)-\widehat{\theta}H_2(\tau,s_3,s_4)-aH_3(\tau,s_5,s_6)\right]
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
f_2(\phi) \;=\; & c_t \exp\left[\lambda_S\tau\left((1+\mu_S)^{i\phi}e^{-\frac{1}{2}(\phi^2+i\phi)\sigma_S^2}-1\right)\right] \\
& \times\exp\left[-\left(i\phi\lambda_S\mu_S\tau+\lambda_v\tau\left(\frac{i\phi\rho\mu_v}{\sigma_v+i\phi\rho\mu_v}\right)\right)\right] \\
& \times\exp\left[-\left(\frac{(i\phi)\rho}{\sigma_v}(v_t+\theta\tau)+\frac{(i\phi)\widehat{\rho}}{\sigma_{\widehat{v}}}(\widehat{v}_t+\widehat{\theta}\tau)\right)\right] \\
& \times\exp\left[(1-i\phi)\left(n(t,T)r_t+a\int_t^T n_d(u,T)\,du\right)\right] \\
& \times\exp\left[-G_1(\tau,q_1,q_2)v_t-G_2(\tau,q_3,q_4)\widehat{v}_t-G_3(\tau,q_5,q_6)r_t\right] \\
& \times\exp\left[-\theta H_1(\tau,q_1,q_2)-\widehat{\theta}H_2(\tau,q_3,q_4)-aH_3(\tau,q_5,q_6)\right]
\end{aligned} \tag{4.3}$$

where the functions G_1,G_2,G_3,H_1,H_2,H_3, are given in Lemma 4.3 and c_t equals

$$c_t = \exp\left(i\phi x_t\right) = \exp\left(i\phi\ln F(t,T)\right).$$

Moreover, the constants s_1,s_2,s_3,s_4,s_5,s_6 are given by

$$\begin{aligned}
s_1 &= -\frac{(1+i\phi)\rho}{\sigma_v}, \\
s_2 &= -\frac{(1+i\phi)^2(1-\rho^2)}{2}-\frac{(1+i\phi)\rho\kappa}{\sigma_v}+\frac{1+i\phi}{2}, \\
s_3 &= -\frac{(1+i\phi)\widehat{\rho}}{\sigma_{\widehat{v}}}, \\
s_4 &= -\frac{(1+i\phi)^2(1-\widehat{\rho})^2}{2}-\frac{(1+i\phi)\widehat{\rho}\widehat{\kappa}}{\sigma_{\widehat{v}}}+\frac{1+i\phi}{2}, \\
s_5 &= 0,\; s_6 = -i\phi,
\end{aligned} \tag{4.4}$$

and the constants $q_1, q_2, q_3, q_4, q_5, q_6$ equal

$$
\begin{aligned}
q_1 &= -\frac{i\phi\rho}{\sigma_v},\\
q_2 &= -\frac{(i\phi)^2(1-\rho^2)}{2} - \frac{i\phi\rho\kappa}{\sigma_v} + \frac{i\phi}{2},\\
q_3 &= -\frac{i\phi\widehat{\rho}}{\sigma_{\widehat{v}}},\\
q_4 &= -\frac{(i\phi)^2(1-\widehat{\rho}^2)}{2} - \frac{i\phi\widehat{\rho}\widehat{\kappa}}{\sigma_{\widehat{v}}} + \frac{i\phi}{2},\\
q_5 &= 0,\, q_6 = i\phi - 1.
\end{aligned}
\tag{4.5}
$$

4.1. Auxiliary results

The proof of Theorem 4.1 hinges on a number of lemmas. We start by stating the well known result, which can be easily obtained from Proposition 8.6.3.4 in Jeanblanc et al. [25]. Let us denote $\tau = T - t$ and let us set, for all $0 \le t < T$,

$$
J^S(t,T) := \sum_{k=N_t^S+1}^{N_T^S} \ln(1+J_k^S). \tag{4.6}
$$

Note that we use here Assumptions (A.3)–(A.5). The property (A.3) (resp. (A.4)) implies that the random variable $J^S(t,T)$ (resp. $Z_T^v - Z_t^v$) is independent of the σ-field $\mathscr{F}_t$. Let ν_1 stand for the Gaussian distribution $N\big(\ln(1+\mu_S) - \frac{1}{2}\sigma_S^2, \sigma_S^2\big)$ and let ν_2 stand for the exponential distribution with the mean μ_v.

Lemma 4.2. *(i) Under Assumptions (A.3) and (A.5), the following equalities are valid*

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}\Big\{\exp\big(i\phi J^S(t,T)\big)\Big\} &= \mathbb{E}_t^{\mathbb{P}}\Big\{\exp\Big(i\phi \textstyle\sum_{k=N_t^S+1}^{N_T^S} \ln\big(1+J_k^S\big)\Big)\Big\}\\
&= \exp\big[\lambda_S\tau \textstyle\int_{-\infty}^{+\infty} \big(e^{i\phi z}-1\big)\,\nu_1(dz)\big]\\
&= \exp\Big[\lambda_Q\tau\Big(\big(1+\mu_S\big)^{i\phi} e^{-\frac{1}{2}\sigma_S^2(\phi^2+i\phi)} - 1\Big)\Big].
\end{aligned}
$$

(ii) Under Assumptions (A.4) and (A.5), the following equalities are valid for $c = a + bi$ with $a \le 0$

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}\Big\{\exp\big(c\,(Z_T^v - Z_t^v)\big)\Big\} &= \mathbb{E}_t^{\mathbb{P}}\Big\{\exp\Big(c \textstyle\sum_{k=N_t^v+1}^{N_T^v} J_k^v\Big)\Big\}\\
&= \exp\big[\lambda_v\tau \textstyle\int_0^{+\infty} (e^{cz}-1)\,\nu_2(dz)\big]\\
&= \exp\Big[\lambda_v\tau\Big(\frac{c\mu_v}{1-c\mu_v}\Big)\Big].
\end{aligned}
$$

The next result which is crucial for the derivation of the pricing formula in the main Theorem 4.1 extends Lemma 4.2 in Ahlip and Rutkowski [22] (see also Duffie *et al.* [8]) where the model without the jump component in the dynamics of v was examined.

Lemma 4.3. *Let the dynamics of processes $v, \widehat{v}$ and r be given by SDEs (2.1) with independent Brownian motions W^v, $\widehat{W}^v$ and W^r. For any complex numbers μ_1, λ_1, μ_2, λ_2, $\widetilde{\mu}$, $\widetilde{\lambda}$, we set*

$$
\begin{aligned}
F(\tau, v_t, \widehat{v}_t, r_t) \;=\; & \mathbb{E}_t^{\mathbb{P}}\Big\{\exp\Big(-\lambda_1 v_T - \mu_1\int_t^T v_u\,du - \lambda_2\widehat{v}_T - \mu_2\int_t^T \widehat{v}_u\,du\\
& -\widetilde{\lambda} r_T - \widetilde{\mu}\int_t^T r_u\,du\Big)\Big\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
F(\tau, v_t, \widehat{v}_t, r_t) = \;& \exp\big[-G_1(\tau,\lambda_1,\mu_1)v_t - G_2(\tau,\lambda_2,\mu_2)\widehat{v}_t - (G_3(\tau,\widetilde{\lambda},\widetilde{\mu})r_t\big]\\
& \times \exp[-\theta H_1(\tau,\lambda_1,\mu_1) - \widehat{\theta} H_2(\tau,\lambda_2,\mu_2) - aH_3(\tau,\widetilde{\lambda},\widetilde{\mu})]
\end{aligned}
$$

