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SOME RESULTS ABOUT HOM-COMODULE ALGEBRAS WITH HOM-HOPF MODULE STRUCTURE

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Abstract. The main subject of this paper is Hom-comodule algebras with Hom-Hopf module structure. First, we give the factorization of a class of Hom-bialgebras, which is not only Hom-module coalgebras but also Hom-comodule algebras with Hom-Hopf module structure. Next, we obtain the factorization of this class of Hom-comodule algebras. Finally, we discuss the relation between this class of Hom-comodule algebas and cleft extensions.

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1. Introduction

The study of Hom-associative algebras originates with work by Hartwig, Larsson and Silvestrov in the Lie case [4], where a notion of Hom-Lie algebra was introduced in the context of studying deformations of Witt and Virasoro algebras. Later, it was extended to the associative case by Makhlouf and Silvestrov in [7]. Now the associativity is replaced by Hom-associativity $\alpha(a)(bc) = (ab)\alpha(c)$. Hom-coassociativity for a Hom-coalgebra can be considered in a similar way, see [8].

The crossed product algebra was introduced in [1], which is a generalization of the smash product algebra. In [1], Blattner, Cohen and Montgomery showed the equivalence of crossed products and cleft extensions. In [2], Blattner and Montgomery gave several characterizations of crossed products. Lu and Wang [5] generalized the results in [1] to the case of Hom-Hopf algebras. Hopf modules (see [9]) are vector spaces with both a comodule and module structure which are related in a natural way. The theory of Hopf modules accounts for some of the deeper results for Hopf algebras. Hom-Hopf modules, as the generalization of Hopf modules, are also studied by many people. Motivated by this, the main subject of this paper

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is the Hom-comodule algebras with Hom-Hopf module structure. In this paper we not only discuss the relation between this class of Hom-algebas and Hom-crossed product algebras, but also discuss the relation between this class of Hom-algebas and cleft extensions.

This paper is organized as follows. In Section 2, we recall some basic definitions and results, such as Hom-Hopf algebra, Hom-Hopf module, Hom-(co)module (co)algebra, Hom-crossed product algebra and so on. Next, we always assume that H is a Hom-Hopf algebra, B is a right H-Hom-Hopf module and (B, ρ) is a right H-Hom-comodule algebra, set $A = B^{coH}$. In Section 3, let B be also a Hom-module coalgebra and a Hom-bialgebra, we provide the factorization of Hom-bialgebra B, that is $B \simeq A\Box^{\cdot}_{\rho}H$ as Hom-bialgebras (see Theorem 3.3). In Section 4, we define a weak action of H on A, thus obtain the Hom-crossed product algebra $A\sharp_{\sigma}H$ and $B \simeq A\sharp_{\sigma}H$ as Hom-algebras (see Theorem 4.2), that is, we obtain the factorization of this class of Hom-comodule algebas. Furthermore, we discuss the relation between this class of Hom-comodule algebas and cleft extensions (see Theorem 4.7).

2. Preliminaries

Throughout this paper, k is a fixed field. Unless otherwise stated, all vector spaces, algebras, coalgebras, maps and unadorned tensor products are over k.

We now recall from [7,8,10] some definitions and results about Hom-Hopf algebras, Hom-(co)modules and so on.

2.1. Hom-Hopf algebra. A Hom-algebra is a quadruple $(A, \mu, 1_A, \alpha)$ (abbr. (A, α)), where A is a linear space, $\mu : A \otimes A \to A$ is a linear map, with notation $\mu(a \otimes a') = aa'$, $1_A \in A$ and $\alpha \in Aut_k(A)$, such that for any $a, a', a'' \in A$,

$$\alpha(aa') = \alpha(a)\alpha(a'),\ a1_A = 1_A a = \alpha(a),$$
$$\alpha(a)(a'a'') = (aa')\alpha(a''),\ \alpha(1_A) = 1_A.$$

A Hom-coalgebra is a quadruple $(C, \Delta, \varepsilon, \beta)$ (abbr. (C, β)), where C is a linear space, $\Delta : C \to C \otimes C$, $\varepsilon : C \to k$ are linear maps, and $\beta \in Aut_k(C)$, such that for any $c \in C$,

$$\beta(c_1) \otimes \beta(c_2) = \beta(c)_1 \otimes \beta(c)_2,\ \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c),$$
$$\beta(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta(c_2),\ \varepsilon\beta = \varepsilon.$$

A Hom-bialgebra is a sextuple $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$ (abbr. (H, γ)), where $(H, \mu, 1_H, \gamma)$ is a Hom-algebra, and $(H, \Delta, \varepsilon, \gamma)$ is a Hom-coalgebra, such that Δ, ε are morphisms of Hom-algebra, i.e.

$$\Delta(hh') = \Delta(h)\Delta(h'),\ \Delta(1_H) = 1_H \otimes 1_H,\ \varepsilon(hh') = \varepsilon(h)\varepsilon(h'),\ \varepsilon(1_H) = 1.$$

Furthermore, if there exists a linear map $S : H \to H$ such that

$$S(h_1)h_2 = h_1 S(h_2) = \varepsilon(h)1_H,\ S(\gamma(h)) = \gamma(S(h)),$$

then we call $(H, \mu, 1_H, \Delta, \varepsilon, S, \gamma)$ (abbr. (H, S, γ)) a Hom-Hopf algebra.

2.2. The fundamental theorem of Hom-Hopf module. Let (A, β) be a Hom-algebra, a right (A, β)-Hom-module is a triple $(M, \cdot, \alpha)$, where M is a linear space, $\cdot : M \otimes A \to M$ is a linear map, and α is an automorphism of M, such that for any $a, a' \in A$ and $m \in M$,

$$\alpha(m) \cdot (aa') = (m \cdot a) \cdot \beta(a'),\ m \cdot 1_A = \alpha(m),\ \alpha(m \cdot a) = \alpha(m) \cdot \beta(a).$$

Let (C, β) be a Hom-coalgebra, a right (C, β)-Hom-comodule is a triple (M, ρ, α), where M is a linear space, $\rho : M \to M \otimes C$ is a linear map (write $\rho(m) = m_{(0)} \otimes m_{(1)}$), and α is an automorphism of M, such that for any $m \in M$,

$$\alpha(m_{(0)}) \otimes m_{(1)1} \otimes m_{(1)2} = m_{(0)(0)} \otimes m_{(0)(1)} \otimes \beta(m_{(1)}),$$
$$m_{(0)}\varepsilon(m_{(1)}) = \alpha(m),\ \alpha(m)_{(0)} \otimes \alpha(m)_{(1)} = \alpha(m_{(0)}) \otimes \beta(m_{(0)}).$$

Let (H, α) be a Hom-Hopf algebra, a right H-Hom-Hopf module is a quadruple $(M, \cdot, \rho, \beta)$, where M is a right H-Hom-module and a right H-Hom-comodule, such that for all $m \in M$, $h \in H$,

$$\rho(m \cdot h) = m_{(0)} \cdot h_1 \otimes m_{(1)}h_2.$$

2.3. Hom-module coalgebra and Hom-comodule algebra. Recall from [6], let (H, α) be a Hom-Hopf algebra and (C, β) be a Hom-coalgebra, if $(C, \cdot, \alpha)$ is a left (H, β)-Hom-module, for all $c \in C$, $h \in H$ the following conditions hold

$$(h \cdot c)_1 \otimes (h \cdot c)_2 = h_1 \cdot c_1 \otimes h_2 \cdot c_2,\ \varepsilon(h \cdot c) = \varepsilon(h)\varepsilon(c),$$

then $(C, \cdot, \beta)$ is called an H-Hom-module coalgebra.

