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ON LATTICES OF INTEGRAL GROUP ALGEBRAS AND SOLOMON ZETA FUNCTIONS

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Abstract. We investigate integral forms of certain simple modules over group algebras in characteristic 0 whose p-modular reductions have precisely three composition factors. As a consequence we, in particular, complete the description of the integral forms of the simple $\mathbb{Q}\mathfrak{S}_n$-module labelled by the hook partition $(n-2,1^2)$. Moreover, we investigate the integral forms of the Steinberg module of finite special linear groups $\mathrm{PSL}_2(q)$ over suitable fields of characteristic 0. In the second part of the paper we explicitly determine the Solomon zeta functions of various families of modules and lattices over group algebra, including Specht modules of symmetric groups labelled by hook partitions and the Steinberg module of $\mathrm{PSL}_2(q)$.

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1. Introduction

In this paper we continue our study of integral representations of symmetric groups, begun in [5]. Let $\mathfrak{S}_n$ be the symmetric group of degree $n \geqslant 0$. Moreover, for every partition λ of n, let $S^\lambda_{\mathbb{Q}}$ be the corresponding Specht $\mathbb{Q}\mathfrak{S}_n$-module. As λ varies over the set of partitions of n, the Specht modules $S^\lambda_{\mathbb{Q}}$ yield representatives of the isomorphism classes of (absolutely) simple $\mathbb{Q}\mathfrak{S}_n$-modules. Every Specht module $S^\lambda_{\mathbb{Q}}$ is already equipped with a particular integral form, the *Specht $\mathbb{Z}\mathfrak{S}_n$-lattice $S^\lambda_{\mathbb{Z}}$*. In light of the celebrated Jordan–Zassenhaus Theorem, it is, therefore, natural to ask for a description of all isomorphism classes of $\mathbb{Z}\mathfrak{S}_n$-lattices that are $\mathbb{Z}$-forms of a given Specht module $S^\lambda_{\mathbb{Q}}$, or at least for the number of these. To do so, a possible strategy is to consider, for each prime p, the p-adic completion $S^\lambda_{\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Q}} S^\lambda_{\mathbb{Q}}$ and determine the $\mathbb{Z}_p\mathfrak{S}_n$-lattices that are $\mathbb{Z}_p$-forms of $S^\lambda_{\mathbb{Q}_p}$. In general, this is way too difficult a task.

In [5] we investigated the case where λ is a *hook partition* of $n \geqslant 3$, that is, a partition of the form $(n-r, 1^r)$, for some $r \in \{1, \ldots, n-2\}$. Specht modules (and Specht lattices) labelled by hook partitions have been studied a lot and much is known about their structure. Nevertheless, as far as the determination of the $\mathbb{Z}$-forms of $S_{\mathbb{Q}}^{(n-r,1^r)}$ as concerned, we so far only have complete information in the case where $r = 1$: By work of Plesken [13] and Craig [3], the number of isomorphism classes of $\mathbb{Z}$-forms of $S_{\mathbb{Q}}^{(n-1,1)}$ equals the number of positive divisors of n, and one can give explicit representatives.

So, one may focus on the case where $r > 1$. If p is an odd prime, then the $\mathbb{Q}_p\mathfrak{S}_n$-module $S_{\mathbb{Q}_p}^{(n-r,1^r)}$ admits precisely $\nu_p(n)+1$ isomorphism classes of $\mathbb{Z}_p$-forms, where $\nu_p(n)$ denotes the p-adic valuation of n. Explicit representatives of these isomorphism classes have been determined in [5, Theorem 6.1]; see also the work of Plesken in [14, Satz (III.8)] and [15, Theorem (VI.2)], who studied these modules using different methods.

The case where $p = 2$ turned out to be considerably more difficult. In [5, Section 7], we were only able to give explicit representatives of the isomorphism classes of $\mathbb{Z}_2$-forms of the $\mathbb{Q}_2\mathfrak{S}_n$-module $S_{\mathbb{Q}_2}^{(n-2,1^2)}$, and only if $n \not\equiv 0 \pmod 4$. One aim of the present paper is to settle the remaining case $n \equiv 0 \pmod 4$. This will be achieved in Theorem 3.7, which then entails Corollary 3.9 on the number of isomorphism classes of $\mathbb{Z}$-forms of $S_{\mathbb{Q}}^{(n-2,1^2)}$.