where

$$
\begin{aligned}
G_1(\tau,\lambda_1,\mu_1) &= \frac{\lambda_1[(\gamma_1+\kappa) + e^{\gamma_1\tau}(\gamma_1-\kappa)] + 2\mu_1(e^{\gamma_1\tau}-1)}{\sigma_v^2\,\lambda_1\,(e^{\gamma_1\tau}-1) + \gamma - \kappa + e^{\gamma_1\tau}(\gamma_1+\kappa)},\\
G_2(\tau,\lambda_2,\mu_2) &= \frac{\lambda_2[(\gamma_2+\widehat{\kappa}) + e^{\gamma_2\tau}(\gamma_2-\widehat{\kappa})] + 2\mu_2(e^{\gamma_2\tau}-1)}{\sigma_{\widehat{v}}^2\,\lambda_2\,(e^{\gamma_2\tau}-1) + \gamma_2 - \widehat{\kappa} + e^{\gamma_2\tau}(\gamma_2+\widehat{\kappa})},\\
G_3(\tau,\widetilde{\lambda},\widetilde{\mu}) &= \frac{\widetilde{\lambda}[(\widetilde{\gamma}+b) + e^{\widetilde{\gamma}\tau}(\widetilde{\gamma}-b)] + 2\widetilde{\mu}(e^{\widetilde{\gamma}\tau}-1)}{\sigma_r^2\,\widetilde{\lambda}\,\big(e^{\widetilde{\gamma}\tau}-1\big) + \widetilde{\gamma} - b + e^{\widetilde{\gamma}\tau}(\widetilde{\gamma}+b)},
\end{aligned}
$$

and

$$\begin{aligned}
H_1(\tau,\lambda_1,\mu_1) = &-\frac{2}{\sigma_v^2}\ln\left(\frac{2\gamma_1 e^{[(\gamma_1+\kappa)\tau]/2}}{\sigma_v^2\lambda_1\left(e^{\gamma_1\tau}-1\right)+\gamma_1-\kappa+e^{\gamma_1\tau}(\gamma_1+\kappa)}\right)\\
&+\frac{2\lambda_v\mu_v\sigma_v^2}{\theta(\sigma_v^2+2\mu_v\alpha_1)(\sigma_v^2+2\mu_v\beta_1)}\ln\left(\frac{(\sigma_v^2+2\beta_1\mu_v)+\Gamma_1(\sigma_v^2+2\alpha_1\mu_v)e^{\gamma_1\tau}}{(\sigma_v^2+2\beta_1\mu_v)+\Gamma_1(\sigma_v^2+2\alpha_1\mu_v)}\right)\\
&+\frac{2\lambda_v\mu_v\beta_1}{\theta(\sigma_v^2+2\beta_1\mu_v)}\tau,\\
H_2(\tau,\lambda_2,\mu_2) = &-\frac{2}{\sigma_{\widehat{v}}^2}\ln\left(\frac{2\gamma_2 e^{[(\gamma_2+\widehat{\kappa})\tau]/2}}{\sigma_{\widehat{v}}^2\lambda_2\left(e^{\gamma_2\tau}-1\right)+\gamma_2-\widehat{\kappa}+e^{\gamma_2\tau}(\gamma_2+\widehat{\kappa})}\right),\\
H_3(\tau,\widetilde{\lambda},\widetilde{\mu}) = &-\frac{2}{\sigma_r^2}\ln\left(\frac{2\widetilde{\gamma}e^{\frac{(\widetilde{\gamma}+b)\tau}{2}}}{\sigma_r^2\widetilde{\lambda}\left(e^{\widetilde{\gamma}\tau}-1\right)+\widetilde{\gamma}-b+e^{\widetilde{\gamma}\tau}(\widetilde{\gamma}+b)}\right),
\end{aligned}$$

where we denote $\gamma_1=\sqrt{\kappa^2+2\sigma_v^2\mu_1}$, $\gamma_2=\sqrt{\hat{\kappa}^2+2\sigma_{\hat{v}}^2\mu_2}$, $\widetilde{\gamma}=\sqrt{b^2+2\sigma_r^2\widetilde{\mu}}$,
$\alpha_1=\frac{-\kappa+\gamma_1}{2}$, $\beta_1=\frac{-\kappa-\gamma_1}{2}$, $\Gamma_1=\frac{2\beta_1-\lambda_1\sigma_v^2}{\lambda_1\sigma_v^2-2\alpha_1}$.

Proof. For the reader's convenience, we sketch the proof of the lemma. Let us set, for $t\in[0,T]$,

$$M_t = F(\tau,v_t,\widehat{v}_t,r_t)\exp\left(-\mu_1\int_0^t v_u\,du-\mu_2\int_0^t\widehat{v}_u\,du-\widetilde{\mu}\int_0^t r_u\,du\right). \tag{4.7}$$

Then the process $M=(M_t)_{t\in[0,T]}$ satisfies

$$M_t=\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left(-\lambda_1 v_T-\mu_1\int_0^T v_u\,du-\lambda_2\widehat{v}_T-\mu_2\int_0^T\widehat{v}_u\,du-\widetilde{\lambda}r_T-\widetilde{\mu}\int_0^T r_u\,du\right)\right\}$$

and thus it is an $\mathbb{F}$-martingale under $\mathbb{P}$. By applying the Itô formula to the right-hand side in (4.7) and by setting the drift term in the dynamics of M to be zero, we deduce that the function $F(\tau,v,\widehat{v},r,\widehat{r})$ satisfies the following partial integro-differential equation (PIDE)

$$\begin{aligned}
&-\frac{\partial F}{\partial\tau}+\frac{1}{2}\sigma_v^2 v\frac{\partial^2F}{\partial v^2}+\lambda_v\int_0^\infty\big(F(\tau,v+z,r)-F(\tau,v,r)\big)\nu_2(dz)\\
&+\frac{1}{2}\sigma_{\widehat{v}}^2\widehat{v}\frac{\partial^2F}{\partial\widehat{v}^2}+\frac{1}{2}\sigma_r^2 r\frac{\partial^2F}{\partial r^2}+(\theta-\kappa v)\frac{\partial F}{\partial v}+(\widehat{\theta}-\widehat{\kappa}\widehat{v})\frac{\partial F}{\partial\widehat{v}}\\
&+(a-br)\frac{\partial F}{\partial r}-(\mu_1 v+\mu_2\widehat{v}+\widetilde{\mu}r)F=0
\end{aligned}$$

with the initial condition $F(0,v,\widehat{v},r)=\exp(-\lambda_1 v-\lambda_2\widehat{v}-\widetilde{\lambda}r)$. We search for a solution to this PIDE in the form

$$\begin{aligned}
F(\tau,v,r,\widehat{r}) = \quad &\exp\big[-G_1(\tau,\lambda_1,\mu_1)v-G_2(\tau,\lambda_2,\mu_2)\widehat{v}-G_3(\tau,\widetilde{\lambda},\widetilde{\mu})r\\
&-\theta H_1(\tau,\lambda_1,\mu_1)-\widehat{\theta}H_2(\tau,\lambda_2,\mu_2)-aH_3(\tau,\widetilde{\lambda},\widetilde{\mu})\big]
\end{aligned}$$

with

$$G_1(0,\lambda_1,\mu_1)=\lambda_1,\quad G_2(0,\lambda_2,\mu_2)=\lambda_2,\quad G_3(0,\widetilde{\lambda},\widetilde{\mu})=\widetilde{\lambda},$$

and

$$H_1(0,\lambda_1,\mu_1)=H_2(0,\lambda_2,\mu_2)=H_3(0,\widetilde{\lambda},\widetilde{\mu})=0.$$