Recall from [10], let (H, α) be a Hom-Hopf algebra and (A, β) be a Hom-algebra, if (A, ρ, β) is a left (H, α)-Hom-comodule, for all $a, b \in A$ the following conditions hold

$$\rho(ab) = a_{-1}b_{-1} \otimes a_0 b_0,\ \rho(1_A) = 1_H \otimes 1_A,$$

then (A, ρ, β) is called an H-Hom-comodule algebra.

2.4. Hom-crossed product algebra and cleft extension. Recall from [5], let (H,α) be a Hom-Hopf algebra and (A,β) be a Hom-algebra. We say that H acts weakly on A from the left if there is a linear map given by $\rightharpoonup$: $H\otimes A\to A$, such that for all $a,b\in A$ and $h\in H$,

$$\beta(h\rightharpoonup a)=\alpha(h)\rightharpoonup\beta(a),\quad h\rightharpoonup 1=\varepsilon(h)1,$$
$$\alpha^2(h)\rightharpoonup(ab)=(h_1\rightharpoonup a)(h_2\rightharpoonup b),\quad 1\rightharpoonup a=\beta(a).$$

Let (H,α) be a Hom-Hopf algebra and (A,β) be a Hom-algebra. Assume that H acts weakly on A from the left; Let $\sigma\in Hom(H\otimes H,A)$ be a linear map. For all $a,b\in A$ and $h,g\in H$, define $A\sharp_\sigma H$ whose underlying vector space is $A\otimes H$ with the multiplication given by

$$(a\otimes h)(b\otimes g)=a((\alpha^{-4}(h_{11})\rightharpoonup\beta^{-2}(b))\sigma(\alpha^{-3}(h_{12}),\alpha^{-2}(g_1)))\otimes\alpha^{-1}(h_2g_2).$$

We say that $(A\sharp_\sigma H,\beta\otimes\alpha)$ is a Hom-crossed product algebra if and only if

(C1) $\sigma(1,h)=\sigma(h,1)=\varepsilon(h)1,\quad \sigma(\alpha\otimes\alpha)=\beta\sigma$;

(C2) $(h_1\rightharpoonup(\alpha^{-1}(g_1)\rightharpoonup a))\sigma(\alpha(h_2),\alpha(g_2))=\sigma(\alpha(h_1),\alpha(g_1))(\alpha^{-1}(h_2g_2)\rightharpoonup\beta(a))$;

(C3) $(h_1\rightharpoonup\sigma(g_1,l_1))\sigma(\alpha(h_2),g_2l_2)=\sigma(\alpha(h_1),\alpha(g_1))\sigma(h_2g_2,\alpha^2(l))$.

Let (H,α) be a Hom-Hopf algebra and (B,ρ,β) be a left H-Hom-comodule algebra. Denote by $A=B^{coH}=\{a\in B\mid\rho(a)=\beta(a)\otimes 1\}$, then $A\subset B$ is said to be a cleft extension if there exists a left H-Hom-comodule map $\gamma:H\to B$ which is convolution invertible.

3. The structure of Hom-bialgebras with the Hom-Hopf module structure

In this section, we mainly provide the factorization of Hom-bialgebras with Hom-Hopf module structure. On the one hand, we give the algebra factorization for right H-Hom-comodule algebras with Hom-Hopf module structure.

Proposition 3.1. *Let (H,S,α) be a Hom-Hopf algebra and $(B,\cdot,\rho,\beta)$ be a right H-Hom-Hopf module. We assume that (B,ρ,β) is a right H-Hom-comodule algebra (write $\rho(b)=b_{(0)}\otimes b_{(1)}$), and set $A=B^{coH}=\{a\in B\mid\rho(a)=\beta(a)\otimes 1\}$. Define a multiplication on $A\otimes H$ as follows*

$$(a\otimes h)(b\otimes g)=((\beta^{-2}(a)\cdot\alpha^{-4}(h_{11}))(\beta^{-2}(b)\cdot\alpha^{-4}(g_{11})))\cdot S\alpha^{-3}(h_{12}g_{12})\otimes\alpha^{-1}(h_2g_2),$$

for all $a,b\in A,h,g\in H$, then $(A\otimes H,\beta\otimes\alpha,1_A\otimes 1_H)$ is a Hom-algebra, we write it as $A\Box_\rho H$. Moreover, $B\simeq A\Box_\rho H$ as Hom-algebras.

Proof. We first check that the multiplication is well defined. For all $a, b \in A$, $h, g \in H$, we have

$$
\begin{aligned}
& \rho(((\beta^{-2}(a)\cdot\alpha^{-4}(h_1))(\beta^{-2}(b)\cdot\alpha^{-4}(g_1)))\cdot S\alpha^{-3}(h_2g_2)) \\
= \;& ((\beta^{-2}(a)\cdot\alpha^{-4}(h_1))(\beta^{-2}(b)\cdot\alpha^{-4}(g_1)))_{(0)}\cdot S\alpha^{-3}(h_{22}g_{22}) \\
& \otimes((\beta^{-2}(a)\cdot\alpha^{-4}(h_1))(\beta^{-2}(b)\cdot\alpha^{-4}(g_1)))_{(1)}S\alpha^{-3}(h_{21}g_{21}) \\
= \;& ((\beta^{-2}(a)\cdot\alpha^{-4}(h_1))_{(0)}(\beta^{-2}(b)\cdot\alpha^{-4}(g_1))_{(0)})\cdot S\alpha^{-3}(h_{22}g_{22}) \\
& \otimes((\beta^{-2}(a)\cdot\alpha^{-4}(h_1))_{(1)}(\beta^{-2}(b)\cdot\alpha^{-4}(g_1))_{(1)})S\alpha^{-3}(h_{21}g_{21}) \\
= \;& (((\beta^{-2}(a))_{(0)}\cdot\alpha^{-4}(h_{11}))((\beta^{-2}(b))_{(0)}\cdot\alpha^{-4}(g_{11})))\cdot S\alpha^{-3}(h_{22}g_{22}) \\
& \otimes(((\beta^{-2}(a))_{(1)}\cdot\alpha^{-4}(h_{12}))_{(1)}((\beta^{-2}(b))_{(1)}\cdot\alpha^{-4}(g_{12})))S\alpha^{-3}(h_{21}g_{21}) \\
= \;& ((\beta^{-1}(a)\cdot\alpha^{-4}(h_{11}))(\beta^{-1}(b)\cdot\alpha^{-4}(g_{11})))\cdot S\alpha^{-3}(h_{22}g_{22}) \\
& \otimes\alpha^{-3}(h_{12}g_{12})S\alpha^{-3}(h_{21}g_{21}) \\
= \;& (((\beta^{-1}(a)\cdot\alpha^{-3}(h_1))(\beta^{-1}(b)\cdot\alpha^{-3}(g_1))))\cdot S\alpha^{-3}(h_{22}g_{22}) \\
& \otimes\alpha^{-4}(h_{211}g_{211})S\alpha^{-4}(h_{212}g_{212}) \\
= \;& (((\beta^{-1}(a)\cdot\alpha^{-3}(h_1))(\beta^{-1}(b)\cdot\alpha^{-3}(g_1))))\cdot S\alpha^{-2}(h_2g_2)\otimes 1_H \\
= \;& \beta(((\beta^{-2}(a)\cdot\alpha^{-4}(h_1))(\beta^{-2}(b)\cdot\alpha^{-4}(g_1)))\cdot S\alpha^{-3}(h_2g_2)))\otimes 1_H.
\end{aligned}
$$

Thus, $((\beta^{-2}(a)\cdot\alpha^{-4}(h_1))(\beta^{-2}(b)\cdot\alpha^{-4}(g_1)))\cdot S\alpha^{-3}(h_2g_2)\in A$.

