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# STRONGLY GRADED RINGS WHICH ARE KRULL RINGS

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Abstract. Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a strongly graded ring of type $\mathbb{Z}$ and $R_0$ is a prime Goldie ring. It is shown that the following three conditions are equivalent: (i) $R_0$ is a $\mathbb{Z}$-invariant Krull ring, (ii) $R$ is a Krull ring and (iii) $R$ is a graded Krull ring. We completely describe all $v$-invertible $R$-ideals in $Q$, where $Q$ is a quotient ring of $R$.

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## 1. Introduction

Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a strongly graded ring of type $\mathbb{Z}$, that is, $R_nR_m = R_{n+m}$ for all $n, m \in \mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers and $R_0$ is a prime Goldie ring with quotient ring $Q_0$ and $R_0 \subset Q_0$. Let $\mathcal{C}_0 = \{c_0 \in R_0 \mid c_0 \text{ is regular}\}$ is a regular Ore set of $R$ and $Q^g = R{\mathcal{C}_0}^{-1}$, the quotient ring of $R$ at $\mathcal{C}_0$, which is of the form $Q^g = \oplus_{n\in\mathbb{Z}} Q_0R_n$ $(Q_0R_n = R_nQ_0)$. It follows that $Q^g = Q_0[X, X^{-1}, \sigma]$, a skew Laurent polynomial ring over $Q_0$, where $\sigma$ is an automorphism of $Q_0$ and $X$ is a unit in $Q^g$ with $X \in R_1$ (see [2] or [7]). We denote by $Q$ the quotient ring of $R$.

A graded right $R$-submodule $I$ of $Q^g$ is called a *graded right $R$-ideal* if $I$ contains a regular homogeneous element in $Q^g$ and $aI \subseteq R$ for some regular homogeneous element $a$ in $Q^g$. In a similar way, we define a *graded left $R$-ideal* in $Q^g$. Note that if $I$ is a graded right $R$-ideal in $Q^g$, then $I = I_0R$, where $I_0 = I \cap Q_0$ is a right $R_0$-ideal in $Q_0$ since $Q^g$ is a strongly graded ring of type $\mathbb{Z}$ (see [5, Corollary I.3.8]). We refer the readers to [3] and [5] for some properties and definitions of order theory and graded rings which are not mentioned in this paper, respectively.

## 2. Main results

We use the following notation: for a right $R$-ideal $I$ in $Q$ define $(R : I)_l = \{q \in Q \mid qI \subseteq R\}$, a left $R$-ideal in $Q$ and for a left $R$-ideal $J$ in $Q$ $(R : J)_r = \{q \in Q \mid Jq \subseteq R\}$, a right $R$-ideal in $Q$. Let $I$ be a right ideal of $R$ and $r \in R$. We define

$r^{-1} \cdot I = \{s \in R \mid rs \in I\}$. We first study some properties of graded right ideals of $R$.

**Lemma 2.1.** *Let $I = I_0R$ be a graded right $R$-ideal in $Q^g$. Then*

(1) $(R : I_0R)_l = R(R_0 : I_0)_l$ *and is a graded left $R$-ideal in $Q^g$.*

(2) *If $I$ is a graded right ideal of $R$ and $r_n \in R_n$, then $r_n^{-1} \cdot I$ is a graded right ideal and $r_n^{-1} \cdot I = (r_n^{-1} \cdot (I_0R_n))R$, where $r_n^{-1} \cdot (I_0R_n) = \{s_0 \in R_0 \mid r_ns_0 \in I_0R_n\}$. In particular, for $r_0 \in R_0$, $r_0^{-1} \cdot (I_0R) = (r_0^{-1} \cdot I_0)R$.*