By substituting this expression in the PIDE and using part (ii) in Lemma 4.2, we obtain the following system of ODEs for the functions G_1,G_2,G_3,H_1,H_2,H_3 (for brevity, we suppress the last three arguments)

$$\begin{aligned}
\frac{\partial G_1(\tau)}{\partial\tau}&=-\frac{1}{2}\sigma_v^2G_1^2(\tau)-\kappa G_1(\tau)+\mu_1,\\
\frac{\partial H_1(\tau)}{\partial\tau}&=G_1(\tau)+\frac{\lambda_v}{\theta}\left(\frac{\mu_vG_1}{1+\mu_vG_1(\tau)}\right)\\
\frac{\partial G_2(\tau)}{\partial\tau}&=-\frac{1}{2}\sigma_{\widehat{v}}^2G_2^2(\tau)-\widehat{\kappa}G_2(\tau)+\mu_2,\\
\frac{\partial H_2(\tau)}{\partial\tau}&=G_2(\tau),\\
\frac{\partial G_3(\tau)}{\partial\tau}&=-\frac{1}{2}\sigma_r^2G_3^2(\tau)-bG_3(\tau)+\widetilde{\mu}.\\
\frac{\partial H_3(\tau)}{\partial\tau}&=G_3(\tau),
\end{aligned}$$

By solving these equations, we obtain the stated formulae. □

Under the assumptions of Lemma 4.3, it is possible to factorise F as a product of two conditional expectations. This means that the functions G_1 (H_1), G_2 (H_2) and G_3 (H_3) are of the same form, except that they correspond to different sets of parameters.
We now introduce a convenient change of the underlying probability measure, from the spot martingale measure $\mathbb{P}$ to the forward martingale measure $\mathbb{P}_T$.

Definition 4.4. The *The T- forward martingale measure* $\mathbb{P}_T$, equivalent to $\mathbb{P}$ on $(\Omega, \mathscr{F}_T)$, is defined by the Radon-Nikodým derivative process $\eta = (\eta_t)_{t\in[0,T]}$ where

$$\eta_t = \frac{d\mathbb{P}_T}{d\mathbb{P}}\Big|_{\mathscr{F}_t} = \exp\left(-\int_0^t \sigma_r n(u,T)\sqrt{r_u}\, dW_u^r - \frac{1}{2}\int_0^t \sigma_r^2 n^2(u,T) r_u\, du\right). \tag{4.8}$$

An application of Girsanov's theorem shows that the process $W^T = (W_t^T)_{t\in[0,T]}$, which is given by the equality

$$W_t^T = W_t^r + \int_0^t \sigma_r n(u,T)\sqrt{r_u}\, du, \tag{4.9}$$

is the Brownian motion under the domestic forward martingale measure $\mathbb{P}_T$. Using the standard change of a numéraire technique, one can check that the price of the European foreign exchange call option admits the following representation under the probability measure $\mathbb{P}_T$

$$C_t(T,K) = B_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\left(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\right) - KB_d(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\left(\mathbb{1}_{\{F(T,T)>K\}}\right). \tag{4.10}$$

The following auxiliary result is easy to establish and thus its proof is omitted. Recall that $J^S(t,T)$ is given by equality (4.6).

Lemma 4.5. *Under Assumptions (A.1)–(A.6), the dynamics of the forward stock price dynamics $F(t,T)$ under the forward martingale measure $\mathbb{P}_T$ are given by the SDE*

$$dF(t,T) = F(t,T)\left(dZ_t^S - \lambda_S\mu_S dt + \sqrt{v_t}\, dW_t^S + \sqrt{\widehat{v}_t}\, d\widehat{W}_t^S + \sigma_d n_d(t,T)\sqrt{r_t}\, dW_t^T\right)$$

or, equivalently,

$$F(T,T) = F(t,T)\exp\left(J^S(t,T) - \lambda_S\mu_S(T-t) + \int_t^T \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2 du\right)$$

where the dot $\cdot$ denotes the inner product in $\mathbb{R}^3$, $(\widetilde{\sigma}_F(t,T))_{t\in[0,T]}$ is the $\mathbb{R}^3$-valued process (row vector) given by

$$\widetilde{\sigma}_F(t,T) = \left[\sqrt{v_t},\, \sqrt{\widehat{v}_t},\, \sigma_r n(t,T)\sqrt{r_t}\right]$$

and $\widetilde{W}^T = (\widetilde{W}_t^T)_{t\in[0,T]}$ is the $\mathbb{R}^3$-valued process (column vector) given by $\widetilde{W}^T = \left[W^S, \widehat{W}^S, W^T\right]^*$.

Under Assumptions (A.1)–(A.6), the process $\widetilde{W}^T$ is the three-dimensional standard Brownian motion under $\mathbb{P}_T$. In view of Lemma 4.5, we have that

$$\begin{aligned}
&B(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\left(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\right)\\
&= B(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\Bigg\{F(t,T)\exp\Bigg(J^S(t,T) - \lambda_S\mu_S(T-t)\\
&\qquad + \int_t^T \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2 du\Bigg)\mathbb{1}_{\{F(T,T)>K\}}\Bigg\}\\
&= S_t\,\mathbb{E}_t^{\mathbb{P}_T}\Bigg\{\exp\Bigg(J^S(t,T) - \lambda_S\mu_S(T-t)\\
&\qquad + \int_t^T \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2 du\Bigg)\mathbb{1}_{\{F(T,T)>K\}}\Bigg\}.
\end{aligned}$$

To deal with the first term in the right-hand side of (4.10), we introduce another auxiliary probability measure.

Definition 4.6. The *modified forward martingale measure* $\widehat{\mathbb{P}}_T$, equivalent to $\mathbb{P}_T$ on $(\Omega, \mathscr{F}_T)$, is defined by the Radon-Nikodým derivative process $\widehat{\eta} = (\widehat{\eta}_t)_{t\in[0,T]}$ where

$$\widehat{\eta}_t = \frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}_T}\Big|_{\mathscr{F}_t} = \exp\left(\int_0^t \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_0^t \|\widetilde{\sigma}_F(u,T)\|^2 du\right).$$

Using Lemma 4.5 and equation (3.1), we obtain

$$B(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\left(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\right) = S_t\,\frac{\mathbb{E}_t^{\mathbb{P}_T}\left(\mathbb{1}_{\{F(T,T)>K\}}\widehat{\eta}_T\right)}{\mathbb{E}_t^{\mathbb{P}_T}\left(\widehat{\eta}_T\right)}$$

and thus the Bayes formula and Definition 4.6 yield

$$B(t,T)\,\mathbb{E}_t^{\mathbb{P}_T}\left(F(T,T)\mathbb{1}_{\{F(T,T)>K\}}\right) = S_t\,\mathbb{E}_t^{\widehat{\mathbb{P}}_T}\left(\mathbb{1}_{\{F(T,T)>K\}}\right).$$

This shows that $\widehat{\mathbb{P}}_T$ is a martingale measure associated with the choice of the price process S_t as a numéraire asset.

Lemma 4.7. *The price of the call option satisfies*

$$C_t(T,K) = S_t\,\widehat{\mathbb{P}}_T\left(S_T > K\,|\,\mathscr{F}_t\right) - KB(t,T)\,\mathbb{P}_T\left(S_T > K\,|\,\mathscr{F}_t\right)$$

or, equivalently,

$$C_t(T,K) = S_t\,\widehat{\mathbb{P}}_T\left(x_T > \ln K\,|\,\mathscr{F}_t\right) - KB(t,T)\,\mathbb{P}_T\left(x_T > \ln K\,|\,\mathscr{F}_t\right). \tag{4.11}$$

To complete the proof of Theorem 4.1, it remains to evaluate the conditional probabilities given in formula (4.11). By another application of Girsanov's theorem, one can check that the process $(S,v,\widehat{v},r)$ has the Markov property under the probability measures $\mathbb{P}_T$ and $\widehat{\mathbb{P}}_T$. In view of Proposition 3.1 and Lemma 3.2, the random variable x_T is a function of S_T and r_T. Hence it follows that

$$C_t(T,K) = S_t\,P_1(t,S_t,v_t,\widehat{v}_t,r_t,K) - KB(t,T)P_2(t,S_t,v_t,\widehat{v}_t,r_t,K) \tag{4.12}$$

where we denote

$$\begin{aligned}
P_1(t,S_t,v_t,\widehat{v}_t,r_t,K) &= \widehat{\mathbb{P}}_T(x_T > \ln K\,|\,S_t,v_t,\widehat{v}_t,r_t),\\
P_2(t,S_t,v_t,\widehat{v}_t,r_t,K) &= \mathbb{P}_T(x_T > \ln K\,|\,S_t,v_t,\widehat{v}_t,r_t).
\end{aligned}$$