Next, we check that the associativity holds. For all $a, b, c \in A, h, g, k \in H$, we can calculate

$$
\begin{aligned}
& ((a\otimes h)(b\otimes g))(\beta(c)\otimes\alpha(k)) \\
= \;& ((\beta^{-2}(((\beta^{-2}(a)\cdot\alpha^{-4}(h_{11}))(\beta^{-2}(b)\cdot\alpha^{-4}(g_{11})))\cdot S\alpha^{-3}(h_{12}g_{12}))\cdot\alpha^{-5}(h_{211}g_{211})) \\
& (\beta^{-1}(c)\cdot\alpha^{-3}(k_{11})))\cdot S\alpha^{-3}(\alpha^{-1}(h_{212}g_{212})\alpha(k_{12}))\otimes\alpha^{-2}(h_{22}g_{22}))k_2 \\
= \;& ((((\beta^{-3}(a)\cdot\alpha^{-5}(h_{11}))(\beta^{-3}(b)\cdot\alpha^{-5}(g_{11})))\cdot S\alpha^{-5}(h_{12}g_{12})\alpha^{-6}(h_{211}g_{211})) \\
& (\beta^{-1}(c)\cdot\alpha^{-3}(k_{11})))\cdot S\alpha^{-4}(h_{212}g_{212})\alpha^{-2}(k_{12})\otimes\alpha^{-2}(h_{22}g_{22}))k_2 \\
= \;& ((((\beta^{-3}(a)\cdot\alpha^{-4}(h_1))(\beta^{-3}(b)\cdot\alpha^{-4}(g_1)))\cdot S\alpha^{-6}(h_{211}g_{211})\alpha^{-6}(h_{212}g_{212})) \\
& (\beta^{-1}(c)\cdot\alpha^{-3}(k_{11})))\cdot S\alpha^{-4}(h_{221}g_{221})\alpha^{-2}(k_{12}))\otimes\alpha^{-3}(h_{222}g_{222}))k_2 \\
= \;& (((\beta^{-2}(a)\cdot\alpha^{-3}(h_1))(\beta^{-2}(b)\cdot\alpha^{-3}(g_1)))(\beta^{-1}(c)\cdot\alpha^{-3}(k_{11}))) \\
& \cdot S\alpha^{-3}(h_{21}g_{21})\alpha^{-2}(k_{12})\otimes\alpha^{-2}(h_{22}g_{22}))k_2.
\end{aligned}
$$

In a similar way, we get

$$(\beta(a)\otimes\alpha(h))((b\otimes g)(c\otimes k))$$

$$\begin{aligned}
&= ((\beta^{-1}(a)\cdot\alpha^{-3}(h_{11}))((\beta^{-2}(b)\cdot\alpha^{-3}(g_1))(\beta^{-2}(c)\cdot\alpha^{-3}(k_1))))\\
&\quad \cdot S(\alpha^{-2}(h_{12})\alpha^{-3}(g_{21}k_{21}))\otimes h_2\alpha^{-2}(g_{22}k_{22})\\
&= (((\beta^{-2}(a)\cdot\alpha^{-3}(h_1))(\beta^{-2}(b)\cdot\alpha^{-3}(g_1)))(\beta^{-1}(c)\cdot\alpha^{-3}(k_{11})))\\
&\quad \cdot S(\alpha^{-3}(h_{21}g_{21})\alpha^{-2}(k_{12}))\otimes\alpha^{-2}(h_{22}g_{22}))k_2.
\end{aligned}$$

So we get $((a\otimes h)(b\otimes g))(\beta(c)\otimes\alpha(k)) = (\beta(a)\otimes\alpha(h))((b\otimes g)(c\otimes k))$. It is easy to see that $(a\otimes h)(1_A\otimes 1_H) = (1_A\otimes 1_H)(a\otimes h) = \beta(a)\otimes\alpha(h)$. So $(A\otimes H, \beta\otimes\alpha, 1_A\otimes 1_H)$ is a Hom-algebra.

Finally, we show that $B\simeq A\Box_\rho H$ as Hom-algebras. Since $(B,\beta,\cdot,\rho)$ is a right H-Hom-Hopf module and $A = B^{coH}$, there is an isomorphism of right H-Hom-Hopf module, which is given by

$$\varphi: A\otimes H\to B,\quad \varphi(a\otimes h) = a\cdot h,\ \forall a,b\in A, h,g\in H.$$

We only need to verify that φ is a Hom-algebra morphism. For all $a,b\in A, h,g\in H$,

$$\begin{aligned}
&\varphi((a\otimes h)(b\otimes g))\\
&= \varphi(((\beta^{-2}(a)\cdot\alpha^{-4}(h_{11}))(\beta^{-2}(b)\cdot\alpha^{-4}(g_{11})))\cdot S\alpha^{-3}(h_{12}g_{12})\otimes\alpha^{-1}(h_2g_2))\\
&= (((\beta^{-2}(a)\cdot\alpha^{-4}(h_{11}))(\beta^{-2}(b)\cdot\alpha^{-4}(g_{11})))\cdot S\alpha^{-3}(h_{12}g_{12}))\cdot\alpha^{-1}(h_2g_2)\\
&= ((\beta^{-1}(a)\cdot\alpha^{-3}(h_{11}))(\beta^{-1}(b)\cdot\alpha^{-3}(g_{11})))\cdot(S\alpha^{-3}(h_{12}g_{12})\alpha^{-2}(h_2g_2))\\
&= ((\beta^{-1}(a)\cdot\alpha^{-2}(h_1))(\beta^{-1}(b)\cdot\alpha^{-2}(g_1)))\cdot\varepsilon(h_2g_2)1_H\\
&= ((\beta^{-1}(a)\cdot\alpha^{-1}(h))(\beta^{-1}(b)\cdot\alpha^{-1}(g)))\cdot 1_H\\
&= (a\cdot h)(b\cdot g) = \varphi(a\otimes h)\varphi(b\otimes g).
\end{aligned}$$

Thus, $B\simeq A\Box_\rho H$ as Hom-algebras. □

On the other hand, we have obtained the coalgebra factorization of right H-Hom-module coalgebras with Hom-Hopf module structure (see [3], Theorem 4.1).

