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Finite Difference Solution to the Space-Time Fractional Partial Differential-Difference Toda Lattice Equation

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Abstract

This paper deals with the numerical solution of space-time fractional partial differential-difference Toda lattice equation $\frac{\partial^{2\alpha}u_n}{\partial x^\alpha \partial t^\alpha} = (1+\frac{\partial^\alpha u_n}{\partial t^\alpha})(u_{n-1}-2u_n+u_{n+1})$, $\alpha \in (0,1)$. The finite differences method (FD-method) is used for numerical solution of this problem. According to the method, we approximate the unknown values u_n of the desired function by finite differences approximation. As an application we demonstrate the capabilities of this method for identification of various values of order of fractional derivative α. Numerical results show that the proposed version of FD-method allows to obtain all data from the initial and boundary conditions with enough high accuracy.

1. Introduction

In this paper, we shall consider the space-time fractional (2+1)-dimensional Toda lattice equation described in equation (1) and (2) below. The importance of Toda lattice is, together with the Korteweg –de Vires equation, one of the most classical and significant completely integrable systems. Several methods have been developed to reveal its philosophical mathematical structure [1]. The (2+1)-dimensional Toda lattice hierarchy has been proposed as an extension of the KP hierarchy. This comprises the (2+1)-dimensional Toda lattice equation as the modest nontrivial differential-difference equation. The Toda lattice equation and the sine-Gordon equation are derived by imposing suitable reductions on the (2+1)-dimensional Toda lattice equation [2]. These type of equations, usually, describe the evolution of certain phenomena over the course of time [3].
This paper studies the space-time fractional differential-difference Toda lattice equation (denote $I=(a,b)$),

$$\frac{\partial^{2\alpha}u^n}{\partial x^\alpha \partial t^\alpha} = (1+\frac{\partial^\alpha u^n}{\partial t^\alpha})(u^{n-1}-2u^n+u^{n+1}), \quad (x,t)\in I\times(0,T] \tag{1.1}$$

from the initial and homogeneous Dirichlet boundary condition

$$\begin{cases} u(x,0)=\phi(x),\ x\in I, \\ u(a,t)=u(b,t)=0,\ t\in(0,T], \end{cases}$$

where the mixed derivative $\frac{\partial^{2\alpha}u^n}{\partial x^\alpha \partial t^\alpha}$ denotes the space-time derivative with fractional order 2α of the function $u=u(x,t)$ at $t=t_n$. The derivative $\frac{\partial^\alpha u^n}{\partial t^\alpha}$ also denotes time derivative with fractional order $\alpha\in(0,1)$. We consider the most frequently used the Riemann–Liouville and the Caputo derivative for fractional derivatives in (1.1). Riemann-Liouville fractional derivative with fractional order α of the function $u=u(x,t)$ is defined by [4, 5], i.e.,

$$\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right]_{RL} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha}d\tau, \quad t>0. \tag{1.2}$$

where $\Gamma(x)$ is the Euler's Gamma Function. Another definition of fractional derivative is Caputo derivative. Caputo fractional derivative with fractional order α of the function $u=u(x,t)$ is defined by [4, 5] as follows:

$$\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right]_{C} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-\tau)^\alpha}\frac{\partial u(x,\tau)}{\partial t}d\tau, \quad t>0. \tag{1.3}$$

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From (1.2) and (1.3), it is clear that definitions of Riemann–Liouville derivative and Caputo derivative are not equivalent. But, there is a fact that, almost all the numerical methods for the Riemann–Liouville derivative can be theoretically extended to the Caputo derivative if the function $u(x,t)$ satisfies suitable smooth conditions. Following equality shows the relation between the Riemann–Liouville and Caputo derivatives for $0<\alpha<1$:

$$\left[\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\right]_{RL}=\left[\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\right]_{C}+\frac{t^{-\alpha}u(x,0)}{\Gamma(1-\alpha)},\quad t>0. \tag{1.4}$$

Hence, a natural way to discretize the Caputo derivative in the equation (1.1) is to use the Grünwald–Letnikov approximation [6].

2. Numerical implementation

One method of the solutions of fractional equations based on numerical methods and solutions are determined by implementing the numerical methods on original (physical) domain. These methods are adapted for fractional integrals (Riemann-Liouville integrals etc.) and derivatives (Caputo derivatives and the Riesz Derivatives etc.) based on polynomial interpolation, Gauss interpolation or linear multistep methods.
For the numerical solution to the considered problem above we construct a uniform grid of mesh points t_n with $t_n=n\Delta t,\ n=0,1,\dots,N_t$ and $\Delta t=T/N_t$. One can define the space step size $\Delta x=(b-a)/N_x$. The space grid point x_k is given by $x_k=a+k\Delta x,\ k=0,1,\dots,N_x$. We denote the exact solution $u(x,t)$ at (x_k,t_n) by $u_k^n=u(x_k,t_n)$ and approximate solution by U_k^n at the same grid point (x_k,t_n).