To obtain explicit formulae for the conditional probabilities above, it suffices to derive the corresponding conditional characteristic functions

$$\begin{aligned}
f_1(\phi,t,S_t,v_t,\widehat{v}_t,r_t) &= \mathbb{E}_t^{\widehat{\mathbb{P}}_T}\left[\exp(i\phi x_T)\right],\\
f_2(\phi,t,S_t,v_t,\widehat{v}_t,r_t) &= \mathbb{E}_t^{\mathbb{P}_T}\left[\exp(i\phi x_T)\right].
\end{aligned}$$

The idea is to use the Radon-Nikodým derivatives in order to obtain convenient expressions for the characteristic functions in terms of conditional expectations under the spot martingale measure $\mathbb{P}$. The following lemma will allow us to achieve this goal.

Lemma 4.8. *The following equality holds*

$$\begin{aligned}
\frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}}\Big|_{\mathscr{F}_t} &= \exp\left(\int_0^t \sqrt{v_u}\,dW_u^S + \int_0^t \sqrt{\widehat{v}_u}\,d\widehat{W}_u^S\right)\\
&\quad \times \exp\left(-\frac{1}{2}\int_0^t (v_u+\widehat{v}_u)\,du\right).
\end{aligned}$$

Proof. Straightforward computations show that

$$\begin{aligned}
\frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}}\Big|_{\mathscr{F}_t} &= \frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}_T}\Big|_{\mathscr{F}_t}\frac{d\mathbb{P}_T}{d\mathbb{P}}\Big|_{\mathscr{F}_t}\\
&= \exp\left(\int_0^t \widetilde{\sigma}_F(u,T)\cdot d\widetilde{W}_u^T - \frac{1}{2}\int_0^t \|\widetilde{\sigma}_F(u,T)\|^2\,du\right)\\
&\quad \times \exp\left(-\int_0^t \sigma_r n(u,T)\sqrt{r_u}\,dW_u^r - \frac{1}{2}\int_0^t \sigma_r^2 n^2(u,T) r_u\,du\right)\\
&= \exp\left(\int_0^t \left(\sqrt{v_u}\,dW_u^S + \sqrt{\widehat{v}_u}\,d\widehat{W}_u^S + \sigma_d n_d(u,T)\sqrt{r_u}\,dW_u^T\right)\right)\\
&\quad \times \exp\left(-\frac{1}{2}\int_t^T \left(v_u+\widehat{v}_u+\sigma_r^2 n^2(u,T) r_u\right)du\right)\\
&\quad \times \exp\left(-\int_0^t \sigma_r n(u,T)\sqrt{r_u}\,dW_u^r - \frac{1}{2}\int_0^t \sigma_r^2 n^2(u,T) r_u\,du\right).
\end{aligned}$$

Using (4.9), we now obtain

$$\begin{aligned}
\frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}}\Big|_{\mathscr{F}_t} &= \exp\left(\int_0^t \sqrt{v_u}\,dW_u^S + \sqrt{\widehat{v}_u}\,d\widehat{W}_u^S\right)\\
&\quad \times \exp\left(-\frac{1}{2}\int_0^t (v_u+\widehat{v}_u)\,du\right),
\end{aligned}$$

which is the desired expression. □

In view of the formula established in Lemma 4.8 and the abstract Bayes formula, to compute $f_1(\phi) = f_1(\phi,t,S_t,v_t,\widehat{v}_t,r_t)$, it suffices to focus on the following conditional expectation under $\mathbb{P}$

$$\begin{aligned}
f_1(\phi) &= \mathbb{E}_t^{\mathbb{P}}\Bigg\{\exp\left(i\phi x_T\right)\exp\bigg(\textstyle\int_t^T \sqrt{v_u}\,dW_u^S + \int_t^T \sqrt{\widehat{v}_u}\,d\widehat{W}_u^S\\
&\qquad -\tfrac{1}{2}\textstyle\int_t^T \left(v_u+\widehat{v}_u\right)du\bigg)\Bigg\}.
\end{aligned} \tag{4.13}$$

Similarly, in view of formula (4.8), we obtain for $f_2(\phi) = f_2(\phi,t,S_t,v_t,\widehat{v}_t,r_t)$

$$f_2(\phi) = \mathbb{E}_t^{\mathbb{P}}\left\{\exp(i\phi x_T)\exp\left[-\int_t^T \sigma_r n(u,T)\sqrt{r_u}\,dW_u^r - \frac{1}{2}\int_t^T \sigma_r^2 n^2(u,T) r_u\,du\right]\right\}. \tag{4.14}$$

To proceed, we will need the following result, which is an immediate consequence of Lemma 4.5.

Corollary 4.9. *Under Assumptions (A.1)–(A.4), the process $x_t = \ln F(t,T)$ admits the following representation under the forward martingale measure $\mathbb{P}_T$*

$$x_T = x_t + \int_t^T \widetilde{\sigma}_F(u,T) \cdot d\widetilde{W}_u^T - \frac{1}{2}\int_t^T \|\widetilde{\sigma}_F(u,T)\|^2 du + J^S(t,T) - \lambda_S \mu_S (T-t)$$

or, more explicitly,

$$\begin{aligned} x_T = x_t \quad & + \int_t^T \sqrt{v_u}\, dW_u^S + \int_t^T \sqrt{\widehat{v}_u}\, d\widehat{W}_u^S + \int_t^T \sigma_r n(u,T)\sqrt{r_u}\, dW_u^T \quad - \frac{1}{2}\int_t^T \left(v_u + \widehat{v}_u + \sigma_r^2 n^2(u,T) r_u\right) du \\ & + \sum_{k=N_t^S+1}^{N_T^S} \ln(1+J_k^S) - \lambda_S \mu_S (T-t). \end{aligned}$$

Using equality (4.13) and Corollary 4.9, we obtain

$$\begin{aligned} f_1(\phi) \quad = \quad & \mathbb{E}_t^{\mathbb{P}}\Bigg\{ \exp\left(i\phi x_T\right) \exp\left[\int_t^T \sqrt{v_u}\, dW_u^S + \int_t^T \sqrt{\widehat{v}_u}\, d\widehat{W}_u^S \right. \\ & \left. - \frac{1}{2}\int_t^T \left(v_u + \widehat{v}_u\right) du \right] \Bigg\} \end{aligned}$$

so that

$$\begin{aligned} f_1(\phi) \quad = \quad & \mathbb{E}_t^{\mathbb{P}}\Bigg\{ \exp\left[i\phi\left(x_t + \int_t^T \sqrt{v_u}\, dW_u^S + \int_t^T \sqrt{\widehat{v}_u}\, d\widehat{W}_u^S\right)\right] \\ & \times \exp\left[i\phi\left(\int_t^T \sigma_r n(u,T)\sqrt{r_u}\, dW_u^T\right)\right] \\ & \times \exp\left[-\frac{i\phi}{2}\int_t^T \left(v_u + \widehat{v}_u + \sigma_r^2 n^2(u,T) r_u\right) du\right] \\ & \times \exp\left[\int_t^T \sqrt{v_u}\, dW_u^S + \int_t^T \sqrt{\widehat{v}_u}\, d\widehat{W}_u^S\right] \\ & \times \exp\left[-\frac{1}{2}\int_t^T \left(v_u + \widehat{v}_u\right) du\right] \\ & \times \exp\left[i\phi J^S(t,T) - i\phi\lambda_S \mu_S (T-t)\right] \Bigg\}. \end{aligned}$$