Proposition 3.2. *Let (H,α) be a Hom-Hopf algebra and $(B,\cdot,\rho,\beta)$ be a right H-Hom-Hopf module. We assume that $(B,\cdot,\beta)$ is a right H-Hom-module coalgebra, and set $A = B^{coH} = \{a\in B\mid \rho(a) = \beta(a)\otimes 1\}$. The comultiplication on $A\otimes H$ is given by*

$$\begin{aligned}
\Delta(a\otimes h) &= \beta^{-3}(a_{1(0)(0)})\cdot S\alpha^{-3}(a_{1(0)(1)})\otimes\alpha^{-2}(a_{1(1)})\alpha^{-1}(h_1)\\
&\quad \otimes\beta^{-3}(a_{2(0)(0)})\cdot S\alpha^{-3}(a_{2(0)(1)})\otimes\alpha^{-2}(a_{2(1)})\alpha^{-1}(h_2),
\end{aligned}$$

and the counit is given by

$$\varepsilon(a\otimes h) = \varepsilon(a)\varepsilon(h),$$

for any $a\in A, h\in H$, then $(A\otimes H,\beta\otimes\alpha)$ is a Hom-coalgebra, we write it as $A\Box^{\cdot}H$. Moreover, $B\simeq A\Box^{\cdot}H$ as Hom-coalgebras.

By Proposition 3.1 and Proposition 3.2, we can obtain the main result of this section.

Theorem 3.3. *Let (H,α) be a Hom-Hopf algebra, (B,β) be a Hom-bialgebra and $(B,\cdot,\rho,\beta)$ be a right H-Hom-Hopf module. We assume that (B,ρ,β) is a right H-Hom-comodule algebra and $(B,\cdot,\beta)$ is a right H-Hom-module coalgebra, set $A=B^{coH}=\{a\in B\mid \rho(a)=\beta(a)\otimes 1\}$. Define the following operations on $A\otimes H$,*

$$\begin{aligned}(a\otimes h)(b\otimes g)&=((\beta^{-2}(a)\cdot\alpha^{-4}(h_{11}))(\beta^{-2}(b)\cdot\alpha^{-4}(g_{11})))\cdot S\alpha^{-3}(h_{12}g_{12})\\&\quad\otimes\alpha^{-1}(h_2g_2),\\ \Delta(a\otimes h)&=\beta^{-3}(a_{1(0)(0)})\cdot S\alpha^{-3}(a_{1(0)(1)})\otimes\alpha^{-2}(a_{1(1)})\alpha^{-1}(h_1)\\&\quad\otimes\beta^{-3}(a_{2(0)(0)})\cdot S\alpha^{-3}(a_{2(0)(1)})\otimes\alpha^{-2}(a_{2(1)})\alpha^{-1}(h_2),\\ \varepsilon(a\otimes h)&=\varepsilon(a)\varepsilon(h),\end{aligned}$$

for all $a,b\in A$, $h,g\in H$, then $(A\otimes H,\beta\otimes\alpha)$ is a Hom-bialgebra, we write it as $A\Box^{\cdot}_{\rho}H$. Moreover, $B\simeq A\Box^{\cdot}_{\rho}H$ as Hom-bialgebras.

Proof. By Proposition 3.1, we have $B\simeq A\Box_{\rho}H$ as Hom-algebras with the isomorphism

$$\varphi: A\otimes H\to B,\quad \varphi(a\otimes h)=a\cdot h,\ \forall a,b\in A,h,g\in H.$$

By Proposition 3.2, we have $B\simeq A\Box^{\cdot}H$ as Hom-coalgebras with the same isomorphism φ. Since B is a Hom-bialgebra, it follows that $A\otimes H$ is a Hom-bialgebra with the multiplication $B\simeq A\Box_{\rho}H$ on and the comultiplication on $B\simeq A\Box^{\cdot}H$. Moreover, $B\simeq A\Box^{\cdot}_{\rho}H$ as Hom-bialgebras. □

4. Hom-crossed product algebras and cleft extensions

In this section, let H be a Hom-Hopf algebra, B be a right H-Hom-Hopf module and (B,ρ) be a right H-Hom-comodule algebra. Set $A=B^{coH}$, we can define a weak action of H on A, thus obtain the Hom-crossed product algebra $A\sharp_{\sigma}H$ and $B\simeq A\sharp_{\sigma}H$ as Hom-algebras. Next, we dicuss the relation between cleft extension and Hom-comodule algebra with Hom-Hopf module structure.

Lemma 4.1. *Let (H,S,α) be a Hom-Hopf algebra, $(B,\cdot,\rho,\beta)$ be a right H-Hom-Hopf module and (B,ρ) be a right H-Hom-comodule algebra, set $A=B^{coH}$, define* *the map*

$$\rightharpoonup: H\otimes A\to A,\ h\rightharpoonup a=((1\cdot\alpha^{-3}(h_1))\beta^{-1}(a))\cdot S\alpha^{-1}(h_2),\ \forall h\in H, a\in A,$$

and an element $\sigma\in Hom(H\otimes H, A)$*, for any* $h, g\in H$*,*

$$\sigma(h,g)=((1\cdot\alpha^{-4}(h_1))(1\cdot\alpha^{-4}(g_1)))\cdot S\alpha^{-3}(h_2g_2).$$

If the following condition holds,

$$(ab)\cdot\alpha(h)=\beta(a)(b\cdot h),\ \forall a\in A, b\in B, h\in H, \tag{4.1}$$

then

(1) $\rightharpoonup$ *is a weak action of* H *on* A*;*

(2) $(C1)$*,* $(C2)$*,* $(C3)$ *are satisfied.*

Proof. (1) We first check that $\rightharpoonup$ is well defined. Since $(B,\cdot,\rho,\beta)$ is a right H-Hom-Hopf module and (B,ρ) is a right H-Hom-comodule algebra, $A=B^{coH}$. It is easy to get

$$\rho(((1\cdot\alpha^{-3}(h_1))\beta^{-1}(a))\cdot S\alpha^{-1}(h_2))=((1\cdot\alpha^{-2}(h_1))a)\cdot S(h_2)\otimes 1.$$

From Eq.(4.1), we can get

$$(h_1\rightharpoonup a)(1\cdot h_2)=(1\cdot\alpha(h))\beta(a),\ \forall h\in H, a\in A, \tag{4.2}$$

In fact

$$\begin{aligned}
&(h_1\rightharpoonup a)(1\cdot h_2)\\
=\ &(((1\cdot\alpha^{-3}(h_{11}))\beta^{-1}(a))\cdot S\alpha^{-1}(h_{12}))(1\cdot h_2)\\
\overset{(4.1)}{=}\ &(((1\cdot\alpha^{-3}(h_{11}))\beta^{-1}(a))\cdot S\alpha^{-1}(h_{12}))\cdot\alpha(h_2)\\
=\ &((1\cdot\alpha^{-1}(h_1))a)\cdot(S\alpha^{-1}(h_{21})\alpha^{-1}(h_{22}))\\
=\ &(1\cdot\alpha(h))\beta(a).
\end{aligned}$$

For all $a\in A, b\in B, h\in H$, we can obtain

$$\begin{aligned}
&(h_1\rightharpoonup a)(h_2\rightharpoonup b)\\
=\ &(h_1\rightharpoonup a)(((1\cdot\alpha^{-3}(h_{21}))\beta^{-1}(b))\cdot S\alpha^{-1}(h_{22}))\\
=\ &(((\alpha^{-3}(h_{11}\rightharpoonup\beta^{-2}(a))(1\cdot\alpha^{-3}(h_{12})))b)\cdot S\alpha(h_2)\\
\overset{(4.2)}{=}\ &(((1\cdot\alpha^{-2}(h_1))\beta^{-1}(a))b)\cdot S\alpha(h_2)\\
=\ &((1\cdot\alpha^{-1}(h_1))\beta^{-1}(ab))\cdot S\alpha(h_2)\\
=\ &\alpha^2(h)\rightharpoonup(ab),
\end{aligned}$$

and $h\rightharpoonup 1=\varepsilon(h)1,\ \ 1\rightharpoonup a=\beta(a)$. Thus $\rightharpoonup$ is a weak action of H on A.

