International Electronic Journal of Algebra
Volume 25 (2019) 77-86
DOI: 10.24330/ieja.504118

ON THE EXTENDED TOTAL GRAPH OF MODULES OVER COMMUTATIVE RINGS

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Received: 23 February 2018; Revised: 7 September 2018; Accepted: 7 September 2018
Communicated by A. Çiğdem Özcan

Abstract. Let M be a module over a commutative ring R and U a nonempty proper subset of M. In this paper, the extended total graph, denoted by $ET_U(M)$, is presented, where U is a multiplicative-prime subset of M. It is the graph with all elements of M as vertices, and for distinct $m, n \in M$, the vertices m and n are adjacent if and only if $rm + sn \in U$ for some $r, s \in R \setminus (U : M)$. We also study the two (induced) subgraphs $ET_U(U)$ and $ET_U(M \setminus U)$, with vertices U and $M \setminus U$, respectively. Among other things, the diameter and the girth of $ET_U(M)$ are also studied.

Mathematics Subject Classification (2010): 13C13, 05C75, 13A15
Keywords: Total graph, prime submodule, multiplicative-prime subset

1. Introduction

Throughout this paper, R is a commutative ring with nonzero identity and M is a unitary R-module. Recently, there has been considerable attention in the literature to associating graphs with algebraic structures (see [1], [2], [3], [5], [6], [7], [8], and [9]). Anderson and Badawi in [4] defined a nonempty proper subset H of R to be a *multiplicative-prime* subset of R if the following two conditions hold: (i) $rs \in H$ for every $r \in H$ and $s \in R$; (ii) if $rs \in H$ for some $r, s \in R$, then either $r \in H$ or $s \in H$. They introduced the notion of the generalized total graph of a commutative ring $GT_H(R)$ with the vertices all elements of R, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in H$, where H is a *multiplicative-prime* subset of R.

Let R be a commutative ring and U be a nonempty subset of an R-module M. The subset $\{r \in R : rM \subseteq U\}$ will be denoted by $(U :_R M)$ or $(U : M)$. It is clear that if U is a submodule of M, then $(U : M)$ is an ideal of R. We define a nonempty subset U of M to be a *multiplicative-prime* subset of M if the following two conditions hold: (i) $rm \in U$ for every $r \in R$ and $m \in U$; (ii) if $sx \in U$ for some $s \in R$ and $x \in M$, then $x \in U$ or $s \in (U : M)$. Note that if U is a *multiplicative-prime* submodule of M, then U is necessarily a prime submodule of M. One can show that if U is a *multiplicative-prime* subset of M, then $(U : M)$ is a *multiplicative-prime* subset of R.

The total graph of a module M with respect to a *multiplicative-prime* subset U (denoted by $GT_U(M)$) was introduced in [10]. The set of vertices of $GT_U(M)$ is all elements of M, and two distinct vertices m and n adjacent whenever $m+n \in U$. In this paper, we introduce an extension of the graph $GT_U(M)$, denoted by $ET_U(M)$, such that its vertex set consists of all elements of M and for distinct $m, n \in M$, the vertices m and n are adjacent if and only if $rm+sn \in U$ for some $r, s \in R \setminus (U : M)$, where U is a *multiplicative-prime* subset of M.

Let $ET_U(U)$ be the (induced) subgraph of $ET_U(M)$ with vertex set U, and let $ET_U(M \setminus U)$ be the (induced) subgraph $ET_U(M)$ with vertices consisting of $M \setminus U$. Obviously, the total graph $GT_U(M)$ is a subgraph of $ET_U(M)$. It follows that each edge (path) of $GT_U(M)$ is an edge (path) of $ET_U(M)$. The study of $ET_U(M)$ breaks naturally into two cases depending on whether or not U is a submodule of M. In the second section, we handle the case when U is a submodule of M; in the third section, we do the case when U is not a submodule of M. For every case, we characterize the girth and diameter of $ET_U(M)$, $ET_U(U)$ and $ET_U(M \setminus U)$.

We begin with some notation, and definitions. For a graph Γ, by $E(\Gamma)$ and $V(\Gamma)$, we mean the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two of its distinct vertices. At the other extreme, we say that a graph is totally disconnected if no two vertices of this graph are adjacent. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$). We also define $d(a, a) = 0$. The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). For a graph Γ, the degree of a vertex v in Γ, denoted by $deg(v)$, is the number of edges of Γ incident with v. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 is adjacent(in Γ) to some vertex of Γ_2.