In fact, Theorem 3.7 will turn out to be a special instance of the more general result in Theorem 3.2. The latter deals with the following situation: suppose that G is a finite group, R is a principal ideal domain with field of fractions K of characteristic 0 and residue field $k := R/\mathbf{J}(R)$ of characteristic $p > 0$. Suppose further that V is an absolutely simple KG-module with an R-form L whose modular reduction $k \otimes_R L$ has, as kG-module, precisely three composition factors satisfying some additional properties. Then we shall determine all R-forms of the KG-module V up to isomorphism.

The hypotheses of Theorem 3.2 might at first seem rather special. In Proposition 3.18, we shall see a second application of this result to the case where G is a finite projective special linear group of degree 2 and V is the *Steinberg module* of KG, for suitable fields K of characteristic 0.

In light of this, we are tempted to ask whether Theorem 3.2 can be used to treat further finite groups and simple KG-modules arising as augmentation kernels of two-transitive permutation representations; see Question 3.19. At the moment we are, however, not able to answer this question.

In Section 4 we then investigate the *(Solomon) zeta functions* of various families of $\mathbb{Z}G$-lattices, where G is a finite group. In [16] L. Solomon introduced a generalization of the Riemann zeta function with the aim to study enumerative problems in integral representation theory. Subsequently, Bushnell and Reiner intensively studied Solomon's zeta functions; see [2] for an overview of their theory.

In Section 4, we shall give a concise summary of Solomon's definitions and the properties of the Solomon zeta functions relevant to our applications. In the case where G is a finite group and L is a $\mathbb{Z}G$-lattice, the zeta function of L is defined as

$$\zeta_{\mathbb{Z}G}(L,s) := \sum_{N\subseteq L} [L:N]^{-s} \quad (s\in\mathbb{C}),$$

where N varies over all $\mathbb{Z}G$-sublattices of L of finite index. This will usually be viewed as a formal Dirichlet series, disregarding questions of convergence.

The concrete computation of zeta functions of $\mathbb{Z}G$-lattices is in general a rather difficult problem, and not too much is known in this direction. The case where L is the regular $\mathbb{Z}G$-lattice has been studied most intensively; for a list of known results see [6]. In [6], the second author determined the zeta functions $\zeta_{\mathbb{Z}\mathfrak{S}_n}(L,s)$, where L is a $\mathbb{Z}$-form of the Specht $\mathbb{Q}\mathfrak{S}_n$-module labelled by the hook partition $(2,1^{n-2})$. In Section 4.5 of the present paper we shall generalize the results of [6], and determine global and local zeta functions of further Specht lattices labelled by hook partitions. As well, in Section 4.6 we again consider the projective special linear group $\mathrm{PSL}_2(q)$, where q is a prime power, and the Steinberg module of $\mathbb{Q}[\mathrm{PSL}_2(q)]$. We shall determine the zeta function of a distinguished $\mathbb{Z}$-form of this module. The key ingredient here will again be Theorem 3.2.

The present paper is organized as follows: In Section 2 we briefly summarize some properties of graduated orders that will be relevant in subsequent sections. Section 3 is then devoted to establishing Theorem 3.2 and its applications to the study of integral forms of the Specht $\mathbb{Q}\mathfrak{S}_n$-module $S_{\mathbb{Q}}^{(n-2,1^2)}$ and the Steinberg module of $\mathbb{Q}[\mathrm{PSL}_2(q)]$, respectively. In Section 4 we recall Solomon's notion of global and local zeta functions of modules over group algebra. We then explicitly compute these zeta functions for various families of modules and lattices, including Specht modules of symmetric groups labelled by hook partitions, and the Steinberg module of $\mathrm{PSL}_2(q)$.

2. Notation and prerequisites

In this section we fix some notation, briefly recall the notion of a graduated order, and summarize the known results that will be relevant in Section 3 later. We follow the work of Plesken on the subject, and refer the reader to [14] and [15] for further background.