(2) It is obvious that $(C1)$ holds. Now we verify that $(C2)$ holds. On the one hand,

$$
\begin{aligned}
&(h_1 \rightharpoonup (\alpha^{-1}(g_1) \rightharpoonup a))\sigma(\alpha(h_2), \alpha(g_2)) \\
= \ &(h_1 \rightharpoonup (\alpha^{-1}(g_1) \rightharpoonup a))(((1 \cdot \alpha^{-3}(h_{21}))(1 \cdot \alpha^{-3}(g_{21}))) \cdot S\alpha^{-2}(h_{22}g_{22})) \\
= \ &(((\alpha^{-3}(h_{11}) \rightharpoonup (\alpha^{-4}(g_{11}) \rightharpoonup \beta^{-2}(a)))(1 \cdot \alpha^{-3}(h_{12})))(1 \cdot \alpha^{-2}(g_{12}))) \cdot S(h_2 g_2) \\
= \ &(((1 \cdot \alpha^{-2}(h_1))(\alpha^{-3}(g_{11}) \rightharpoonup \beta^{-1}(a)))(1 \cdot \alpha^{-2}(g_{12}))) \cdot S(h_2 g_2) \\
= \ &((1 \cdot \alpha^{-1}(h_1))((1 \cdot \alpha^{-2}(g_1))a)) \cdot S(h_2 g_2).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\sigma(\alpha(h_1), \alpha(g_1))(\alpha^{-1}(h_2 g_2) \rightharpoonup \beta(a)) \\
= \ &(((1 \cdot \alpha^{-3}(h_{11}))(1 \cdot \alpha^{-3}(g_{11}))) \cdot S\alpha^{-2}(h_{12}g_{12}))(((1 \cdot \alpha^{-4}(h_{21}g_{21}))a) \cdot \\
&S\alpha^{-2}(h_{22}g_{22})) \\
= \ &(((((1 \cdot \alpha^{-5}(h_{11}))(1 \cdot \alpha^{-5}(g_{11}))) \cdot S\alpha^{-4}(h_{12}g_{12}))(1 \cdot \alpha^{-4}(h_{21}g_{21})))\beta(a)) \cdot \\
&S\alpha^{-1}(h_{22}g_{22}) \\
= \ &(((((1 \cdot \alpha^{-5}(h_{11}))(1 \cdot \alpha^{-5}(g_{11}))) \cdot S\alpha^{-4}(h_{12}g_{12})) \cdot \alpha^{-3}(h_{21}g_{21}))\beta(a)) \cdot \\
&S\alpha^{-1}(h_{22}g_{22}) \\
= \ &(((((1 \cdot \alpha^{-4}(h_{11}))(1 \cdot \alpha^{-4}(g_{11}))) \cdot (S\alpha^{-4}(h_{12}g_{12})\alpha^{-4}(h_{21}g_{21})))\beta(a)) \cdot \\
&S\alpha^{-1}(h_{22}g_{22}) \\
= \ &((1 \cdot \alpha^{-1}(h_1))((1 \cdot \alpha^{-2}(g_1))a)) \cdot S(h_2 g_2).
\end{aligned}
$$

Similarly we can obtain $(C3)$ holds. The proof is completed. □

Theorem 4.2. *Let (H, S, α) be a Hom-Hopf algebra, $(B, \cdot, \rho, \beta)$ be a right H-Hom-Hopf module and (B, ρ) be a right H-Hom-comodule algebra, set $A = B^{coH}$, if for all $a \in A, b \in B$ and $h \in H$, Eq.(4.1) holds. Then we get a Hom-crossed product algebra $A\sharp_\sigma H$ and $B \simeq A\sharp_\sigma H$ as Hom-algebras.*

Proof. By Lemma 4.1, we get a Hom-crossed product algebra $A\sharp_\sigma H$ with the multiplication given by

$$(a \otimes h)(b \otimes g) = a((\alpha^{-4}(h_{11}) \rightharpoonup \beta^{-2}(b))\sigma(\alpha^{-3}(h_{12}), \alpha^{-2}(g_1))) \otimes \alpha^{-1}(h_2 g_2).$$

We note that $B \simeq A \otimes H$ as right H-Hom-Hopf modules, and the isomorphic map is given by

$$g : B \to A \otimes H, \quad g(b) = \beta^{-4}(b_{(0)(0)}) \cdot S\alpha^{-4}(b_{(0)(1)}) \otimes \alpha^{-2}(b_{(1)}).$$

Then we only need to prove that g is a Hom-algebra map. In fact, for all $b, b^{'} \in B$, we have

$$
\begin{aligned}
& g(b)g(b^{'}) \\
= \;& (\beta^{-4}(b_{(0)(0)}) \cdot S(\alpha^{-4}(b_{(0)(1)})) \otimes \alpha^{-2}(b_{(1)}))(\beta^{-4}(b^{'}_{(0)(0)}) \cdot S\alpha^{-4}(b^{'}_{(0)(1)}) \\
& \otimes\alpha^{-2}(b^{'}_{(1)})) \\
= \;& (\beta^{-4}(b_{(0)(0)}) \cdot S(\alpha^{-4}(b_{(0)(1)})))((\alpha^{-6}(b_{(1)11}) \rightharpoonup (\beta^{-6}(b^{'}_{(0)(0)}) \cdot S\alpha^{-6}(b^{'}_{(0)(1)}))) \\
& \sigma(\alpha^{-5}(b_{(1)12}), \alpha^{-4}(b^{'}_{(1)1}))) \otimes \alpha^{-3}(b_{(1)2}b^{'}_{(1)2}) \\
\overset{(4.2)}{=} \;& (\beta^{-4}(b_{(0)(0)}) \cdot S(\alpha^{-4}(b_{(0)(1)})))((((1 \cdot \alpha^{-8}(b_{(1)11}))(\beta^{-7}(b^{'}_{(0)(0)}) \cdot \\
& S\alpha^{-7}(b^{'}_{(0)(1)})))(1 \cdot \alpha^{-7}(b^{'}_{(1)11}))) \cdot S\alpha^{-6}(b_{(1)12}b^{'}_{(1)12})) \otimes \alpha^{-3}(b_{(1)2}b^{'}_{(1)2}) \\
= \;& (\beta^{-4}(b_{(0)(0)}) \cdot S(\alpha^{-4}(b_{(0)(1)})))(((1 \cdot \alpha^{-7}(b_{(1)11}))(\beta^{-6}(b^{'}_{(0)(0)}) \cdot \\
& (S\alpha^{-7}(b^{'}_{(0)(1)})\alpha^{-8}(b^{'}_{(1)11})))) \cdot S\alpha^{-6}(b_{(1)12}b^{'}_{(1)12})) \otimes \alpha^{-3}(b_{(1)2}b^{'}_{(1)2}) \\
= \;& (\beta^{-3}(b_{(0)}b^{'}_{(0)}) \cdot S\alpha^{-4}(b_{(1)1}b^{'}_{(1)1})) \otimes \alpha^{-3}(b_{(1)2}b^{'}_{(1)2}) \\
= \;& g(bb^{'}).
\end{aligned}
$$

The proof is completed. □

Now, we discuss the relation between cleft extensions and Hom-comodule algebras with Hom-Hopf module structure.