2. The case when U is a submodule of M

In this section, we study the case when U is a submodule of M. It is clear that if U is a submodule of M, then U is a prime submodule of M. If $U = M$, then it is clear that $ET_U(M)$ is a complete graph and $ET_U(M)$ is a disconnected graph when $U = 0$ and $|M| \geq 2$. So we may assume that $U \neq 0$ and $U \neq M$.

First, we begin with the following example that shows we may have $ET_U(M) \neq GT_U(M)$.

Example 2.1. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{10}$. Set $U = \{\bar{0}, \bar{5}\}$. It is clear that U is a submodule of M and $(U : M) = 5\mathbb{Z}$. Since $\bar{1} + \bar{3} = \bar{4} \notin U$, so $\bar{1} - \bar{3}$ is not an edge in $GT_U(M)$. But $2(\bar{1}) + 1(\bar{3}) = \bar{5} \in U$ and $2, 1 \in R \setminus (U : M)$. Thus $\bar{1} - \bar{3}$ is an edge in $ET_U(M)$. Hence $ET_U(M) \neq GT_U(M)$.*

The main goal of this section is a general structure theorem (Theorem 2.4) for $ET_U(M \setminus U)$ when U is a submodule of M. But first, we record the trivial observation that if U is a submodule of M, then $ET_U(U)$ is a complete subgraph of $ET_U(M)$ and is disjoint from $ET_U(M \setminus U)$. Thus we will concentrate on the subgraph $ET_U(M \setminus U)$ throughout this section.

Theorem 2.2. *Let M be a module over a commutative ring R and U be a prime submodule of M. Then $ET_U(U)$ is a complete subgraph of $ET_U(M)$ and is disjoint from $ET_U(M \setminus U)$. In particular, $ET_U(U)$ is connected and $ET_U(M)$ is disconnected.*

Proof. Let $m, n \in U$. Then it is clear that $m + n \in U$ since U is a submodule of M. If $x \in U$ is adjacent to $y \in M \setminus U$, then $rx + sy \in U$ for some $r, s \in R \setminus (U : M)$. This implies that $sy \in U$; so $y \in U$ or $s \in (U : M)$ since U is a prime submodule, which is a contradiction. The "in particular" statement is clear. □

Theorem 2.3. *Let M be a module over a commutative ring R and U be a prime submodule of M. Let G be an induced subgraph of $ET_U(M \setminus U)$, and m and m' be distinct nonadjacent vertices of G that are connected by a path in G. Then there exists a path in G of length 2 between m and m'. In particular, if $ET_U(M \setminus U)$ is connected, then $diam(ET_U(M \setminus U)) \leq 2$.*

Proof. (1) Let m_1, m_2, m_3 and m_4 be distinct vertices of G. It suffices to show that if there is a path $m_1 - m_2 - m_3 - m_4$ from m_1 to m_4, then m_1 and m_4 are adjacent. Now, $r_1m_1 + r_2m_2, r'_2m_2 + r'_3m_3, r_3m_3 + r_4m_4 \in U$ for some $r_1, r_2, r'_2, r'_3, r_3, r_4 \in R \setminus (U : M)$. Hence $(r_1r_3r'_2)m_1 + (r_2r'_3r_4)m_4 = r_3r'_2(r_1m_1 + r_2m_2) - r_2r_3(r'_2m_2 +$

$r_3'm_3) + r_2r_3'(r_3m_3 + r_4m_4) \in U$, and $r_1r_3r_2', r_2r_3'r_4 \notin (U : M)$ since $(U : M)$ is a prime ideal of R. Thus m_1 and m_4 are adjacent. So if $ET_U(M \setminus U)$ is connected, then $diam(ET_U(M \setminus U)) \leq 2$. □

Now, we give the main theorem of this section. Since $ET_U(U)$ is a complete subgraph of $ET_U(M)$ by Theorem 2.2, the next theorem gives a complete description of $ET_U(M \setminus U)$. Let $|U| = \alpha$. We allow α to be an infinite cardinal. Compare the next theorem with [10, Theorem 3.5].