**Proof.** (1) It is clear.
(2) Let $s = s_{n_1} + \cdots + s_{n_k} \in r_n^{-1} \cdot (I_0R)$, where $n_1 > \cdots > n_k$. Then $I_0R \ni r_ns = r_ns_{n_1} + \cdots + r_ns_{n_k}$ and so $r_ns_{n_j} \in I_0R_{(n+n_j)} \subseteq I_0R$ for any $j$ $(1 \leq j \leq k)$. Thus $s_{n_j} \in r_n^{-1} \cdot (I_0R)$ and hence $r_n^{-1} \cdot (I_0R)$ is a graded right ideal. Furthermore $s_0 \in (r_n^{-1} \cdot (I_0R))_0$ if and only if $r_ns_0 \in I_0R_n$ if and only if $s_0 \in r_n^{-1} \cdot (I_0R_n)$, and so $(r_n^{-1} \cdot (I_0R))_0 = r_n^{-1} \cdot (I_0R_n)$. Hence $r_n^{-1} \cdot I = (r_n^{-1} \cdot (I_0R_n))R$ follows. The last statement is now clear. □

**Definition 2.2.** A graded right ideal $I$ of $R$ is called *graded essential* if $I \cap J \neq (0)$ for any non-zero graded right ideal $J$ of $R$.

**Lemma 2.3.** *Let $I = I_0R$ and $J = J_0R$ be graded right ideals of $R$. Then*

(1) $I \cap J = (I_0 \cap J_0)R$ *and so $I \cap J$ is also a graded right ideal.*

(2) *$I$ is graded essential if and only if $I_0$ is essential, that is, $I_0 \cap \mathcal{C}_0 \neq \emptyset$.*

**Proof.** The proof is obvious. □

By considering that $R \subset Q$, we define

$$\mathcal{F}_R = \{F : \text{ right ideal of } R \mid (R : r^{-1} \cdot F)_l = R \text{ for all } r \in R\}$$

is a right Gabriel topology on $R$ (see [3], p.116).

Similarly,

$$\mathcal{F}'_R = \{F' : \text{ left ideal of } R \mid (R : F' \cdot r^{-1})_r = R \text{ for all } r \in R\}$$

is a left Gabriel topology on $R$, where $F' \cdot r^{-1} = \{s \in R \mid sr \in F'\}$.

For a right ideal $I$ of $R$, we define the $\tau$-closure of $I$,

$$cl_\tau(I) = \{r \in R \mid rF \subseteq I \text{ for some } F \in \mathcal{F}_R\}$$

and $I$ is called $\tau$-closed if $I = cl_\tau(I)$. Similarly we can define a $\tau$-closed left ideal of $R$. Recall that $R$ is $\tau$-Noetherian if $R$ satisfies the ascending chain conditions on $\tau$-closed right ideals as well as $\tau$-closed left ideals.

Furthermore, we consider that $R_0 \subset Q_0$ and define

$$\mathcal{F}_0 = \{F_0 : \text{ right ideal of } R_0 \mid (R_0 : r_0^{-1} \cdot F_0)_l = R_0 \text{ for all } r_0 \in R_0\},$$

a right Gabriel topology on $R_0$, and

$$\mathcal{F}_0' = \{F_0' : \text{ right ideal of } R_0 \mid (R_0 : F_0' \cdot r_0^{-1})_r = R_0 \text{ for all } r_0 \in R_0\},$$

a left Gabriel topology on $R_0$. Note that any $F_0 \in \mathcal{F}_0$ is an essential right ideal by [3, Proposition 2.2.1].

For graded case we introduce the following topology.

**Definition 2.4.** A non-empty set $\mathcal{F}'$ of graded right ideals of $R$ is called a *graded right Gabriel topology* on $R$ if the following two conditions are satisfied:

a. For any $F \in \mathcal{F}'$, $a_n^{-1} \cdot F \in \mathcal{F}'$ for all $a_n \in R_n$.
b. Let $G = G_0R$ be a graded right ideal of $R$ and $F \in \mathcal{F}'$ such that $r_n^{-1} \cdot G \in \mathcal{F}'$ for any $r_n \in F_n$. Then $G \in \mathcal{F}'$.

If $\mathcal{F}'$ is a graded right Gabriel topology on $R$, then the following conditions are satisfied:

(i) For any $F \in \mathcal{F}'$ and a graded right ideal $G$ such that $F \subseteq G$, then $G \in \mathcal{F}'$.
(ii) If $F$ and $G$ are graded right ideals such that $F,\ G \in \mathcal{F}'$, then so is $F \cap G$.