Toda Lattice Equation for Riemann-Liouville derivative in time: For the numerical solution to the considered problem (1.1), we consider Riemann-Liouville time-fractional derivative:

$$\left[\frac{\partial^{2\alpha}u^n}{\partial x^{\alpha}\partial t^{\alpha}}\right]_{RL}=\left(1+\left[\frac{\partial^{\alpha}u^n}{\partial t^{\alpha}}\right]_{RL}\right)(u^{n-1}-2u^n+u^{n+1}),\quad (x,t)\in I\times(0,T] \tag{2.1}$$

We can discretize the Riemann-Liouville fractional derivative of $u(x,t)$ at $t=t_n$ by the Grünwald–Letnikov formula as follows:

$$\left[\frac{\partial^{\alpha}u(x_k,t^n)}{\partial t^{\alpha}}\right]_{RL}=\frac{1}{\Delta t^{\alpha}}\sum_{j=0}^{n}w_j^{\alpha}u_k^{n-j}+O(\Delta t^{p}),\ t>0$$

where w_j^{α} are the coefficients of the generating function, that is $w_0^{\alpha}=1,\ w_j^{\alpha}=(1-(\alpha+1)/j)w_{j-1}^{\alpha},\ j\geq 1$ and $p=1$ [4, 5]. Then the finite difference approximation of (2.1) is given as follows:

$$\frac{1}{\Delta t^{\alpha}}\sum_{j=0}^{n}w_j^{\alpha}(\delta_x^{\alpha}U_k^{n-j})=\left(1+\frac{1}{\Delta t^{\alpha}}\sum_{j=0}^{n}w_j^{\alpha}U_k^{n-j}\right)(U_k^{n-1}-2U_k^n+U_k^{n+1}),\ n\geq 1, \tag{2.2}$$

where $\delta_x^{\alpha}U_k^{n-j}$ is the approximation of the Riemann-Liouville space-fractional derivative $\frac{\partial^{\alpha}u^n}{\partial x^{\alpha}}$ and defined by the Grünwald–Letnikov formula similarly:

$$\delta_x^{\alpha}U_k^n=\frac{1}{\Delta x^{\alpha}}\sum_{i=0}^{k}w_i^{\alpha}U_{k-i}^n.$$

So (2.2) gives the approximate solution for all points $(x_k,t_n),\ k=\overline{1,N_x-1},\ n=\overline{1,N_t-1}$ as follows:

$$\begin{cases}\frac{1}{\Delta t^{\alpha}}\frac{1}{\Delta x^{\alpha}}\sum_{j=0}^{n}\sum_{i=0}^{k}w_j^{\alpha}w_i^{\alpha}U_{k-i}^{n-j}=\left(1+\frac{1}{\Delta t^{\alpha}}\sum_{j=0}^{n}w_j^{\alpha}U_k^{n-j}\right)(U_k^{n-1}-2U_k^n+U_k^{n+1}),\ n\geq 0,\\ U_k^0=\phi(x_k),\ k=\overline{0,N_x},\quad U_0^n=U_{N_x}^n=0,\ n=\overline{1,N_t}.\end{cases}$$

Example 1. We consider here $\phi(x)=10x(10-x),\ 0\leq x\leq 10$ as initial data and $\alpha=0.75$ as fractional order of derivative. In this example the time step size is $\Delta t=0.001$, number of time nodes is $N_t=41$ and the space step size is $\Delta x=0.5$, number of space nodes is $N_x=21$. The left Figure 2.1 shows numerical solution $U(x,t)$ for $x\in[0,10],\ t\in(0,T],\ T=0.04$. The right Figure 2.1 shows final time profile of numerical solution at $T=0.04$.