Notation 2.1. (a) Let F be any field, and let A be a finite-dimensional F-algebra. An *A-module* is always supposed to be a finitely generated left module. For an A-module V, we denote by $\mathrm{Rad}(V)$ the Jacobson radical of V, and by $\mathrm{Hd}(V) := V/\mathrm{Rad}(V)$ the head of V. The socle of V will be denoted by $\mathrm{Soc}(V)$.

(b) Let R be a principal ideal domain with field of fractions K, let $\mathfrak{m} = (\pi)$ be a maximal ideal in R, and let $k := R/\mathfrak{m}$ be the corresponding residue field. By an *R-order* we understand a finitely generated R-algebra Λ that is free over R of finite R-rank. One has a k-algebra isomorphism $k \otimes_R \Lambda \cong \Lambda/\mathfrak{m}\Lambda$; for convenience, we shall often identify these algebras and denote them simply by $k\Lambda$. A *Λ-lattice* is then a finitely generated left Λ-module L that is R-free of finite R-rank, which we denote by $\mathrm{rk}_R(L)$. The factor module $\bar{L} := L/\mathfrak{m}L$ naturally carries the structure of a $k\Lambda$-module, and $L/\mathfrak{m}L \cong k \otimes_R L$.

If A is a finite-dimensional K-algebra and if an R-order Λ is a subring of A with $K\Lambda = A$, then one calls Λ an *R-order in A*. In this case, we also identify the K-algebras A and $K \otimes_R \Lambda$. If V is an A-module and L is a Λ-lattice such that $K \otimes_R L \cong V$ as A-modules, then one calls L an *R-form* of V. As usual, we shall often work with an R-form L of V such that $L \subseteq V$. Moreover, recall that every R-form of V is isomorphic to a Λ-sublattice of any given R-form L.

(c) With the notation as in (b), suppose that L is a Λ-lattice, and let $L' \subseteq L$ be a Λ-sublattice of L with $\mathrm{rk}_R(L) = \mathrm{rk}_R(L')$. If $L' \subseteq \pi^i L$, for some $i \in \mathbb{N}$, then we denote by L'/π^i the Λ-sublattice $\{\pi^{-i}x : x \in L'\}$ of L, which satisfies $\pi^i(L'/\pi^i) = L'$ and is isomorphic to L'.

(d) Now suppose that R is local, and again let Λ be an R-order. We shall call a Λ-module L *simple* if $L \neq \{0\}$ and if L and $\{0\}$ are the only Λ-submodules of L. If L is a simple Λ-module, then $\mathfrak{m}L = \pi L$ is a Λ-submodule of L, and $\mathfrak{m}L \neq L$, by Nakayama's Lemma [4, (30.2)]. Thus $\mathfrak{m}L = \{0\}$; in particular, every simple Λ-module is a torsion module. This shows that $\mathfrak{m}\Lambda$ is contained in the Jacobson radical $\mathbf{J}(\Lambda)$ of Λ, which entails k-algebra isomorphisms

$$\Lambda/\mathbf{J}(\Lambda) \cong (\Lambda/\mathfrak{m}\Lambda)/(\mathbf{J}(\Lambda)/\mathfrak{m}\Lambda) = (\Lambda/\mathfrak{m}\Lambda)/\mathbf{J}(\Lambda/\mathfrak{m}\Lambda) \cong k\Lambda/\mathbf{J}(k\Lambda)\,;$$

in particular, there are bijections between the isomorphism classes of simple modules of Λ, $\Lambda/\mathbf{J}(\Lambda)$, $k\Lambda$ and $k\Lambda/\mathbf{J}(k\Lambda)$, respectively.

If L is any (finitely generated) Λ-module, then one also has $\pi L = \mathfrak{m}L = \mathfrak{m}\Lambda \cdot L \subseteq \mathbf{J}(\Lambda) \cdot L \subseteq \mathrm{Rad}(L)$, and

$$L/\operatorname{Rad}(L) \cong (L/\mathfrak{m}L)/(\operatorname{Rad}(L)/\mathfrak{m}L) = (L/\mathfrak{m}L)/(\operatorname{Rad}(L/\mathfrak{m}L)),$$

as Λ-modules and $k\Lambda$-modules.