Theorem 2.4. *Let M be a module over a commutative ring R, U be a prime submodule of M, and $|U| = \alpha$.*

(1) *If $r + s \in (U : M)$ for some $r, s \in R \setminus (U : M)$, then $ET_U(M \setminus U)$ is the union of complete subgraphs.*
(2) *If $r + s \notin (U : M)$ for all $r, s \in R \setminus (U : M)$, then $ET_U(M \setminus U)$ is the union of totally disconnected subgraphs and some connected subgraphs.*

Proof. (1) Suppose that $r + s \in (U : M)$ for some $r, s \in R \setminus (U : M)$. For $m, m' \in M \setminus U$, we write $m \sim m'$ if and only if $tm + t'm' \in U$ and $t + t' \in (U : M)$ for some $t, t' \in R \setminus (U : M)$. It is straightforward to check that $\sim$ is an equivalence relation on $M \setminus U$ since U is a prime submodule. For $m \in M \setminus U$, we denote the equivalence class which contains m by $[m]$. Now let $m \in M \setminus U$. If $[m] = \{m\}$, then $r(m + u_1) + s(m + u_2) = (r + s)m + ru_1 + su_2 \in U$ for every $u_1, u_2 \in U$ since $r + s \in (U : M)$. Then $m + U$ is a complete subgraph of $ET_U(M \setminus U)$ with at most α vertices. Now let $|[m]| = \nu$ and $m' \in [m]$. Then $tm + t'm' \in U$ and $t + t' \in (U : M)$ for some $t, t' \in R \setminus (U : M)$. So $t(m + u_1) + t'(m' + u_2) = tm + t'm' + tu_1 + t'u_2 \in U$ for every $u_1, u_2 \in U$. Thus $m + U$ is part of the complete graph k^{μ}, where $\mu \leq \alpha\nu$.
(2) Assume that $r + s \notin (U : M)$ for all $r, s \in R \setminus (U : M)$. Let

$$A_m = \{m' \in M \setminus U : rm + sm' \in U \ for\ some\ r, s \in R \setminus (U : M)\}$$

be the set of all vertices adjacent to m. If $A_m = \emptyset$, then $pm + qm' \notin U$ for every $m' \in M \setminus U$ and every $p, q \in R \setminus (U : M)$. In this case, we show that $m + U$ is a totally disconnected subgraph of $ET_U(M \setminus U)$. If $r(m + m_1) + s(m + m_2) \in U$ for some $r, s \in R \setminus (U : M)$ and $m_1, m_2 \in U$, then $(r + s)m \in U$. Since U is a prime submodule of M and $m \notin U$, then $r + s \in (U : M)$, which is a contradiction. Therefore $m + U$ is a totally disconnected subgraph of $ET_U(M \setminus U)$. Now, we may assume that $A_m \neq \emptyset$. Then $rm + sm' \in U$ for some $r, s \in R \setminus (U : M)$ and $m' \in M \setminus U$. Thus $r(m + u_1) + s(m' + u_2) = rm + sm' + ru_1 + su_2 \in U$ for every $u_1, u_2 \in U$; hence each element of $m + U$ is adjacent to each element of $m' + U$.

If $|A_m| = \nu$, then we have a connected subgraph of $ET_U(M \setminus U)$ with at most $\alpha\nu$ vertices. So $ET_U(M \setminus U)$ is the union of totally disconnected subgraphs and some connected subgraphs. □

Now it is easy to compute the diameter and the girth of $ET_U(M \setminus U)$ using Theorem 2.4.

Theorem 2.5. *Let M be a module over a commutative ring R such that U is a prime submodule of M.*

(1) *$diam(ET_U(M \setminus U)) = 0$ if and only if $U = \{0\}$ and $|M| = 2$.*
(2) *$diam(ET_U(M \setminus U)) = 1$ if and only if either $|M \setminus U| = 1$ and $r+s \notin (U : M)$ for some $r, s \in R \setminus (U : M)$ or $|M \setminus U| = 2$, $r + s \in (U : M)$ for every $r, s \in R \setminus (U : M)$ and $x + y \in U$ for some distinct elements $x, y \in M \setminus U$.*
(3) *$diam(ET_U(M \setminus U)) = 2$ if and only if $|M \setminus U| = 2$, $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$, $x + y \in U$ for some distinct elements $x, y \in M \setminus U$ and $|m + U| \geq 2$ for some $m \in M \setminus U$.*
(4) *Otherwise, $diam(ET_U(M \setminus U)) = \infty$.*