Toda Lattice Equation for Caputo derivative in time: For the numerical solution to the considered problem (1.1), we consider Caputo time-fractional derivative:

$$\left[\frac{\partial^{2\alpha}u^n}{\partial x^{\alpha}\partial t^{\alpha}}\right]_{C}=\left(1+\left[\frac{\partial^{\alpha}u^n}{\partial t^{\alpha}}\right]_{C}\right)(u^{n-1}-2u^n+u^{n+1}),\quad (x,t)\in I\times(0,T]. \tag{2.3}$$

We can discretize the Caputo fractional derivative of $u(x,t)$ at $t=t_n$ by the *L1-method* defined as follows:

$$\left[\frac{\partial^{\alpha}u(x_k,t^n)}{\partial t^{\alpha}}\right]_{C}=\frac{1}{\Delta t^{\alpha}}\sum_{j=0}^{n-1}b_{n-j-1}^{\alpha}(u_k^{j+1}-u_k^{j})+O(\Delta t^{p}),\ t>0$$

where b_{n-j-1}^{α} are the coefficients, that is $b_j^{\alpha}=\frac{1}{\Gamma(2-\alpha)}[(j+1)^{1-\alpha}-(j)^{1-\alpha}]$ and $p=1$ [4, 5]. Then the finite difference approximation of (2.3) is given as follows:

$$\frac{1}{\Delta t^{\alpha}}\sum_{j=0}^{n-1}b_{n-j-1}^{\alpha}[\delta_x^{\alpha}(U_k^{j+1}-U_k^{j})]=\left(1+\frac{1}{\Delta t^{\alpha}}\sum_{j=0}^{n-1}b_{n-j-1}^{\alpha}[\delta_x^{\alpha}(U_k^{j+1}-U_k^{j})]\right)(U_k^{n-1}-2U_k^n+U_k^{n+1}),\ n\geq 1, \tag{2.4}$$

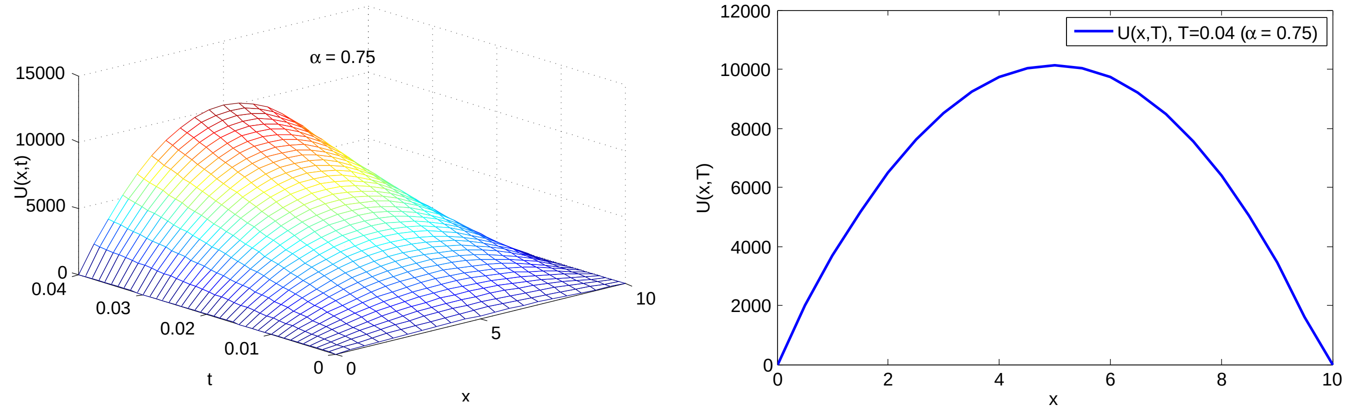

Figure 2.1: Numerical solutions for Riemann-Liouville fractional derivative ($\alpha = 0.75$)

where $\delta_x^\alpha U_k^n$ is the approximation of the Riemann-Liouville space-fractional derivative $\frac{\partial^\alpha u^n}{\partial x^\alpha}$ and defined by the Grünwald–Letnikov formula similarly. So (2.4) gives the approximate solution for all points (x_k, t_n), $k = \overline{1, N_x - 1}$, $n = \overline{1, N_t - 1}$ as follows:

$$\begin{cases} \frac{1}{\Delta t^\alpha} \frac{1}{\Delta x^\alpha} \sum_{j=0}^{n-1} \sum_{i=0}^{k} b_{n-j-1}^\alpha w_i^\alpha (U_{k-i}^{j+1} - U_{k-i}^j) = \left(1 + \frac{1}{\Delta t^\alpha} \sum_{j=0}^{n-1} b_{n-j-1}^\alpha (U_k^{j+1} - U_k^j)\right) (U_k^{n-1} - 2U_k^n + U_k^{n+1}), \; n \geq 1 (U_{k-i}^{j+1} - U_{k-i}^j), \\ U_k^0 = \phi(x_k), \; k = \overline{0, N_x}, \quad U_0^n = U_{N_x}^n = 0, \; n = \overline{1, N_t}. \end{cases} \tag{2.5}$$