For simplicity, for the remainder of this section, R will be a local principal ideal domain with maximal ideal $\mathfrak{m} = (\pi)$, field of fractions K and residue field $k := R/\mathfrak{m}$.

Definition 2.2. Let $n \in \mathbb{N}$. An R-order Λ in the matrix algebra $A := K^{n\times n}$ is called a *graduated R-order* in A if there exist pairwise orthogonal idempotents $e_1, \dots, e_n \in \Lambda$ such that $1_A = 1_\Lambda = e_1 + \cdots + e_n$.

Remark 2.3. *Suppose that A is any semisimple K-algebra, and let ε be a block idempotent of A, that is, a projection onto one of the Wedderburn components of A. If K is a splitting field of the simple K-algebra $\varepsilon A\varepsilon$, then $\varepsilon A\varepsilon \cong K^{n\times n}$, where n is the dimension of the (up to isomorphism uniquely determined) simple $\varepsilon A\varepsilon$-module. This is the situation we shall investigate in the following.*

2.4. Exponent matrices and normal form. (a) Following [14, Definition (I.3)] and [15, Definition (II.1), (II.2)], consider $r, n, d_1, \dots, d_r \in \mathbb{N}$ with $n = d_1 + \cdots + d_r$ as well as a matrix $M = (m_{ij}) \in \mathbb{Z}^{r\times r}$ all of whose entries are non-negative. Then the set of block matrices

$$\Lambda := \Lambda(d_1, \dots, d_r; M) := \{(a_{ij}) \in R^{n\times n} : a_{ij} \in \mathfrak{m}^{m_{ij}} \cdot R^{d_i\times d_j}\} \subseteq R^{n\times n} \subseteq K^{n\times n}$$

is a (graduated) order in $K^{n\times n}$ if and only if, for all $i, j, k \in \{1, \dots, r\}$, one has

$$m_{ii} = 0 \tag{1}$$

and

$$m_{ij} + m_{jk} \geqslant m_{ik}\,. \tag{2}$$

If, moreover, one has

$$m_{ij} + m_{ji} > 0\,, \tag{3}$$

whenever $i \neq j$, one says that Λ is in *standard form*, and calls M the *exponent matrix* of Λ. By [15, Remark (II.3)], every graduated order in $K^{n\times n}$ is isomorphic to a graduated order in standard form.

Theorem 2.5 ([14, Satz (I.26)],[15, Theorem (II.16)])**.** *Suppose that A is a semisimple K-algebra, let Γ be any R-order in A, and let ε be a block idempotent of A. Moreover, let V be an absolutely simple A-module with $\varepsilon V = V$, and let $L \subseteq V$ be* *a Γ-lattice that is an R-form of V. Then $\varepsilon\Gamma\varepsilon$ is a graduated order in $\varepsilon A\varepsilon$ if and only if the following conditions are satisfied:*

(i) *every composition factor of the $k\Gamma$-module $L/\mathfrak{m}L$ occurs with multiplicity 1;*

(ii) *every composition factor of the $k\Gamma$-module $L/\mathfrak{m}L$ is absolutely simple.*

In the course of this paper we shall apply Theorem 2.5 in the case where $A = KG$ is the group algebra of a finite group G over K, and Γ is the R-order RG in A. Therefore, we recall how to obtain a graduated order in standard form in $\varepsilon KG\varepsilon$ that is isomorphic to $\varepsilon RG\varepsilon$.

2.6. Sublattices and exponent matrices. We keep the notation of Theorem 2.5, and suppose that conditions (i) and (ii) are satisfied. Denote the R-order $\varepsilon\Gamma\varepsilon$ of $\varepsilon A\varepsilon$ by Λ. Let $D_1, \ldots, D_r$ be the pairwise non-isomorphic composition factors of the $k\Gamma$-module $L/\mathfrak{m}L$, with k-dimensions $d_1, \ldots, d_r$.