Proposition 4.3. *Let (B, ρ, γ, β) be a cleft extension. We assume that H acts on B to the right by*

$$b \cdot h = (\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h)),$$

then

(1) *$(B, \cdot, \beta)$ is a right H-Hom-module and Eq.*(4.1) *holds;*
(2) *$(B, \cdot, \rho, \beta)$ is a right H-Hom-Hopf module.*

Proof. Note that γ is convolution invertible right H-Hom-comodule map, we have

$$\gamma^{-1}(h)_{(0)} \otimes \gamma^{-1}(h)_{(1)} = \gamma^{-1}(h_2) \otimes S(h_1).$$

(1) We first verify that $(B,\cdot,\beta)$ is a right H-Hom-module. For all $b\in B, h, h'\in H$, we calculate

$$
\begin{aligned}
& (b\cdot h)\cdot\alpha(h') \\
= & ((\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h)))\cdot\alpha(h') \\
= & (\beta^{-2}(((\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h)))_{(0)}) \\
& \gamma^{-1}(\alpha^{-3}(((\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h)))_{(1)1}))) \\
& \gamma(\alpha^{-3}(((\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))\gamma(\alpha^{-3}(b_{(1)2}))\alpha^{-1}(h)))_{(1)2})h') \\
= & (\beta^{-2}((\beta^{-2}(b_{(0)})_{(0)}\gamma^{-1}(\alpha^{-3}(b_{(1)1}))_{(0)})\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h))_{(0)}) \\
& \gamma^{-1}(\alpha^{-3}((\beta^{-2}(b_{(0)})_{(1)}\gamma^{-1}(\alpha^{-3}(b_{(1)1}))_{(1)})\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h))_{(1)})_1)) \\
& \gamma(\alpha^{-3}((\beta^{-2}(b_{(0)})_{(1)}\gamma^{-1}(\alpha^{-3}(b_{(1)1}))_{(1)})\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h))_{(1)})_2h') \\
= & (\beta^{-2}((\beta^{-2}(b_{(0)(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)12})))\gamma(\alpha^{-3}(b_{(1)21})\alpha^{-1}(h_1))) \\
& \gamma^{-1}(\alpha^{-3}((\beta^{-2}(b_{(0)(1)})S\alpha^{-3}(b_{(1)11}))(\alpha^{-3}(b_{(1)22})\alpha^{-1}(h_2)))_1)) \\
& \gamma((\alpha^{-3}((\beta^{-2}(b_{(0)(1)})S\alpha^{-3}(b_{(1)11}))(\alpha^{-3}(b_{(1)22})\alpha^{-1}(h_2)))_2)h') \\
= & (\beta^{-2}((\beta^{-2}(b_{(0)(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)21})))\gamma(\alpha^{-4}(b_{(1)221})\alpha^{-1}(h_1))) \\
& \gamma^{-1}(\alpha^{-3}((\beta^{-2}(b_{(0)(1)})S\alpha^{-2}(b_{(1)1}))(\alpha^{-4}(b_{(1)222})\alpha^{-1}(h_2)))_1)) \\
& \gamma((\alpha^{-3}((\beta^{-2}(b_{(0)(1)})S\alpha^{-2}(b_{(1)1}))(\alpha^{-4}(b_{(1)222})\alpha^{-1}(h_2)))_2)h') \\
= & (\beta^{-2}((\beta^{-1}(b_{(0)})\gamma^{-1}(\alpha^{-4}(b_{(1)221})))\gamma(\alpha^{-5}(b_{(1)2221})\alpha^{-1}(h_1))) \\
& \gamma^{-1}(\alpha^{-3}((\beta^{-2}(b_{(1)1})S\alpha^{-3}(b_{(1)21}))(\alpha^{-5}(b_{(1)2222})\alpha^{-1}(h_2)))_1)) \\
& \gamma((\alpha^{-3}((\beta^{-2}(b_{(1)1})S\alpha^{-3}(b_{(1)21}))(\alpha^{-5}(b_{(1)2222})\alpha^{-1}(h_2)))_2)h') \\
= & (\beta^{-2}((\beta^{-1}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)21})))\gamma(\alpha^{-4}(b_{(1)221})\alpha^{-1}(h_1))) \\
& \gamma^{-1}(\alpha^{-3}((\beta^{-3}(b_{(1)11})S\alpha^{-3}(b_{(1)12}))(\alpha^{-4}(b_{(1)222})\alpha^{-1}(h_2)))_1)) \\
& \gamma((\alpha^{-3}((\beta^{-3}(b_{(1)11})S\alpha^{-3}(b_{(1)12}))(\alpha^{-4}(b_{(1)222})\alpha^{-1}(h_2)))_2)h') \\
= & (((\beta^{-3}(b_{(0)})\gamma^{-1}(\alpha^{-5}(b_{(1)21})))\gamma(\alpha^{-6}(b_{(1)221})\alpha^{-3}(h_1))) \\
& \gamma^{-1}((\varepsilon(b_{(1)1})1_H(\alpha^{-7}(b_{(1)222})\alpha^{-4}(h_2))_1))\gamma(1_H(\alpha^{-7}(b_{(1)222})\alpha^{-4}(h_2)))_2)h') \\
= & (((\beta^{-3}(b_{(0)})\gamma^{-1}(\alpha^{-4}(b_{(1)1})))\gamma(\alpha^{-5}(b_{(1)21})\alpha^{-3}(h_1)))\gamma^{-1}((\alpha^{-5}(b_{(1)22})\alpha^{-3}(h_2))_1)) \\
& \gamma(((\alpha^{-5}(b_{(1)22})\alpha^{-3}(h_2))_2)h') \\
= & ((\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))(\gamma(\alpha^{-5}(b_{(1)21})\alpha^{-3}(h_1))\gamma^{-1}(\alpha^{-6}(b_{(1)221})\alpha^{-4}(h_{21})))) \\
& \gamma((\alpha^{-5}(b_{(1)222})\alpha^{-3}(h_{22}))h') \\
= & ((\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-4}(b_{(1)11})))(\gamma(\alpha^{-5}(b_{(1)12})\alpha^{-4}(h_{11}))\gamma^{-1}(\alpha^{-5}(b_{(1)21})\alpha^{-4}(h_{12})))) \\
& \gamma((\alpha^{-4}(b_{(1)22})\alpha^{-2}(h_2))h')
\end{aligned}
$$

$$
\begin{aligned}
&= ((\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))(\varepsilon(b_{(1)21})\varepsilon(h_1)1_H))\gamma((\alpha^{-4}(b_{(1)22})\alpha^{-2}(h_2))h') \\
&= (\beta^{-1}(b_{(0)})\gamma^{-1}(\alpha^{-2}(b_{(1)1})))\gamma((\alpha^{-3}(b_{(1)2})\alpha^{-1}(h))h') \\
&= \beta(b)\cdot(hh'),
\end{aligned}
$$

and $b\cdot 1 = (\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))\gamma(\alpha^{-2}(b_{(1)2})) = \beta(b)$. Thus $(B,\beta,\cdot)$ is a right H-Hom-module.