We denote $\alpha = 1 + i\phi$, $\beta = i\phi$ and $c_t = \exp(i\phi x_t)$. After simplifications and rearrangement, the formula above becomes

$$\begin{aligned} f_1(\phi) \quad = \quad & c_t\, \mathbb{E}_t^{\mathbb{P}}\Bigg\{ \exp\left[\alpha\left(\int_t^T \sqrt{v_u}\, dW_u^S + \int_t^T \sqrt{\widehat{v}_u}\, d\widehat{W}_u^S - \frac{1}{2}\int_t^T v_u\, du - \frac{1}{2}\int_t^T \widehat{v}_u\, du\right)\right] \\ & \times \exp\left[\beta\left(\int_t^T \sigma_r n(u,T)\sqrt{r_u}\, dW_u^T - \frac{1}{2}\int_t^T \sigma_r^2 n^2(u,T) r_u\, du\right)\right] \\ & \times \exp\left[\beta J^S(t,T) - \beta\lambda_S \mu_S (T-t)\right] \Bigg\}. \end{aligned}$$

In view of Assumptions (A.1)–(A.6), we may use the following representation for the Brownian motion W^Q

$$W_t^S = \rho_1 W_t^v + \sqrt{1-\rho^2}\, W_t \tag{4.15}$$

where $W = (W_t)_{t\in[0,T]}$ is a Brownian motion under $\mathbb{P}$ independent of the Brownian motions $W^S, W^v, \widehat{W}^v$ and W^r.

$$\widehat{W}_t^S = \rho_2 \widehat{W}_t^{\widehat{v}} + \sqrt{1-\widehat{\rho}^2}\, \widehat{W}_t$$

where $\widehat{W} = (\widehat{W}_t)_{t\in[0,T]}$ is a Brownian motion under $\mathbb{P}$ independent of the Brownian motions $W^v, \widehat{W}^v, W^S$ and W^r . Consequently, the conditional characteristic function $f_1(\phi)$ can be represented in the following way

$$\begin{aligned} f_1(\phi) \quad = \quad & c_t\, \mathbb{E}_t^{\mathbb{P}}\Bigg\{ \exp\left[\alpha\rho\int_t^T \sqrt{v_u}\, dW_u^v + \alpha\sqrt{1-\rho^2}\int_t^T \sqrt{v_u}\, dW_u - \frac{\alpha}{2}\int_t^T v_u\, du\right] \\ & \times \exp\left[\alpha\widehat{\rho}\int_t^T \sqrt{\widehat{v}_u}\, d\widehat{W}_u^{\widehat{v}} + \alpha\sqrt{1-\widehat{\rho}^2}\int_t^T \sqrt{\widehat{v}_u}\, d\widehat{W}_u - \frac{\alpha}{2}\int_t^T \widehat{v}_u\, du\right] \\ & \times \exp\left[\beta\left(\int_t^T \sigma_r n(u,T)\sqrt{r_u}\, dW_u^T - \frac{1}{2}\int_t^T \sigma_r^2 n^2(u,T) r_u\, du\right)\right] \\ & \times \exp\left[\beta J^S(t,T) - \beta\lambda_S (T-t)\mu_S\right] \Bigg\}. \end{aligned} \tag{4.16}$$

By combining Proposition 3.1 with Definition 4.4, we obtain the following auxiliary result, which will be helpful in the proof of Theorem 4.1.

Lemma 4.10. *Given the dynamics (2.1) of processes $v, \widehat{v}$ and r and formula (4.9), we obtain the following equalities*

$$
\begin{aligned}
&\int_t^T \sqrt{v_u}\,dW_u^v = \frac{1}{\sigma_v}\Big(v_T - v_t - \theta\tau + \kappa\int_t^T v_u\,du - (Z_T^v - Z_t^v)\Big),\\
&\int_t^T \sqrt{\widehat{v}_u}\,d\widehat{W}_u^{\widehat{v}} = \frac{1}{\sigma_{\widehat{v}}}\Big(\widehat{v}_T - \widehat{v}_t - \widehat{\theta}\tau + \widehat{\kappa}\int_t^T \widehat{v}_u\,du\Big),\\
&\int_t^T \sigma_r n_d(u,T)\sqrt{r_u}\,dW_u^T - \frac{1}{2}\int_t^T \sigma_r^2 n^2(u,T) r_u\,du = -n(t,T)r_t - \int_t^T an(u,T)\,du + \int_t^T r_u\,du.
\end{aligned}
$$

Proof. The first asserted formula is an immediate consequence of (2.1). For the second, we recall that the function $n(t,T)$ is known to satisfy the following differential equation, for any fixed $T>0$,

$$
\frac{\partial n(t,T)}{\partial t} - \frac{1}{2}\sigma_r^2 n^2(t,T) - bn(t,T) + 1 = 0
$$

with the terminal condition $n(T,T)=0$. Therefore, using the Itô formula and equality (4.9), we obtain

$$
\begin{aligned}
d(n(t,T)r_t) &= r_t\,dn(t,T) + n(t,T)\,dr_t\\
&= r_t\Big(\frac{1}{2}\sigma_r^2 n^2(t,T) + bn(t,T) - 1\Big)dt + n_d(t,T)(a - br_t)\,dt + n(t,T)\sigma_d\sqrt{r_t}\,dW_t^r\\
&= \frac{1}{2}\sigma_r^2 n^2(t,T) r_t\,dt - r_t\,dt + n(t,T)a\,dt + n(t,T)\sigma_r\sqrt{r_t}\,dW_t^r\\
&= -\frac{1}{2}\sigma_r^2 n^2(t,T) r_t\,dt - r_t\,dt + n(t,T)a\,dt + n(t,T)\sigma_r\sqrt{r_t}\,dW_t^T.
\end{aligned}
$$

This yields the second asserted formula, upon integration between t and T. The derivation of the last one is based on the same arguments and thus it is omitted. □

4.2. Proof of theorem 4.1

The proof of Theorem 4.1 is split into two steps in which we deal with $f_1(\phi)$ and $f_2(\phi)$, respectively.
Step 1. We will first compute $f_1(\phi)$. By combining (4.16) with the equalities derived in Lemma 4.10, we obtain the following representation for $f_1(\phi)$

$$
\begin{aligned}
f_1(\phi) \;=\; c_t\,\mathbb{E}_t^{\mathbb{P}}\Big\{ &\exp\Big[-\frac{\alpha\rho}{\sigma_v}\Big[(v_t+\theta\tau) + \Big(\widehat{v}_t + \widehat{\theta}\tau\Big)\Big]\\
&\times\exp\Big[\Big(\frac{\alpha\rho\kappa}{\sigma_v} - \frac{\alpha}{2}\Big)\int_t^T v_u\,du + \Big(\frac{\alpha\widehat{\rho}\widehat{\kappa}}{\sigma_{\widehat{v}}} - \frac{\alpha}{2}\Big)\int_t^T \widehat{v}_u\,du\Big]\\
&\times\exp\Big[\alpha\sqrt{1-\rho^2}\int_t^T \sqrt{v_u}\,dW_u + \frac{\alpha\rho}{\sigma_v}v_T\Big]\\
&\times\exp\Big[\alpha\sqrt{1-\widehat{\rho}^2}\int_t^T \sqrt{\widehat{v}_u}\,d\widehat{W}_u + \frac{\alpha\widehat{\rho}}{\sigma_{\widehat{v}}}\widehat{v}_T\Big]\\
&\times\exp\Big[-\beta\Big(n(t,T)r_t + \int_t^T an(u,T)\,du\Big) + \beta\int_t^T r_u\,du\Big]\\
&\times\exp\Big[\beta J^S(t,T) - \beta\lambda_S\mu_S(T-t) - \frac{\alpha\rho}{\sigma_v}(Z_T^v - Z_t^v)\Big]\Big\}.
\end{aligned}
$$