(a) As mentioned in 2.1, one has bijections between the isomorphism classes of simple modules of Λ, $\Lambda/\mathbf{J}(\Lambda)$, $k\Lambda$ and $k\Lambda/\mathbf{J}(k\Lambda)$, respectively. Analogously, $\mathfrak{m}\Gamma \subseteq \mathbf{J}(\Gamma)$, and one has bijections between the isomorphism classes of simple modules of Γ, $\Gamma/\mathbf{J}(\Gamma)$, $k\Gamma$ and $k\Gamma/\mathbf{J}(k\Gamma)$, respectively.

Since $L \subseteq V$ and ε acts as the identity on V, it also acts as the identity on L and all its sublattices. In particular, $L = \varepsilon L$ is also a Λ-lattice, and the Γ-sublattices of L are just the inflations of the Λ-sublattices of L, along the surjective R-algebra homomorphism $\Gamma \to \Lambda = \varepsilon\Gamma\varepsilon\,, a \mapsto \varepsilon a$.

We may also view $L/\mathfrak{m}L$ both as $k\Gamma$ and $k\Lambda$-module. Since V is, up to isomorphism, the only simple $\varepsilon A\varepsilon$-module and since Λ is an R-order in $\varepsilon A\varepsilon$, the simple $k\Lambda$-modules arise precisely as the composition factors of the $k\Lambda$-module $L/\mathfrak{m}L$.

On the other hand, if D is a simple Λ-module, then D also becomes a simple Γ-module via inflation along the surjective R-algebra homomorphism $\Gamma \to \Lambda = \varepsilon\Gamma\varepsilon\,, a \mapsto \varepsilon a$.

Thus, altogether, the simple $k\Gamma$-modules $D_1, \ldots, D_r$ may also be viewed as simple $k\Lambda$-modules. As such they are the composition factors of the $k\Lambda$-module $L/\mathfrak{m}L$. Moreover, $D_1, \ldots, D_r$ are representatives of the isomorphism classes of simple $k\Lambda$-modules.

(b) As observed in (a), every Γ-sublattice of L is also a Λ-sublattice of L, and conversely. So consider the lattice of Γ-sublattices of L of full R-rank. By [14, Folgerung (I.24)], [15, Remark (II.4); p. 14], for each $i \in \{1, \ldots, r\}$, there is a unique sublattice L_i of L such that $L_i \not\subseteq \mathfrak{m}L$ and $L_i/\operatorname{Rad}(L_i) \cong D_i$ as $k\Gamma$-modules.

Then, for $i, j \in \{1, \dots, r\}$, let $m_{ij} \in \mathbb{N}_0$ be the multiplicity of D_i as a composition factor of L/L_j. Setting $M_L := (m_{ij})$, one deduces $\Lambda \cong \Lambda(d_1, \dots, d_r; M_L)$ as R-orders, and $\Lambda(d_1, \dots, d_r; M_L)$ is in standard form. The definition of M_L of course depends on L as well as on the ordering of the simple modules $D_1, \dots, D_r$. However, once the latter ordering has been fixed and L and L' are R-forms of V, [14, Satz (I.7)], [15, Proposition (II.6)] show that $M_L = M_{L'}$ if and only of $L \cong L'$ as Γ-lattices.

By [14, Satz (I.23)], [15, Remark (II.4)], every projective indecomposable Λ-module is isomorphic to a Λ-sublattice of L of full rank. More precisely, if, for $i \in \{1, \dots, r\}$, P_i denotes a projective cover of the simple Λ-module D_i, then $P_i \cong L_i$ as Λ-modules. Note that, as Γ-module, L_i is, in general, not projective.