(2) Now we prove $(B,\cdot,\rho,\beta)$ is a right H-Hom-Hopf module.

$$
\begin{aligned}
& \rho(b\cdot h) \\
&= \rho((\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h))) \\
&= (\beta^{-2}(b_{(0)(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)12})))\gamma(\alpha^{-3}(b_{(1)21})\alpha^{-1}(h_1)) \\
&\quad \otimes(\alpha^{-2}(b_{(0)(1)})S\alpha^{-3}(b_{(1)11}))(\alpha^{-3}(b_{(1)22})\alpha^{-1}(h_2)) \\
&= (\beta^{-1}(b_{(0)})\gamma^{-1}(\alpha^{-4}(b_{(1)212})))\gamma(\alpha^{-4}(b_{(1)121})\alpha^{-1}(h_1)) \\
&\quad \otimes(\alpha^{-2}(b_{(1)1})S\alpha^{-4}(b_{(1)211}))(\alpha^{-4}(b_{(1)222})\alpha^{-1}(h_2)) \\
&= (\beta^{-1}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)12})))\gamma(\alpha^{-3}(b_{(1)21})\alpha^{-1}(h_1)) \\
&\quad \otimes(\alpha^{-4}(b_{(1)111})S\alpha^{-4}(b_{(1)112}))(\alpha^{-3}(b_{(1)22})\alpha^{-1}(h_2)) \\
&= (\beta^{-1}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)12})))\gamma(\alpha^{-3}(b_{(1)21})\alpha^{-1}(h_1))\otimes\varepsilon(b_{(1)11})(\alpha^{-2}(b_{(1)22})h_2) \\
&= (\beta^{-1}(b_{(0)})\gamma^{-1}(\alpha^{-2}(b_{(1)1})))\gamma(\alpha^{-3}(b_{(1)21})\alpha^{-1}(h_1))\otimes\alpha^{-2}(b_{(1)22})h_2 \\
&= (\beta^{-1}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)11})))\gamma(\alpha^{-3}(b_{(1)12})\alpha^{-1}(h_1))\otimes\alpha^{-1}(b_{(1)2})h_2 \\
&= (\beta^{-2}(b_{(0)(0)})\gamma^{-1}(\alpha^{-3}(b_{(0)(1)1})))\gamma(\alpha^{-3}(b_{(0)(1)2})\alpha^{-1}(h_1))\otimes b_1h_2 \\
&= b_{(0)}\cdot h_1\otimes b_{(1)}h_2.
\end{aligned}
$$

The proof is completed. □

By Theorem 4.2 and Proposition 4.3, we can get the following result.

Corollary 4.4. *Let (B,ρ,γ,β) be a cleft extension. Then we get a Hom-crossed product algebra $A\sharp_\sigma H$ and $B\simeq A\sharp_\sigma H$ as Hom-algebras.*

In Theorem 4.2, we get B is isomorphic to a Hom-crossed product algebra, that is $B\simeq A\sharp_\sigma H$ as Hom-algebras. Now, if we slightly change the condition (4.1), we can get B is isomorphic to a Hom-smash product algebra.

Lemma 4.5. *Let (H,S,α) be a Hom-Hopf algebra, $(B,\cdot,\rho,\beta)$ be a right H-Hom-Hopf module and (B,ρ) be a right H-Hom-comodule algebra satisfying the following condition*

$$(bb')\cdot\alpha(h) = \beta(b)(b'\cdot h),\ \forall b,b'\in B, h\in H. \tag{4.3}$$

Set $A = B^{coH}$, define the map

$$\rightharpoonup: H \otimes A \to A, \quad h \rightharpoonup a = ((1 \cdot \alpha^{-3}(h_1))\beta^{-1}(a)) \cdot S\alpha^{-1}(h_2).$$

Then $(A, \rightharpoonup)$ is a left H-Hom-module algebra.

Proof. We only need prove $(A, \rightharpoonup)$ is a left H-Hom-module. $\forall a \in A, h, g \in H$,

$$\begin{aligned} & \alpha(h) \rightharpoonup (g \rightharpoonup a) \\ = \;& \alpha(h) \rightharpoonup (((1 \cdot \alpha^{-3}(g_1))\beta^{-1}(a)) \cdot S\alpha^{-1}(g_2)) \\ = \;& ((1 \cdot \alpha^{-2}(h_1))(((1 \cdot \alpha^{-4}(g_1))\beta^{-2}(a)) \cdot S\alpha^{-2}(g_2)))) \cdot S(h_2) \\ = \;& (((1 \cdot \alpha^{-4}(h_1 g_1))\beta^{-1}(a)) \cdot S\alpha^{-1}(g_2)) \cdot S(h_2) \\ = \;& ((1 \cdot \alpha^{-3}(h_1 g_1))a) \cdot S\alpha^{-1}(h_2 g_2) \\ = \;& (hg) \rightharpoonup \beta(a). \end{aligned}$$

It is easy to see $1 \rightharpoonup a = \beta(a)$, so $(A, \rightharpoonup)$ is a left H-Hom-module. □

Theorem 4.6. *Let (H, S, α) be a Hom-Hopf algebra and $(B, \cdot, \rho, \beta)$ be a right H-Hom-Hopf module. Assume (B, ρ) is a right H-Hom-comodule algebra satisfying Eq.(4.3). Set $A = B^{coH}$, then we get a Hom-smash product $A\sharp H$ with the multiplication given by*

$$(a \otimes h)(a' \otimes h') = a(h_1 \rightharpoonup \beta^{-1}(a')) \otimes \alpha^{-1}(h_2)h',$$

and $B \simeq A\sharp H$ as Hom-algebras.

Proof. It is easy to prove by Lemma 4.5. □

From Corollary 4.4, we know that if (B, ρ, γ, β) is a cleft extension, then we get B is isomorphic to the Hom-crossed product algebra $A\sharp_\sigma H$. Next, we give an equivalent characterization about cleft extension.