Moreover, we consider that $R \subset Q^g$, and we define

$$\begin{aligned}\mathcal{F}_g \quad = \quad & \{F : \text{ graded essential right ideal of } R \mid (R : r_n^{-1} \cdot F)_l = R \\ & \text{for any } r_n \in R_n \text{ and } n \in \mathbb{Z}\}.\end{aligned}$$

We prove in Lemma 2.6 that $\mathcal{F}_g$ is a graded right Gabriel topology on $R$.

**Lemma 2.5.** *Let $F_0$ be a right ideal of $R_0$. Then $F_0 \in \mathcal{F}_0$ if and only if $F_0R \in \mathcal{F}_g$.*

**Proof.** Suppose $F_0 \in \mathcal{F}_0$. Let $r_n \in R_n$. Then we claim that $r_n^{-1} \cdot (F_0R) \cap \mathcal{C}_0 \neq \emptyset$. Let $c_0 \in F_0 \cap \mathcal{C}_0$ (see [3, Proposition 2.2.1]). Then there are $d_0 \in \mathcal{C}_0$ and $s_n \in R_n$ such that $r_nd_0 = c_0s_n$. Thus $d_0 \in r_n^{-1} \cdot (F_0R)$ as claimed. It follows that $(r_n^{-1} \cdot (F_0R))Q^g = Q^g$. Let $q \in (R : r_n^{-1} \cdot (F_0R))_l$. Then $q \in Q^g$. By Lemma 2.1, $(R : r_n^{-1} \cdot (I_0R))_l$ is a graded left $R$-ideal and so we may assume that $q$ is homogeneous, say, $q \in Q_0R_l = R_lQ_0$. Write $1 = \sum b_ia_i \in R_{-n} \cdot R_n = R_0$, where $b_i \in R_{-n}$ and $a_i \in R_n$. For fixed $b_i$, $qb_i(r_nb_i)^{-1} \cdot F_0R \subseteq q(r_n^{-1} \cdot (F_0R)) \subseteq R$. So $qb_i(r_nb_i)^{-1} \cdot F_0 \subseteq R_lR_{-n} = R_{l-n}$ and $R_{n-l}qb_i(r_nb_i)^{-1} \cdot F_0 \subseteq R_0$ follows. Thus $R_{n-l}qb_i \subseteq (R_0 : (r_nb_i)^{-1} \cdot F_0)_l = R_0$ since $F_0 \in \mathcal{F}_0$, that is, $qb_i \in R_{l-n}$ and so $qb_ia_i \in R_{l-n} \cdot R_n = R_l$ for all $i$. Thus $q \in R_l$ and hence $(R : r_n^{-1} \cdot (F_0R))_l = R$, that is, $F_0R \in \mathcal{F}_g$.

Suppose $F = F_0R \in \mathcal{F}_g$. Then, by Lemma 2.3, $F_0 \cap \mathcal{C}_0 \neq \emptyset$. For any $r_0 \in R_0$, $R = (R : r_0^{-1} \cdot (F_0R))_l = (R : (r_0^{-1} \cdot F_0)R)_l = R(R_0 : r_0^{-1} \cdot F_0)_l$ by Lemma 2.1. Hence $R_0 = (R_0 : r_0^{-1} \cdot F_0)_l$, that is, $F_0 \in \mathcal{F}_0$. $\square$

**Lemma 2.6.** *$\mathcal{F}_g$ is a graded right Gabriel topology on $R$.*

**Proof.** It is enough to prove the condition (b). Let $G = G_0R$ and $F = F_0R$ be graded right ideals such that $F \in \mathcal{F}_g$ and $r_n^{-1} \cdot (G_0R) \in \mathcal{F}_g$ for any $r_n \in F_n$. By Lemma 2.5, $F_0 \in \mathcal{F}_0$ and for any $r_0 \in F_0$, since $r_0^{-1} \cdot G \in \mathcal{F}_g$, $R = (R : r_0^{-1} \cdot G)_l = (R : (r_0^{-1} \cdot G_0)R)_l = R(R_0 : r_0^{-1} \cdot G_0)_l$ by Lemma 2.5.