(c) Again let $L \subseteq V$ be a Λ-lattice that is an R-form of V, and let $M_L = (m_{ij})$ be the corresponding exponent matrix, so that $\Lambda \cong \Lambda(d_1, \dots, d_r; M_L)$. By [14, Satz (I.8)], [15, Remark (II.4)], there is a bijection between the set of isomorphism classes of Λ-lattices that are R-forms of V and the set of r-tuples $(m_1, \dots, m_r) \in \mathbb{N}_0^r$ satisfying

$$m_{ij} + m_j \geqslant m_i \,, \text{ for } i, j \in \{1, \dots, r\} \tag{4}$$

and

$$m_k = 0 \,, \text{ for some } k \in \{1, \dots, r\} \,. \tag{5}$$

More precisely, to each such r-tuple $(m_1, \dots, m_r)$, one associates the unique full-rank Λ-sublattice $L(m_1, \dots, m_r)$ of L such that, for $i \in \{1, \dots, r\}$, the simple Λ-module D_i occurs with multiplicity m_i as a composition factor of $L/L(m_1, \dots, m_r)$.

(d) Now suppose that $A = KG$ and $\Gamma = RG$, for a finite group G. Then $k\Gamma \cong kG$. Consider the simple A-module V^*, that is, the K-linear dual of V, let ε^* be the block idempotent of A with $\varepsilon^* V^* = V^*$, and let $T \subseteq V^*$ be an R-form of V^*. Then V^* and T also satisfy the hypotheses of Theorem 2.5. The composition factors of the kG-module $T/\mathfrak{m}T$ are isomorphic to the simple kG-modules $D_1^*, \dots, D_r^*$. If V is a self-dual KG-module, and if $D_1, \dots, D_r$ are self-dual kG-modules, then we may take $T \cong L$ and, by [14, Satz (III.1)] and [15, Proposition (IV.1)], we obtain

$$m_{ij} + m_{jk} + m_{ki} = m_{ji} + m_{kj} + m_{ik} \,, \tag{6}$$

for $i, j \in \{1, \dots, r\}$. It should be mentioned that, at the beginning of [14, Chapter III], L is assumed to be projective, when viewed as $\varepsilon RG\varepsilon$-module. In our applications, we shall usually work with lattices that do not have this property. The assertion of [14, Satz(III.1)(i)] is, however, valid without any restrictions on L.

3. On simple KG-modules with three modular composition factors

Throughout this section, let R be a principal ideal domain with maximal ideal $\mathfrak{m} = (\pi)$, residue field k of characteristic $p > 0$, and field of fractions K of characteristic 0. Moreover, let G be a finite group.

3.1. Submodule lattices. In this subsection, we shall investigate R-forms of particular absolutely simple KG-modules with three modular composition factors. Theorem 3.2 below will subsequently be applied to two examples on finite symmetric and projective special linear groups, respectively. Throughout this subsection suppose that R is local.

Hypotheses 3.1. Let V be an absolutely simple KG-module, and let $S_1 := L \subseteq V$ be an R-form of V satisfying the following properties:

(a) there are pairwise non-isomorphic simple kG-modules D_1, D_2 and D_3 with k-dimensions d_1,d_2 and d_3, respectively, such that $d_2 \neq d_1 \neq d_3$ and $d_1 \neq d_2 + d_3$, and such that the kG-module $S_1/\pi S_1$ has radical isomorphic to D_1 and head isomorphic to $D_2 \oplus D_3$;

(b) $D_1 \cong D_1^*$ and $(D_2 \oplus D_3)^* \cong D_2 \oplus D_3$;

(c) $\mathrm{Soc}(S_1/\pi S_1) = \mathrm{Rad}(S_1/\pi S_1)$;

(d) S_2 and S_3 are the maximal sublattices of S_1 with $S_1/S_2 \cong D_2$ and $S_1/S_3 \cong D_3$ as kG-modules;

(e) there is some $t \in \mathbb{N}$ and, for each $i \in \{1, \dots, t\}$, there is some sublattice S_{3i+1} of S_1 such that
  (i) for each $i \in \{0, \dots, t-1\}$, $\pi S_{3i+1} \subseteq S_{3(i+1)+1} \subseteq S_{3i+1}$ and the dimension satisfies $\dim_k(S_{3(i+1)+1}/\pi S_{3i+1}) = d_1$;
  (ii) $\forall i \in \{1, \dots, t-1\}$: $S_{3i+1}/\pi S_{3i+1} \cong D_1 \oplus D_2 \oplus D_3$;
  (iii) $S_{3t+1} \cong S_1^*$.