Proof. (1) If $diam(ET_U(M \setminus U)) = 0$, then $ET_U(M \setminus U)$ is a complete graph K^1, and so $|U| = |M/U| = 1$ by Theorem 2.4. Hence $U = \{0\}$ and $|M| = 2$. Now, let $U = \{0\}$ and $M = \{0, m\}$. Then $m + U$ is a single graph K^1. So $diam(ET_U(M \setminus U)) = 0$.
(2) It is clear that $ET_U(M \setminus U)$ is a complete graph if and only if $diam(ET_U(M \setminus U)) = 1$. So the proof is clear by Theorem 2.4.
(3) If $diam(ET_U(M \setminus U)) = 2$, then $ET_U(M \setminus U)$ is a complete bipartite graph $K^{m,n}$ such that $m \geq 2$ or $n \geq 2$. Thus $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ by Theorem 2.4. Therefore $|M \setminus U| = 2$ and $x + y \in U$ for some $x, y \in M \setminus U$. Since $m \geq 2$ or $n \geq 2$, we have $|x + U| \geq 2$ or $|y + U| \geq 2$. Conversely, let $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $|M \setminus U| = 2$. Then $M = U \cup (x+U) \cup (y+U)$ and $ET_U(M \setminus U)$ is a complete bipartite graph since $x+y \in U$. Hence $diam(ET_U(M \setminus U)) = 2$, since $|x + U| \geq 2$ or $|y + U| \geq 2$. □

Theorem 2.6. *Let M be a module over a commutative ring R such that U is a prime submodule of M. Then $gr(ET_U(M \setminus U)) = 3, 4$, or ∞. In particular, $gr(ET_U(M \setminus U)) \leq 4$ if $ET_U(M \setminus U)$ contains a cycle.*

Proof. Assume that $ET_U(M \setminus U)$ contains a cycle. Then $ET_U(M \setminus U)$ is not a totally disconnected graph; so by the proof of Theorem 2.4, $ET_U(M \setminus U)$ has either a complete or a complete bipartite subgraph. Therefore it must contain either a 3-cycle or 4-cycle. Thus $gr(ET_U(M \setminus U)) \leq 4$. □

Theorem 2.7. *Let M be a module over a commutative ring R such that U is a prime submodule of M.*

1. $gr(ET_U(M \setminus U)) = 3$ *if and only if* $r + s \in (U : M)$ *and* $|y + U| \geq 3$ *for some* $r, s \in R \setminus (U : M)$ *and* $y \in M \setminus U$.
2. $gr(ET_U(M \setminus U)) = 4$ *if and only if* $r + s \notin (U : M)$ *for every* $r, s \in R \setminus (U : M)$ *and* $pm + qm' \in U$ *for some* $m, m' \in M \setminus U$ *and* $p, q \in R \setminus (U : M)$.
3. *Otherwise,* $gr(ET_U(M \setminus U)) = \infty$.

Proof. (1) Assume that $gr(ET_U(M \setminus U)) = 3$. Then by Theorem 2.4, $ET_U(M \setminus U)$ is a complete graph K^λ, where $\lambda \geq 3$. Then $r + s \in (U : M)$ for some $r, s \in R \setminus (U : M)$ and $|y + U| \geq 3$ for some $y \in M \setminus U$ by Theorem 2.4.
(2) If $gr(ET_U(M)) = 4$, then $ET_U(M \setminus U)$ has a complete bipartite subgraph. So $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$ by Theorem 2.4.
The other implications of (1) and (2) follows directly from Theorem 2.4. □

We end this section with the following theorem.

Theorem 2.8. *Let M be a module over a commutative ring R such that U is a prime submodule of M.*

1. $gr(ET_U(M)) = 3$ *if and only if* $|U| \geq 3$.
2. $gr(ET_U(M)) = 4$ *if and only if* $r + s \notin (U : M)$ *for every* $r, s \in R \setminus (U : M)$, $|U| < 3$, *and* $pm + qm' \in U$ *for some* $m, m' \in M \setminus U$ *and* $p, q \in R \setminus (U : M)$.
3. *Otherwise,* $gr(ET_U(M)) = \infty$.