Thus $R_0 = (R_0 : r_0^{-1} \cdot G_0)_l$ for all $r_0 \in F_0$, that is, $G_0 \in \mathcal{F}_0$. Hence $G = G_0R \in \mathcal{F}_g$ by Lemma 2.5. $\square$

Let $I = I_0R$ be a graded right ideal. As in ungraded case, we define

$$\mathrm{cl}_{\tau_g}(I) = \{r \in R \mid rF \subseteq I \text{ for some } F \in \mathcal{F}_g\}.$$

**Lemma 2.7.** *Let $I = I_0R$ be a graded right ideal. Then*

(1) *$cl_{\tau_g}(I)$ is a graded right ideal.*

(2) *$cl_{\tau_g}(I) = cl_{\tau_0}(I_0)R$, where $cl_{\tau_0}(I_0) = \{r_0 \in R_0 \mid r_0F_0 \subseteq I_0$ for some $F_0 \in \mathcal{F}_0\}$.*

**Proof.** (1) $\mathrm{cl}_{\tau_g}(I)$ is closed under addition since $\mathcal{F}_g$ is a graded right Gabriel topology on $R$. For any $r_n \in R_n$ and $x \in \mathrm{cl}_{\tau_g}(I)$, there is an $F = F_0R \in \mathcal{F}_g$ such that $xF \subseteq I$. Then $I \supseteq xF_0R \supseteq xr_nr_n^{-1} \cdot (F_0R)$ implies $xr_n \in \mathrm{cl}_{\tau_g}(I)$. Thus for any $r = r_{n_1} + \cdots + r_{n_k} \in R$, $xr_{n_j} \in \mathrm{cl}_{\tau_g}(I)$ for any $j$ $(1 \leq j \leq k)$ and so $xr \in \mathrm{cl}_{\tau_g}(I)$. Hence $\mathrm{cl}_{\tau_g}(I)$ is a right ideal of $R$. To prove that $\mathrm{cl}_{\tau_g}(I)$ is graded, let $x = x_{n_1} + \cdots + x_{n_k} \in \mathrm{cl}_{\tau_g}(I)$. Then there is an $F_0 \in \mathcal{F}_0$ such that $xF_0R \subseteq I = I_0R$ by Lemma 2.5. Since $xF_0 \subseteq I_0R$, $x_{n_j}F_0 \subseteq I_0R_{n_j}$ for any $j$ $(1 \leq j \leq k)$ and $x_{n_j}F_0R \subseteq I_0R_{n_j}R = I_0R$. Hence $x_{n_j} \in \mathrm{cl}_{\tau_g}(I)$, that is, $\mathrm{cl}_{\tau_g}(I)$ is graded.
(2) is easy to prove by using Lemma 2.5 and (1). $\square$

A graded right ideal $I$ is called $\tau_g$-*closed* if $\mathrm{cl}_{\tau_g}(I) = I$. Similarly

$$\mathcal{F}_g' = \{G : \text{ graded essential left ideal of } R \mid (R : G \cdot r_n^{-1})_r = R\}$$

is a graded left Gabriel topology on $R$, where $G \cdot r_n^{-1} = \{s \in R \mid sr_n \in G\}$. For any graded left ideal $J$, we define

$$\mathrm{cl}_{\tau_g}(J) = \{r \in R \mid Gr \subseteq J \text{ for some } G \in \mathcal{F}_g'\}$$

and $J$ is $\tau_g$-*closed* if $\mathrm{cl}_{\tau_g}(J) = J$.

$R$ is called a $\tau_g$-*Noetherian* if $R$ satisfies the ascending chain conditions on $\tau_g$-closed right ideals as well as $\tau_g$-closed left ideals.

**Proposition 2.8.** *The following three conditions are equivalent:*

(1) $R_0$ *is* $\tau$*-Noetherian.*
(2) $R$ *is* $\tau_g$*-Noetherian.*
(3) $R$ *is* $\tau$*-Noetherian.*