Example 2. We consider same data in Example 1 to compare the numerical solutions corresponding to the two type of fractional derivatives. Thus, $\phi(x) = 10x(10 - x)$, $0 \leq x \leq 10$ is initial data and $\alpha = 0.75$ is fractional order of derivative. The time step size is $\Delta t = 0.001$, number of time nodes is $N_t = 41$ and the space step size is $\Delta x = 0.5$, number of space nodes is $N_x = 21$. The left Figure 2.2 shows numerical solution $U(x,t)$ for $x \in [0, 10]$, $t \in (0, T]$, $T = 0.04$. The right Figure 2.2 shows final time profile of numerical solution at $T = 0.04$. Figure 2.3 shows a slight differences difference on the solutions with Riemann–Liouville fractional derivative and Caputo fractional derivatives for ($\alpha = 0.75$). This slight difference, may be interpreted as, that is, due to the second term on r.h.s of equation (1.4) which states the relation between the Riemann–Liouville and Caputo derivatives for $0 < \alpha < 1$.

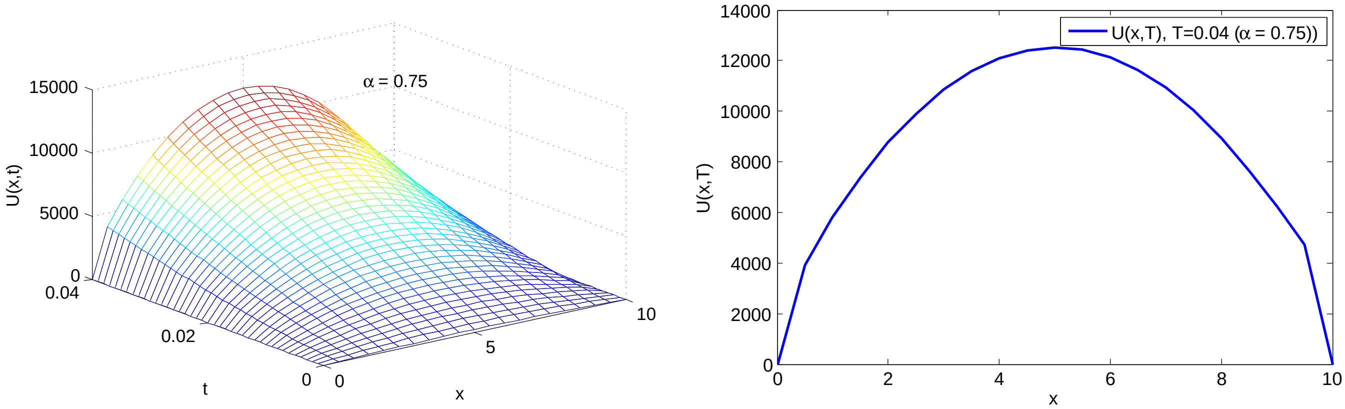

Figure 2.2: Numerical solutions for Caputo fractional derivative ($\alpha = 0.75$)

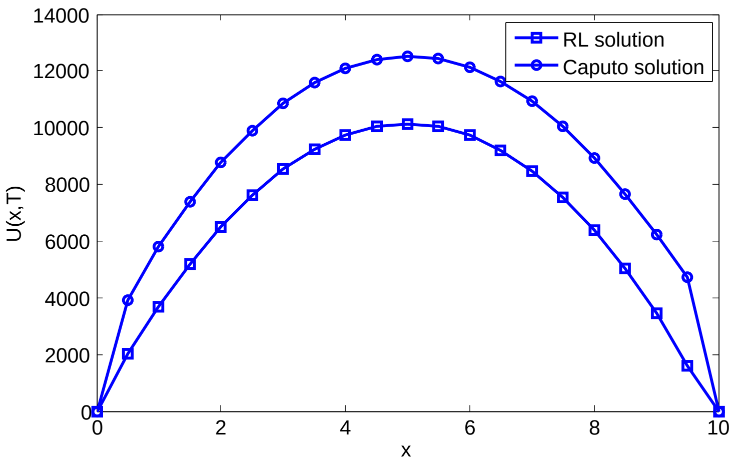

Figure 2.3: Numerical solutions for both Riemann-Liouville and Caputo fractional derivative ($\alpha = 0.75$)

3. Conclusion

In this study the space-time fractional partial differential-difference Toda lattice equation is considered. We use the finite differences method for numerical solution of the problem and present computational results for the case of two type of time fractional derivative (Riemann Liouville and Caputo) with fractional order $\alpha = 0.75$. Numerical experiments show that any of the fractional (Riemann–Liouville and Caputo) derivatives may be used for any physical problem without any reluctance and the choice of the fractional derivative is negligible at least the problem considered in this study.

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