Proof. (1) This follows from Theorem 2.2.
(2) Assume that $gr(ET_U(M)) = 4$. Since $gr(ET_U(U)) = 3$ or ∞, then $gr(ET_U(M \setminus U)) = 4$. Therefore $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$ by Theorem 2.7. On the other hand, $gr(ET_U(M)) \neq 3$; so $|U| < 3$. The other implication follows from Theorem 2.4. □

3. The case when U is not a submodule of M

In this section, we study $ET_U(M)$ when the multiplicative-prime subset U is not a submodule of M. Since U is always closed under multiplication by elements of R, this just means that $0 \in U$ and there are distinct $x, y \in U$ such that $x + y \in M \setminus U$.

First, we begin with the following example that shows we may have $ET_U(M) \neq GT_U(M)$.

Example 3.1. *Let $R = M = \mathbb{Z}$. Set $U = 4\mathbb{Z} \cup 6\mathbb{Z}$. It is clear that $(U : M) = U$ and U is not a submodule of M since $4, 6 \in U$, but $4 + 6 = 10 \notin U$. So $4 - 6$ is not an edge in $GT_U(M)$. But $2(4) + 2(6) = 20 \in U$ and $2 \in R \setminus U$. Thus $4 - 6$ is an edge in $ET_U(M)$. Hence $ET_U(M) \neq GT_U(M)$.*

Now, we have the following theorem that shows $ET_U(U)$ is always connected (but never complete), $ET_U(U)$ and $ET_U(M \setminus U)$ are never disjoint subgraphs of $ET_U(M)$, and $ET_U(M)$ is connected when $ET_U(M \setminus U)$ is connected.

Theorem 3.2. *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M. Then the following hold:*

(1) *$ET_U(U)$ is connected with $diam(ET_U(U)) = 2$.*
(2) *Some vertex of $ET_U(U)$ is adjacent to a vertex of $ET_U(M \setminus U)$. In particular, the subgraphs $ET_U(U)$ and $ET_U(M \setminus U)$ are not disjoint.*
(3) *If $ET_U(M \setminus U)$ is connected, then $ET_U(M)$ is connected.*

Proof. (1) Let $u \in U^* = U \setminus \{0\}$. Then u is adjacent to 0. Thus $u - 0 - u'$ is a path in $ET_U(U)$ of length two between any two distinct $u, u' \in U^*$. Moreover, there exist nonadjacent $u, u' \in U^*$ since U is not a submodule of M; thus $diam(ET_U(U)) = 2$.
(2) Since U is not a submodule of M, there exist distinct $m, n \in U^*$ such that $m + n \notin U$. Then $-m \in U$ and $m + n \notin U$ are adjacent vertices in $ET_U(M)$. Finally, the "in particular" statement is clear.
(3) $ET_U(U)$ and $ET_U(M \setminus U)$ are connected, and there is an edge between $ET_U(U)$ and $ET_U(M \setminus U)$; thus $ET_U(M)$ is connected. □

We determine when $ET_U(M)$ is connected and compute $diam(ET_U(M))$ with the following theorem. Compare the next theorem with [10, Theorem 4.2].

Theorem 3.3. *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M. Then $ET_U(M)$ is connected if and only if for every $m \in M$ there exists $r \in R \setminus (U : M)$ such that $rm \in \langle U \rangle$.*

Proof. Suppose that $ET_U(M)$ is connected and $m \in M$. Then there exists a path $0 - x_1 - x_2 - \cdots - x_n - m$ from 0 to m in $ET_U(M)$. Thus

$$r_1x_1, r_2x_1 + r_3x_2, \ldots, r_{2n-2}x_{n-1} + r_{2n-1}x_n, r_{2n}x_n + sm \in U$$

for some $r_1, r_2, \ldots, r_{2n}, s \in R \setminus (U : M)$. Then

$$sr_1r_3r_5 \cdots r_{2n-1}m = (r_1r_3r_5 \cdots r_{2n-1})(sm + r_{2n}x_n) -$$

$$(r_1r_3r_5\cdots r_{2n-3}r_{2n})(r_{2n-2}x_{n-1}+r_{2n-1}x_n)+\cdots$$

$$-(r_1r_3\cdots r_{2n-2k-5}r_{2n-2k-3}r_{2n-2k}r_{2n-2k-2}\cdots r_{2n})(r_{2n-2k-1}x_{n-k}+r_{2n-2k-2}x_{n-(k+1)})$$