Theorem 4.7. *Let (H, α) be a Hom-Hopf algebra and (B, ρ, β) be a right H-Hom-comodule algebra, and set $A = B^{coH} = \{a \in B \mid \rho(a) = \beta(a) \otimes 1\}$, then $A \subset B$ is a cleft extension if and only if $(B, \cdot, \rho, \beta)$ is a right H-Hom-Hopf module, and there exists a convolution invertible linear map $\gamma : H \to B$ satisfying $\gamma(h) = 1 \cdot \alpha^{-1}(h)$.*

Proof. $\Rightarrow$): Assume that $A \subset B$ is a cleft extension, then there exists a convolution invertible Hom-comodule map $\gamma : H \to B$ such that $\gamma(1) = 1$, so we have $\gamma^{-1}(1) = 1$. Define a map

$$\cdot : B \otimes H \to B,\ b \cdot h = (\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h)),\ \forall b \in B, h \in H,$$

clearly, $\gamma(h) = 1 \cdot \alpha^{-1}(h)$. By Proposition 4.3, we see that $(B, \cdot, \rho, \beta)$ is a right H-Hom-Hopf module.

$\Leftarrow$): It is sufficient to show γ is a right Hom-comodule map, using the fact that $(B, \cdot, \rho, \beta)$ is a right H-Hom-Hopf module and $\gamma(h) = 1 \cdot \alpha^{-1}(h)$, we get

$$\rho\gamma(h) = \rho(1 \cdot \alpha^{-1}(h)) = 1 \cdot \alpha^{-1}(h_1) \otimes h_2 = \gamma(h_1) \otimes h_2, \quad \forall h \in H.$$

The proof is completed. □

By the above results, we have the following conclusion.

Theorem 4.8. *Let (H, α) be a Hom-Hopf algebra and (B, ρ, γ, β) be a cleft extension, set $A = B^{coH}$, then*

$$B \simeq A\Box_\rho H = A\sharp_\sigma H,$$

as Hom-algebras.

Proof. By Proposition 3.1 and Theorem 4.2, we have $B \simeq A\Box_\rho H$ and $B \simeq A\sharp_\sigma H$ as Hom-algebras. Since (B, ρ, γ, β) is a cleft extension, we get $(B, \cdot, \rho, \beta)$ is a right H-Hom-Hopf module by Proposition 3.3, where the module structure is defined by

$$b \cdot h = (\beta^{-2}(b_{(0)})\gamma^{-1}(\alpha^{-3}(b_{(1)1})))\gamma(\alpha^{-3}(b_{(1)2})\alpha^{-1}(h)).$$

Now we prove $A\Box_\rho H$ has the same multiplication with $A\sharp_\sigma H$.

$$\begin{aligned}
& (a \otimes h)(b \otimes g) \\
= \ & ((\beta^{-2}(a) \cdot \alpha^{-4}(h_{11}))(\beta^{-2}(b) \cdot \alpha^{-4}(g_{11}))) \cdot S\alpha^{-3}(h_{12}g_{12}) \otimes \alpha^{-1}(h_2g_2) \\
= \ & (((\beta^{-2}(\beta^{-2}(a)_{(0)})\gamma^{-1}(\alpha^{-3}(\beta^{-2}(a)_{(1)1})))\gamma(\alpha^{-3}(\beta^{-2}(a)_{(1)2})\alpha^{-5}(h_{11}))) \\
& (\beta^{-2}(b) \cdot \alpha^{-4}(g_{11}))) \cdot S\alpha^{-3}(h_{12}g_{12}) \otimes \alpha^{-1}(h_2g_2) \\
= \ & ((\beta^{-2}(a)\gamma(\alpha^{-4}(h_{11})))(\beta^{-2}(b)\gamma(\alpha^{-4}(g_{11})))) \cdot S\alpha^{-3}(h_{12}g_{12}) \otimes \alpha^{-1}(h_2g_2) \\
= \ & (\beta^{-2}((\beta^{-2}(a)_{(0)}\gamma(\alpha^{-4}(h_{11}))_{(0)})(\beta^{-2}(b)_{(0)}\gamma(\alpha^{-4}(g_{11}))_{(0)})) \\
& \gamma^{-1}(\alpha^{-3}(((\beta^{-2}(a)_{(1)}\gamma(\alpha^{-4}(h_{11}))_{(1)})(\beta^{-2}(b)_{(1)}\gamma(\alpha^{-4}(g_{11}))_{(1)}))_1))) \\
& \gamma(\alpha^{-3}(((\beta^{-2}(a)_{(1)}\gamma(\alpha^{-4}(h_{11}))_{(1)})(\beta^{-2}(b)_{(1}\gamma(\alpha^{-4}(g_{11}))_{(1)}))_2)S\alpha^{-4}(h_{12}g_{12})) \\
& \otimes\alpha^{-1}(h_2g_2) \\
= \ & (((\beta^{-3}(a)\gamma(\alpha^{-6}(h_{111})))(\beta^{-3}(b)\gamma(\alpha^{-6}(g_{111}))))\gamma^{-1}(\alpha^{-6}(h_{1121}g_{1121}))) \\
& \gamma(\alpha^{-6}(h_{1122}g_{1122})S\alpha^{-4}(h_{12}g_{12})) \otimes \alpha^{-1}(h_2g_2) \\
= \ & (((\beta^{-3}(a)\gamma(\alpha^{-6}(h_{111})))(\beta^{-3}(b)\gamma(\alpha^{-6}(g_{111}))))\gamma^{-1}(\alpha^{-5}(h_{112}g_{112}))) \\
& \gamma(\alpha^{-4}(h_{12}g_{12})S\alpha^{-4}(h_{21}g_{21})) \otimes \alpha^{-2}(h_{22}g_{22})
\end{aligned}$$

$$\begin{aligned}
&= (((\beta^{-3}(a)\gamma(\alpha^{-5}(h_{11})))(\beta^{-3}(b)\gamma(\alpha^{-5}(g_{11}))))\gamma^{-1}(\alpha^{-4}(h_{12}g_{12})))\\
&\quad \gamma(\alpha^{-5}(h_{211}g_{211})S\alpha^{-5}(h_{212}g_{212}))\otimes\alpha^{-2}(h_{22}g_{22})\\
&= ((\beta^{-2}(a)\gamma(\alpha^{-4}(h_{11})))(\beta^{-2}(b)\gamma(\alpha^{-4}(g_{11}))))\gamma^{-1}(\alpha^{-3}(h_{12}g_{12}))\otimes\alpha^{-1}(h_2g_2)\\
&= a(\gamma(\alpha^{-3}(h_{11}))((\beta^{-3}(b)\gamma(\alpha^{-5}(g_{11})))\gamma^{-1}(\alpha^{-5}(h_{12}g_{12}))))\otimes\alpha^{-1}(h_2g_2)\\
&= a((\alpha^{-4}(h_{11})\rightharpoonup\beta^{-2}(b))\sigma(\alpha^{-3}(h_{12}),\alpha^{-2}(g_1)))\sharp\alpha^{-1}(h_2g_2),
\end{aligned}$$

for all $h, g \in H, a, b \in A$, we define

$$\rightharpoonup: H\otimes A\to A,\quad h\rightharpoonup a = (\gamma(\alpha^{-2}(h_1))\beta^{-1}(a))\gamma^{-1}(\alpha^{-1}(h_2)),$$

and

$$\sigma: H\otimes H\to A,\quad \sigma(h,g)=\gamma(\alpha^{-2}(h_1))(\gamma(\alpha^{-3}(g_1))\gamma^{-1}(\alpha^{-4}(h_2g_2))).$$

Therefore, $B\simeq A\square_\rho H = A\sharp_\sigma H$ as Hom-algebras. $\square$

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