**Proof.** (1) $\Leftrightarrow$ (2) This follows from Lemma 2.7.
(1) $\Rightarrow$ (3) This follows exactly from the same proof of [7, Theorem 4.2].
(3) $\Rightarrow$ (1) Let $E_R(Q/R)$ be the injective hull of $Q/R$ as a right $R$-module, $I_0$ be a right ideal of $R_0$ and $\varphi_0$ be a right $R_0$-homomorphism from $I_0$ to $E_R(Q/R)$. We define a map $\varphi : I_0R \to E_R(Q/R)$ by

$$\varphi(x) = \sum_{i=0}^{l} \varphi_0(a_i)r_i, \text{ where } x = \sum a_i r_i \in I_0R \ (a_i \in I_0 \text{ and } r_i \in R),$$

which extends $\varphi_0$ to $I_0R$. If $\varphi$ is well-defined, then it is easy to see that $\varphi$ is a right $R$-homomorphism. To prove $\varphi$ is well-defined, for any $r_i$, let $r_i = r_{in_1} + \cdots + r_{in_k}$, where $r_{in_j} \in R_{n_j}$ ($n_1 > n_2 > \cdots > n_k$, including $r_{in_j} = 0$). Then $x = 0$ if and only if $\sum_{j=1}^{l} a_j r_{jn_1} = 0, \ldots, \sum_{j=1}^{l} a_j r_{jn_k} = 0$. If $x = 0$, then

$$\begin{aligned} \varphi(x) = \sum_{i=1}^{l} \varphi_0(a_i)r_i &= \varphi_0(a_1)r_{1n_1} + \cdots + \varphi_0(a_1)r_{1n_k} \\ &+ \varphi_0(a_2)r_{2n_1} + \cdots + \varphi_0(a_2)r_{2n_k} \\ &\vdots \\ &+ \varphi_0(a_l)r_{ln_1} + \cdots + \varphi_0(a_l)r_{ln_k}. \end{aligned}$$

For any $s_{-n_1} \in R_{-n_1}$,

$$\begin{aligned} (\sum_{j=1}^{l} \varphi_0(a_j)r_{jn_1})s_{-n_1} &= \sum_{j=1}^{l} \varphi_0(a_j)r_{jn_1}s_{-n_1} = \sum_{j=1}^{l} \varphi_0(a_j r_{jn_1} s_{-n_1}) \\ &= \varphi_0(\sum_{j=1}^{l} a_j r_{jn_1} s_{-n_1}) = \varphi_0((\sum_{j=1}^{l} a_j r_{jn_1})s_{-n_1}) = \varphi_0(0) = 0. \end{aligned}$$

Thus $0 = (\sum_{j=1}^{l} \varphi_0(a_j)r_{jn_1})R_{-n_1}$ and $\sum_{j=1}^{l} \varphi_0(a_j)r_{jn_1} = 0$ follows. Similarly, $0 = \sum_{j=1}^{l} \varphi_0(a_j)r_{jn_p}$ for any $p$ ($1 \le p \le k$). Hence $\varphi(x) = 0$, that is, $\varphi$ is well-defined.

Hence $\varphi$ is extended to $\varphi'$ from $R$ to $E_R(Q/R)$ and so there is a $y \in E_R(Q/R)$ such that $\varphi'(1) = y$, that is, $\varphi'(r) = yr$ for all $r \in R$. In particular, $\varphi(r_0) = yr_0$ for all $r_0 \in R_0$. Hence $E_R(Q/R)$ is injective as a right $R_0$-module. Since $Q_0/R_0 \subseteq Q/R \subseteq E_R(Q/R)$, it follows that $E_{R_0}(Q_0/R_0) \subseteq E_R(Q/R)$. Hence $R_0$ is right

$\tau$-Noetherian (by [6, Proposition 2.4] p. 264). Similarly $R_0$ is left $\tau$-Noetherian. Hence $R_0$ is $\tau$-Noetherian. □

Recall that an $R_0$-ideal $A_0$ in $Q_0$ is called $\mathbb{Z}$-*invariant* if $R_n A_0 = A_0 R_n$ for all $n \in \mathbb{Z}$. it is easy to see that $A_0$ is $\mathbb{Z}$-invariant if and only if $A = A_0 R$ is an $R$-ideal in $Q^g$ ([8, Lemma 2]).

$R_0$ is called a $\mathbb{Z}$-*invariant maximal order* in $Q_0$ if $O_l(A_0) = R_0 = O_r(A_0)$ for any non-zero $\mathbb{Z}$-invariant ideal $A_0$ of $R_0$. We say that $R_0$ is a $\mathbb{Z}$-*invariant Krull ring* if it is a $\mathbb{Z}$-invariant maximal order and $\tau$-Noetherian.