Theorem 3.2. *Suppose that Hypotheses 3.1 hold. Then one has the following:*

(a) *S_4 is the unique common maximal sublattice of S_2 and S_3; moreover, $S_2/S_4 \cong D_3$ and $S_3/S_4 \cong D_2$;*

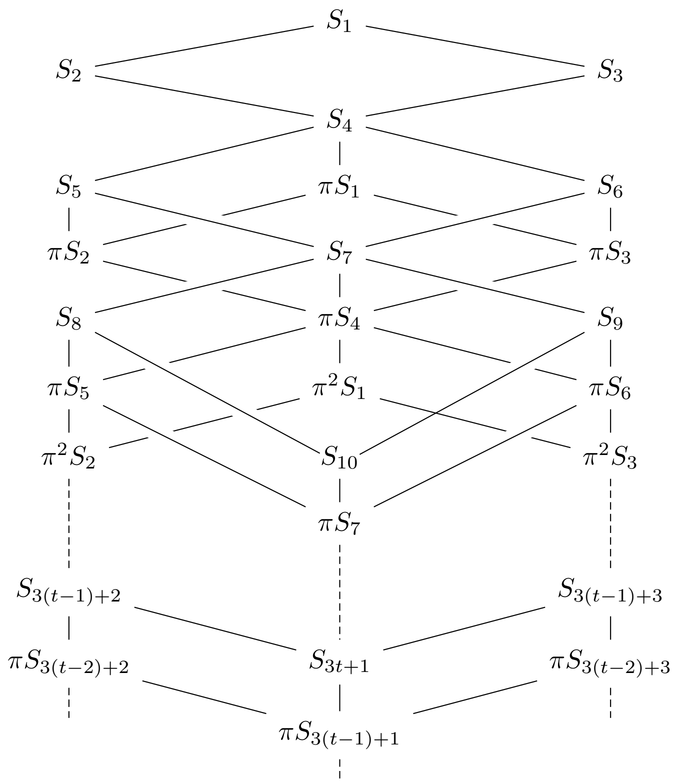
(b) *for $i \in \{1, \dots, t-1\}$, the lattice S_{3i+1} has precisely three maximal sublattices S_{3i+2}, S_{3i+3} and $\pi S_{3(i-1)+1}$, where $S_{3i+1}/S_{3i+2} \cong D_2$, $S_{3i+1}/S_{3i+3} \cong D_3$ and $S_{3i+1}/\pi S_{3(i-1)+1} \cong D_1$;*

(c) *for $i \in \{1, \dots, t-1\}$, the lattice S_{3i+2} has at least two maximal sublattices $S_{3(i+1)+1}$ and $\pi S_{3(i-1)+2}$; moreover, $S_{3i+2}/S_{3(i+1)+1} \cong D_3$ and $S_{3i+2}/pS_{3(i-1)+2} \cong D_1$;*

(d) *for* $i \in \{1, \ldots, t-1\}$, *the lattice* S_{3i+3} *has at least two maximal sublattices* $S_{3(i+1)+1}$ *and* $\pi S_{3(i-1)+3}$; *moreover,* $S_{3i+3}/S_{3(i+1)+1} \cong D_2$ *and* $S_{3i+3}/\pi S_{3(i-1)+3} \cong D_1$;

(e) $\pi S_{3(t-1)+1}$ *is the unique maximal sublattice of* S_{3t+1}, *and* $S_{3t+1}/\pi S_{3(t-1)+1} \cong D_1$;

(f) *the RG-lattices* $S_1, \ldots, S_{3t+1}$ *are pairwise non-isomorphic R-forms of* V. *If* S_4 *is the unique maximal sublattice of* S_2 *as well as the unique maximal sublattice of* S_3, *then* $S_1, \ldots, S_{3t+1}$ *are representatives of the isomorphism classes of R-forms of* V, *and* S_1 *has the following full-rank sublattices:*



(7)