$$+(r_1r_3\cdots r_{2n-2k-5}r_{2n-2k-2}r_{2n-2k}r_{2n-2k-2}\cdots r_{2n})(r_{2n-2k-3}x_{n-(k+1)}+r_{2n-2k-4}x_{n-(k+2)})$$

$$\cdots-(r_2r_4r_6\cdots r_{2n})(r_1x_1)\in\langle U\rangle$$

Since $(U:M)$ is a *multiplicative-prime* subset of R, we have $r=sr_1r_3r_5\cdots r_{2n-1}\in R\setminus(U:M)$ and $rm\in\langle U\rangle$. Conversely, suppose that for every $m\in M$ there exists $r\in R\setminus(U:M)$ such that $rm\in\langle U\rangle$. We show that for each $0\neq m\in M$, there exists a path in $ET_U(M)$ from 0 to m. By assumption, there are elements $u_1,u_2,\ldots,u_n\in U$ such that $rm=u_1+u_2+\cdots+u_n$. Set $y_0=0$ and $y_k=(-1)^{n+k}(u_1+u_2+\cdots+u_k)$ for each integer k with $1\leq k\leq n$. Then $y_k+y_{k+1}=(-1)^{n+k+1}u_{k+1}\in U$ for each integer $1\leq k\leq n-1$. Also, $y_{n-1}+rm=y_{n-1}+y_n=u_n\in U$. Thus $0-y_1-y_2-\cdots-y_{n-1}-m$ is a path from 0 to m in $ET_U(M)$. Now, let $0\neq x,y\in M$. Then by the preceding argument, there are paths from x to 0 and 0 to y in $ET_U(M)$. Hence there is a path from x to y in $ET_U(M)$. So $ET_U(M)$ is connected. □

Theorem 3.4. *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M and for every $m\in M$ there exists $r\in R\setminus(U:M)$ such that $rm\in\langle U\rangle$. Let $n\geq 2$ be the least integer such that $\langle U\rangle=<m_1,m_2,...,m_n>$ for some $m_1,m_2,...,m_n\in U$. Then $diam(ET_U(M))\leq n$.*

Proof. Let m and m' be distinct elements in M. We show that there exists a path from m to m' in $ET_U(M)$ with length at most n. By hypothesis, $rm,r'm'\in\langle U\rangle$ for some $r,r'\in R\setminus(U:M)$; so we can write $rm=\sum_{i=1}^n r_im_i$ and $r'm'=\sum_{i=1}^n s_im_i$ for some $r_i,s_i\in R$. Define $x_0=m$ and $x_k=(-1)^k(\sum_{i=k+1}^n r_im_i+\sum_{i=1}^k s_im_i)$; so $x_k+x_{k+1}=(-1)^k(r_{k+1}-s_{k+1})m_{k+1}\in U$ for each integer k with $1\leq k\leq n-1$. On the other hand, $rm+x_1=(r_1-s_1)m_1\in U$ and $r'm'+(-1)^nx_{n-1}=(s_n-r_n)m_n\in U$. So $m-x_1-x_2-\cdots-x_{n-1}-m'$ is a path from m to m' in $ET_U(M)$ with length at most n since $1,(-1)^n\notin(U:M)$. □

We end the paper with the following theorem.

Theorem 3.5. *Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M. Then the following hold:*

(1) *Either $gr(ET_U(U))=3$ or $gr(ET_U(U))=\infty$.*
(2) *If $gr(ET_U(M))=4$, then $gr(ET_U(U))=\infty$.*

Proof. (1) If $rm+sm' \in U$ for some distinct $m, m' \in U \setminus \{0\}$ and $r, s \in R \setminus (U : M)$, then $0-m-m'-0$ is a cycle of length 3 in $ET_U(U)$; so $gr(ET_U(U)) = 3$. Otherwise, $rm+sm' \in M \setminus U$ for all distinct $m, m' \in U \setminus \{0\}$ and all elements $r, s \in R \setminus (U : M)$. Therefore in this case, each nonzero element $m \in U$ is adjacent to 0, and no two distinct $m, m' \in U \setminus \{0\}$ are adjacent. Thus $gr(ET_U(U)) = \infty$.
(2) If $gr(ET_U(M)) = 4$, then it is clear $gr(ET_U(U)) \neq 3$. So $gr(ET_U(U)) = \infty$ by part (1) above. □

Acknowledgement. The authors are deeply grateful to the referee for a careful reading of this manuscript and his valuable suggestions.

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