Recall the well-known property that if ζ has the standard normal distribution then $\mathbb{E}\big(e^{z\zeta}\big) = e^{z^2/2}$ for any complex number $z\in\mathbb{C}$. Consequently, by conditioning first on the sample path of the process $(v,\widehat{v},r)$ and using the independence of the processes $(v,\widehat{v},r)$ and W under $\mathbb{P}$ and Lemma 4.2, we obtain

$$
\begin{aligned}
f_1(\phi) \;=\; &c_t\exp\Big[\lambda_S\tau\Big((1+\mu_S)^\beta e^{-\frac{1}{2}\beta\gamma\sigma_S^2} - 1\Big)\Big]\\
&\times\exp\Big[-\Big(\beta\lambda_S\mu_S\tau + \lambda_v\tau\frac{\rho\alpha\mu_v}{\sigma_v+\rho\alpha\mu_v} + \frac{\alpha\rho}{\sigma_v}(v_t+\theta\tau) + \frac{\alpha\widehat{\rho}}{\sigma_{\widehat{v}}}(\widehat{v}_t+\widehat{\theta}\tau)\Big)\Big]\\
&\times\exp\Big[-\beta\Big(n(t,T)r_t + \int_t^T an(u,T)\,du\Big)\Big]\\
&\times\mathbb{E}_t^{\mathbb{P}}\Big\{\exp\Big[\frac{\alpha\rho}{\sigma_v}v_T + \Big(\frac{\alpha^2(1-\rho^2)}{2} + \frac{\alpha\rho\kappa}{\sigma_v} - \frac{\alpha}{2}\Big)\int_t^T v_u\,du\Big]\\
&\times\exp\Big[\frac{\alpha\widehat{\rho}}{\sigma_{\widehat{v}}}\widehat{v}_T + \Big(\frac{\alpha^2(1-\widehat{\rho}^2)}{2} + \frac{\alpha\widehat{\rho}\widehat{\kappa}}{\sigma_{\widehat{v}}} - \frac{\alpha}{2}\Big)\int_t^T \widehat{v}_u\,du\Big]\\
&\times\exp\Big[\beta\int_t^T r_u\,du\Big]\Big\}.
\end{aligned}
$$

where we denote $\gamma = 1 - i\phi$. This in turn implies that the following equality holds

$$\begin{aligned} f_1(\phi) &= c_t \exp\left[\lambda_S \tau\left((1+\mu_S)^{\beta} e^{-\frac{1}{2}\beta\gamma\sigma_S^2} - 1\right)\right] \\ &\times \exp\left[-\left(\beta\lambda_S\mu_S\tau + \lambda_v\tau\frac{\rho\alpha\mu_v}{\sigma_v + \rho\alpha\mu_v} + \frac{\alpha\rho}{\sigma_v}(v_t + \theta\tau) + \frac{\alpha\widehat{\rho}}{\sigma_{\widehat{v}}}(\widehat{v}_t + \widehat{\theta}\tau)\right)\right] \\ &\times \exp\left[-\beta\left(n(t,T)r_t + \int_t^T an(u,T)\,du\right)\right] \\ &\times \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left[-s_1 v_T - s_2\int_t^T v_u\,du - s_3\widehat{v}_T - s_4\int_t^T \widehat{v}_u\,du\right]\right. \\ &\times \left.\exp\left[-s_5 r_T - s_6\int_t^T r_u\,du\right]\right\} \end{aligned}$$

where the constants $s_1, s_2, s_3, s_4, s_5, s_6$ are given by (4.4). A direct application of Lemma 4.3 furnishes an explicit formula for $f_1(\phi)$, as reported in the statement of Theorem 4.1.

Step 2. In order to compute the conditional characteristic function

$$f_2(\phi) = f_2(\phi, t, S_t, v_t, \widehat{v}_t, r_t) = \mathbb{E}_t^{\mathbb{P}_T}\left[\exp(i\phi x_T)\right]$$

we proceed in an analogous manner as for $f_1(\phi)$. We first recall that (see (4.14))

$$f_2(\phi) = \mathbb{E}_t^{\mathbb{P}}\left\{\exp(i\phi x_T)\exp\left[-\int_t^T \sigma_r n(u,T)\sqrt{r_u}\,dW_u^r - \frac{1}{2}\int_t^T \sigma_r^2 n^2(u,T) r_u\,du\right]\right\}.$$

Therefore, using Corollary 4.9, we obtain

$$\begin{aligned} f_2(\phi) &= c_t\,\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left[i\phi\left(\int_t^T \sqrt{v_u}\,dW_u^S + \int_t^T \sqrt{\widehat{v}_u}\,d\widehat{W}_u^S + J^S(t,T)\right)\right]\right. \\ &\times \exp\left[i\phi\left(\int_t^T \sigma_r n(u,T)\sqrt{r_u}\,dW_u^T\right)\right] \\ &\times \exp\left[-i\phi\left(\frac{1}{2}\int_t^T (v_u + \sigma_r^2 n^2(u,T) r_u)\,du\right)\right] \\ &\times \left.\exp\left[-\int_t^T \sigma_r n(u,T)\sqrt{r_u}\,dW_u^r - \frac{1}{2}\int_t^T \sigma_r^2 n^2(u,T) r_u\,du\right]\right\}. \end{aligned}$$

Consequently, using formulae (4.9), (4.15) and Lemma 4.2, we obtain the following expression for $f_2(\phi)$

$$\begin{aligned} f_2(\phi) &= c_t \exp\left[\lambda_S\tau\left((1+\mu_S)^{\beta} e^{-\frac{1}{2}\beta\gamma\sigma_S^2} - 1\right) - \beta\lambda_S\mu_S\tau\right] \\ &\times \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left[\beta\left(\rho\int_t^T \sqrt{v_u}\,dW_u^v + \sqrt{1-\rho^2}\int_t^T \sqrt{v_u}\,dW_u\right)\right]\right. \\ &\times \exp\left[\beta\left(\widehat{\rho}\int_t^T \sqrt{\widehat{v}_u}\,d\widehat{W}_u^v + \sqrt{1-\widehat{\rho}^2}\int_t^T \sqrt{\widehat{v}_u}\,d\widehat{W}_u\right)\right] \\ &\times \exp\left[-\beta\left(\frac{1}{2}\int_t^T (\widehat{v}_u + \widehat{v}_u)\,du\right)\right] \\ &\times \left.\exp\left[-\gamma\left(\int_t^T \sigma_r n(u,T)\sqrt{r_u}\,dW_u^r + \frac{1}{2}\int_t^T \sigma_r^2 n^2(u,T) r_u\,du\right)\right]\right\}. \end{aligned}$$

Similarly as in the case of $f_1(\phi)$, we condition on the sample path of the process $(v, \widehat{v}, r)$ and we use the postulated independence of the processes $(v, \widehat{v}, r)$ and W under $\mathbb{P}$. By invoking also Lemma 4.2, we obtain

$$\begin{aligned} f_2(\phi) &= c_t \exp\left[\lambda_S\tau\left((1+\mu_S)^{\beta} e^{-\frac{1}{2}\beta\gamma\sigma_S^2} - 1\right) - \beta\lambda_S\mu_S\tau\right] \\ &\mathbb{E}_t^{\mathbb{P}}\left\{\exp\left[\beta\rho\int_t^T \sqrt{v_u}\,dW_u^v + \frac{\beta^2(1-\rho^2) - \beta}{2}\int_t^T v_u\,du\right]\right. \\ &\times \exp\left[\beta\widehat{\rho}\int_t^T \sqrt{\widehat{v}_u}\,d\widehat{W}_u^v + \frac{\beta^2(1-\widehat{\rho}^2) - \beta}{2}\int_t^T \widehat{v}_u\,du\right] \\ &\times \left.\exp\left[-\gamma\left(\int_t^T \sigma_r n(u,T)\sqrt{r_u}\,dW_u^r + \frac{1}{2}\int_t^T \sigma_r^2 n(u,T) r_u\,du\right)\right]\right\}. \end{aligned}$$