$R$ is called a *graded Krull ring* if $R$ is a graded maximal order in $Q^g$ and is $\tau_g$-Noetherian (see [7] p. 205, for definition of graded maximal order). In [7] they obtained that if $R_0$ is a Krull ring, then $R$ is a Krull ring. Combining Proposition 2.8 with [4, Theorem 1], we have the following theorem.

**Theorem 2.9.** *Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a strongly graded ring of type $\mathbb{Z}$ and $R_0$ be a prime Goldie ring. Then the following conditions are equivalent:*

(1) *$R_0$ is a $\mathbb{Z}$-invariant Krull ring.*
(2) *$R$ is a graded Krull ring.*
(3) *$R$ is a Krull ring.*

Next, in case $R$ is a Krull ring, we describe all $v$-invertible $R$-ideals in $Q$ by using the $v$-invertible $R_0$-ideals in $Q_0$ and the propoerties of $Q^g$. Here an $R$-ideal $A$ in $Q$ is called $v$-invertible if ${}_v((R : A)_l A) = R = ((R : A)_r)_v$.

Let $I$ be a right $R$-ideal in $Q$. We define $I_v = (R : (R : I)_l)_r$ containing $I$. $I$ is called a *right $v$-ideal* in $Q$ if $I = I_v$. Similarly we can define a *left $v$-ideal* in $Q$.

In particular, an $R$-ideal $A$ in $Q$ is called a $v$-ideal if $A_v = A = {}_vA$. Note that for each $R$-ideal $A$, $A_v = {}_vA$ if $R$ is a maximal order ([3] p.110).

The following lemma was obtained in [8].

**Lemma 2.10.** (1) [8, Lemma 2] *Let $A_0$ be a $\mathbb{Z}$-invariant ideal of $R_0$. Then $(A_0 R)_v = (A_0)_v R$.*

(2) [8, Lemma 3] *Let $P$ be a prime ideal of $R$ and $P_0 = P \cap R_0$. Then $P_0 R$ is a prime ideal of $R$.*

(3) [8, Lemma 6] *Let $A$ be a graded $R$-ideal in $Q^g$ with $A_0 = A \cap Q_0 \neq (0)$. Then $A = A_0 R = R A_0$ and $A_0$ is $\mathbb{Z}$-invariant.*

In the remainder of this section, let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a Krull ring. Let $D(R)$ and $D(R_0)$ be the set of all $v$-$R$-ideals in $Q$ and the set of all $\mathbb{Z}$-invariant $v$-$R_0$-ideals in $Q_0$, respectively. Then $D(R)$ is a free Abelian group generated by maximal $v$-ideals of $R$ with respect to multiplication $A \circ B = (AB)_v$ for $A,\ B \in D(R)$ by (see the proof of [3, Theorem 2.1.2]). Similarly $D(R_0)$ is a free Abelian group generated by maximal $\mathbb{Z}$-invariant $v$-ideals of $R_0$. (see the proof of [3, Theorem 2.1.2]. Note that the intersection of $\mathbb{Z}$-invariant ideals is also $\mathbb{Z}$-invariant.)

Let $D^g(R)$ be the set of all graded $v$-$R$-ideals in $Q^g$.

**Lemma 2.11.** *The mapping $\varphi : D(R_0) \to D^g(R)$ given by $\varphi(A_0) = A_0R$ is isomorphic as an Abelian group, where $A_0 \in D(R_0)$.*

**Proof.** Let $A_0 \in D(R_0)$. Then by Lemma 2.10, $(A_0R)_v = (A_0)_vR = A_0R$ and so $\varphi(A_0) \in D^g(R)$. It is clear that $\varphi$ is one-to-one since $A_0R \cap R_0 = A_0$ for any $R_0$-ideal $A_0$ in $Q_0$. For $A_0, B_0 \in D(R_0)$, $\varphi(A_0 \circ B_0) = \varphi((A_0B_0)_v) = (A_0B_0)_vR = (A_0B_0R)_v = (A_0RB_0R)_v = (A_0R) \circ (B_0R) = \varphi(A_0) \circ \varphi(B_0)$. That is $\varphi$ is a semigroup homomorphism. To prove $\varphi$ is onto, let $A \in D^g(R)$. Then $A = A_0R = RA_0$ and $A_0$ is a $\mathbb{Z}$-invariant $v$-$R_0$-ideal in $Q_0$ by Lemma 2.10, that is $A_0 \in D(R_0)$. □