Using Lemma 4.10, we conclude that

$$
\begin{aligned}
f_2(\phi) \quad = \quad & c_t \exp\left[\lambda_S \tau \left((1+\mu_S)^{\beta} e^{-\frac{1}{2}\beta\gamma\sigma_S^2} - 1\right)\right] \\
& \times \exp\left[-\left(\beta\lambda_S\mu_S\tau + \lambda_v\tau\frac{\rho\beta\mu_v}{\sigma_v+\rho\beta\mu_v} + \frac{\beta\rho}{\sigma_v}(v_t+\theta\tau) + \frac{\beta\widehat{\rho}}{\sigma_{\widehat{v}}}(\widehat{v}_t+\widehat{\theta}\tau)\right)\right] \\
& \times \exp\left[-\gamma\left(n(t,T)r_t + \int_t^T an(u,T)\,du\right)\right] \\
& \times \mathbb{E}_t^{\mathbb{P}}\left\{\exp\left[-q_1 v_T - q_2\int_t^T v_u\,du - q_3\widehat{v}_T - q_4\int_t^T \widehat{v}_u\,du\right]\right. \\
& \left.\times \exp\left[-q_5 r_T - q_6\int_t^T r_u\,du\right]\right\}
\end{aligned}
$$

with the coefficients $q_1, q_2, q_3, q_4, q_5, q_6$ reported in formula (4.5). Another straightforward application of Lemma 4.3 yields the closed-form expression (4.3) for the conditional characteristic function $f_2(\phi)$.
To complete the proof of Theorem 4.1, it suffices to combine formula (4.12) with the standard inversion formula (4.1) providing integral representations for the conditional probabilities

$$P_1(t,S_t,v_t,\widehat{v}_t,r_t,K) = \widehat{\mathbb{P}}_T(x_T > \ln K \,|\, S_t,v_t,\widehat{v}_t,r_t)$$

and

$$P_2(t,S_t,v_t,\widehat{v}_t,,r_t,K) = \mathbb{P}_T(x_T > \ln K \,|\, S_t,v_t,\widehat{v}_t,r_t).$$

This ends the derivation of the pricing formula for the call option. The price of the corresponding put option is readily available as well, due to the put-call parity relationship (4.17).

$$C_t(T,K) - P_t(T,K) = S_t - KB(t,T) \tag{4.17}$$

where $C_t(T,K)$ and $P_t(T,K)$ are prices of the call and put options, respectively. □

5. Model calibration and empirical analysis

In this section we estimate the parameters for DHJDH model considered in this paper using Dow Jones Industrial implied volatilities(IV) quoted May 10, 2012 [26] and compare the model's empirical performance with that of the Double Heston Model considered by Christoffersen et.al [2] and the Heston model. In this analysis we have assumed constant interest rates. Calibration of DHJDH model parameters

$$\Theta = \{\lambda_S, \mu_S, \lambda_V, \mu_V, \theta, \widehat{\theta}, \kappa, \widehat{\kappa}, \sigma_v, \sigma_{\widehat{v}}, \rho, \widehat{\rho}, \sigma_S, v_1, v_2\}$$

was performed using Interior Point optimisation. Further, the US treasury yield curve rates for one, three, six and twelve -months have been used as a proxy for the initial interest rates for the different maturities. To fit the model to market implied volatilities we use the approximation implied volatility root mean squared error(`IVRMSE`) loss function considered by Christoffersen et.al.[2], also Carr and Wu [13] and Trolle and Schwartz [27].

$$\mathtt{IVRMSE} \approx \sqrt{\frac{1}{N}\sum_{t,k}\left(\frac{C_{t,k}^M - C_{t,k}^{\Theta}}{\mathtt{BSVega}_{t,k}}\right)^2} \tag{5.1}$$

where $C_{t,k}^M$ is the market price, $C_{t,k}^{\Theta}$ is the model price,and BSVega(t,k) is the Black Scholes sensitivity of the option computed using the implied volatility from the market price of the option, $C_{t,k}^M$. Interior point optimization is used to obtain the set of parameters that minimise the objective function in equation (5.1).
Using the data from Table 1, the parameter estimates Θ for the univariate, double Heston and Double Heston Jump-Diffusion Hybrid models, along with their estimation error are found in Table 2. If we compare the calibrated parameters for the Double Heston and DHJDH models, we notice that κ, σ and v_0 are similar, implying that the calibrated Double Heston parameters can be used as a seed for when calibrating the DHJDH Model. One practical consequence of this is that the Double Heston parameters can be fitted fairly robustly using longer dated options and then jump parameters can be found to generate the extra skew for short-dated options.
The panels in figure 5.1 show the implied volatility surfaces for the double Heston and DHJDH Models for all strikes and across all times to maturities. These figures show that theoretical implied volatilities of the DHJDH model provide satisfactory approximation for the observed implied volatilities across all maturities and across all strikes but particularly outperforms out-of-sample calls for the double Heston Model across all expiries ranging from 37 to 226 days(short dated options). This improvement is achieved through the inclusion of jumps in the dynamics of the stock price and the volatility processes and using only one set of model parameters.
To visualise how well the DHJDH fits the market IV, we have provided contour plots in Figure 5.2, of the Market IV and the predicted market IV using the DHJDH model. Note that the market IV contour plot was generated using the data from Table 1 and the model contour plot was generated using the DHJDH model, therefore the resolution of the model contour is much finer since we can compute many points of the contour, while the resolution of the market contour is coarse since we are only able to use the provided data points. The difference in resolution can be seen from the straight contour lines in the market IV contour plot, while the model contour lines are much smoother due to the abundance of generated contour points from the model. Other than this, the contour plots are very similar, implying that the DHJDH model provides a good fit to the market data.