**Lemma 2.12.** *Let $M$ be an ideal of $R$. Then $M$ is a maximal $v$-ideal of $R$ with $M \cap R_0 = (0)$ if and only if $M = M' \cap R$, where $M'$ is a maximal ideal of $Q^g$.*

**Proof.** Note $Q^g$ is a principal ideal ring. Let $A'$ be a non-zero proper ideal of $Q^g$ and $A = A' \cap R$. Then we prove $A' = AQ^g$, $A \cap R_0 = (0)$ and $A$ is a $v$-ideal. For $a' \in A'$, there is a $c_0 \in C_0$ with $a'c_0 \in A$ and so $a' \in AQ^g$. Thus $A' = AQ^g$. Because $A' \subset Q^g$, $A \cap R_0 = (0)$ holds. Since $A' = A'_v = (AQ^g)_v = A_vQ^g$ by [1, Lemma 3.2], $A$ is a $v$-ideal.

Thus it follows that if $M$ is a maximal $v$-ideal of $R$ with $M \cap R_0 = (0)$, then $M' = MQ^g$ is a maximal ideal of $Q^g$ with $M = M' \cap R$.

Conversely let $M'$ be a maximal ideal of $Q^g$. Then it is clear that $M = M' \cap R$ is a maximal $v$-ideal of $R$ with $M \cap R_0 = (0)$. □

Let $D_0(R)$ be the free Abelian subgroup of $D(R)$ generated by maximal $v$-ideals $M$ of $R$ with $M \cap R_0 = (0)$. By Lemma 2.12, $D_0(R)$ is isomorphic to $D(Q^g)$.

**Lemma 2.13.** (1) *Let $M$ be a maximal $v$-ideal of $R$ with $M_0 = M \cap R_0 \neq (0)$. Then $M_0$ is a maximal $\mathbb{Z}$-invariant $v$-ideal of $R_0$ and $M = M_0R$.*

(2) *Let $M_0$ be a maximal $\mathbb{Z}$-invariant $v$-ideal of $R_0$. Then $M = M_0R$ is a maximal $v$-ideal of $R$.*

**Proof.** (1) By Lemma 2.10, $M_0$ is $\mathbb{Z}$-invariant. Since $M = M_v \supseteq (M_0R)_v = M_{0v}R \supseteq M_{0v}$ by Lemma 2.10, $M_0$ is a $v$-ideal of $R_0$. Since $R$ is a Krull ring, $(R : M)_l \subseteq (R : M_0R)_l = (R : M_0R)_r$. Hence $M_0R(R : M)_l \subseteq M_0R(R : M_0R)_r \subseteq R$ and $M_0R(R : M)_lM \subseteq M_0R$. If $M \supset M_0R$, then $M_0R(R : M)_l \subseteq M_0R$ because

$M_0R$ is a prime ideal of $R$ by Lemma 2.10, and so $(R : M)_l \subseteq O_r(M_0R) = R$, that is, $M_v = R$, a contradiction. Thus $M = M_0R$. Now it is easily proved that $M_0$ is a maximal $\mathbb{Z}$-invariant $v$-ideal of $R$.

(2) This is clear from the proof of (1). □

Let $M$ be a maximal $v$-ideal of $R$. Then either $M = M_0R$, where $M_0$ is a $\mathbb{Z}$-invariant maximal $v$-ideal of $R_0$ or $M = M' \cap R$, where $M'$ is a maximal ideal of $Q^g$, which follows from Lemma 2.11 and Lemma 2.12. Thus we have the following theorem.

**Theorem 2.14.** *Suppose $R = \oplus_{n\in\mathbb{Z}} R_n$ is a Krull ring. Then*

$$D(R) = D^g(R) \times D_0(R) \cong D(R_0) \times D(Q^g).$$

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