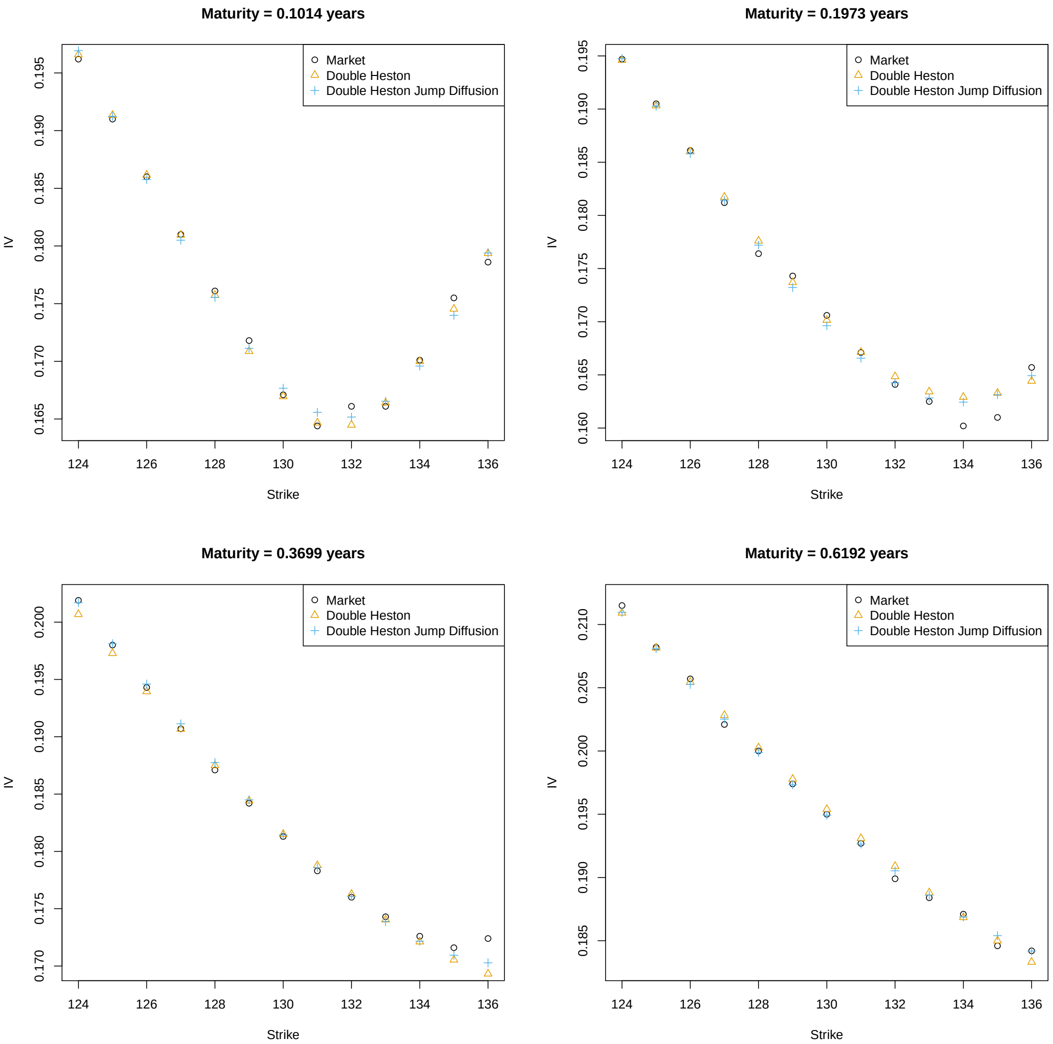

Figure 5.1: The implied volatility for various strike prices at four maturity times. Each plot shows the market IV, the calibrated Double Heston IV and the calibrated DHJDH IV.

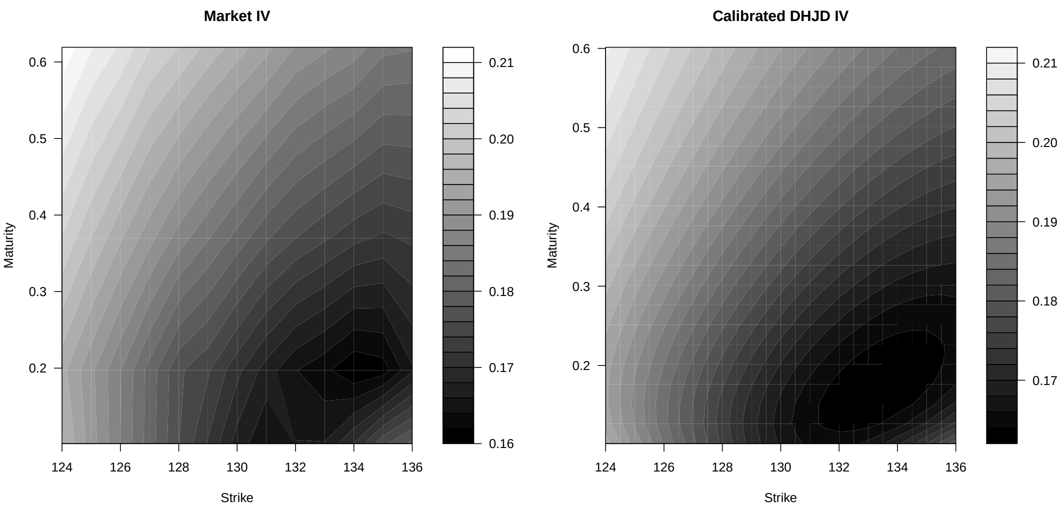

Figure 5.2: Contour plots showing the implied volatility for the given strike prices and maturities from market data (left) and the calibrated DHJDH Model (right).

Strike	Maturity			
	37	72	135	226
124	19.62	19.47	20.19	21.15
125	19.10	19.05	19.8	20.82
126	18.60	18.61	19.43	20.57
127	18.10	18.12	19.07	20.21
128	17.61	17.64	18.71	20.00
129	17.18	17.43	18.42	19.74
130	16.71	17.06	18.13	19.50
131	16.44	16.71	17.83	19.27
132	16.61	16.41	17.60	18.99
133	16.61	16.25	17.43	18.84
134	17.01	16.02	17.26	18.71
135	17.55	16.10	17.16	18.46
136	17.86	16.57	17.24	18.42

Table 1: S&P 500 index Implied Volatilities for strike prices ranging from 124 to 136 and maturities from 37 to 226 days.

Method	κ	θ	σ	v_0	ρ	IVMSE
Univariate	0.8998	0.1721	1.3390	0.0325	-0.3716	3.951×10^{-4}
Double Heston	2.7994	0.0716	0.9565	0.0179	-0.8510	1.227×10^{-4}
	18.4552	0.0074	1.8167	0.0221	0.7557	
DHJDH	2.2336	0.1642	0.5424	0.0092	-0.8372	1.039×10^{-4}
	18.9014	0.0179	1.8764	0.0287	0.1547	
	λ_V	λ_S	μ_S	μ_V	σ_S	
	0.0047	0.0617	2.0541	0.7108	2.2827	

Table 2: Calibrated parameters of the Double Heston Jump-Diffusion Hybrid Model, along with the Single and Double Heston model calibrated parameters. The last column shows the model mean square error.

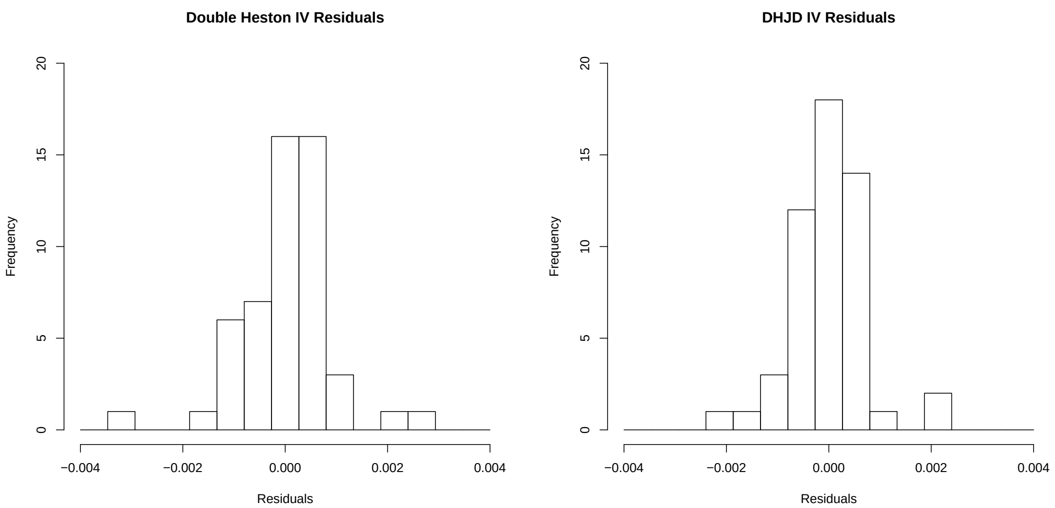

Figure 5.3: Histogram of the residuals of the Double Heston (left) and DHJDH (right) models.

Finally, we will examine the model residuals. Figure 5.3 contains the histograms of the Double Heston residuals and the Double DHJDH residuals. We can see from the histograms that the majority of the residuals for the Double DHJDH model are located near zero, with only a few residuals located further than ± 0.001, while the Double Heston residuals are more widely spread between -0.002 and 0.002. The smaller residuals from the DHJDH model is a clear indication it having a smaller IVRMSE then the Double Heston model.

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