(g) *if* D_1, D_2 *and* D_3 *are absolutely simple, let* ε *be the block idempotent of* KG *corresponding to* V. *Then* $\Lambda := \varepsilon RG\varepsilon$ *is a graduated R-order in* $A := \varepsilon KG\varepsilon$. *One* *has* $\Lambda \cong \Lambda(d_1, d_2, d_3; M_L)$*, where*

$$M_L = \begin{pmatrix} 0 & 0 & 0 \\ t & 0 & a \\ t & b & 0 \end{pmatrix},$$

for some $a, b \in \mathbb{N}$*. One has* $a = 1 = b$ *if and only if* S_4 *is the unique maximal sublattice of* S_2 *and the unique maximal sublattice of* S_3*.*

Proof. Assertion (a) is clear from our hypotheses. To prove assertions (b)–(d) we argue by induction on i.

Suppose first that $i = 1$. The lattice S_4 has three maximal sublattices, since $S_4/\pi S_4 \cong D_1 \oplus D_2 \oplus D_3$ as kG-module. By our hypotheses, $\pi S_1 \subseteq S_4 \subseteq S_1$ and $\dim_k(S_4/\pi S_1) = d_1$. By Hypotheses 3.1(a), this forces $S_4/\pi S_1 \cong D_1$; in particular, πS_1 is maximal in S_4. Now let S_5 and S_6 be the maximal sublattices of S_4 such that $S_4/S_5 \cong D_2$ and $S_4/S_6 \cong D_3$. Then we have $S_5/\pi S_4 \cong D_1 \oplus D_3$ and $S_6/\pi S_4 \cong D_1 \oplus D_2$. Moreover, from $\pi S_4 \subseteq S_7 \subseteq S_4$ and $\dim_k(S_7/\pi S_7) = d_1$ we deduce that $S_7/\pi S_4 \cong D_1$, so that $S_7/\pi S_4$ is the unique submodule of $S_4/\pi S_4$ isomorphic to D_1. In particular, $S_7/\pi S_4 \subseteq S_5/\pi S_4$ and $S_7/\pi S_4 \subseteq S_6/\pi S_4$. This implies $S_7 \subseteq S_5$, $S_7 \subseteq S_6$, $S_5/S_7 \cong D_3$, and $S_6/S_7 \cong D_2$. Hence S_7 is maximal in both S_5 and S_6.

Next we show that πS_2 is maximal in S_5, πS_3 is maximal in S_6 and $S_5/\pi S_2 \cong D_1 \cong S_6/\pi S_3$. First assume that $\pi S_2 \not\subseteq S_5$. Since $\pi S_2 \subseteq \pi S_1 \subseteq S_4$ and since S_5 is maximal in S_4, this gives $S_4 = S_5 + \pi S_2 = S_5 + \pi S_1$ and

$$S_4/\pi S_4 = (S_5/\pi S_4) + (\pi S_2/\pi S_4) \cong (D_1 \oplus D_3) + D_3\,.$$

So either $\pi S_2/\pi S_4 \subseteq S_5/\pi S_4$ or $S_4/\pi S_4 = S_5/\pi S_4 \oplus \pi S_2/\pi S_4$. Since we are assuming $\pi S_2 \not\subseteq S_5$, the first case cannot occur, implying $S_4/\pi S_4 \cong D_1 \oplus D_3 \oplus D_3$, a contradiction, since $D_2 \not\cong D_3$. Consequently, $\pi S_2 \subseteq S_5$ and

$$S_5/\pi S_2 \cong (S_5/\pi S_4)/(\pi S_2/\pi S_4) \cong (D_1 \oplus D_3)/D_3 \cong D_1\,;$$

in particular, πS_2 is maximal in S_5 and $\pi S_2 \neq S_7$.

Analogously, we deduce that $\pi S_3 \neq S_7$ is maximal in S_6 with $S_6/\pi S_3 \cong D_2$.

So suppose now that $i > 1$. As above, we deduce that S_{3i+1} has precisely three maximal sublattices $\pi S_{3(i-1)+1}$, S_{3i+2} and S_{3i+3}, where $S_{3i+1}/\pi S_{3(i-1)+1} \cong D_1$, $S_{3i+1}/S_{3i+2} \cong D_2$ and $S_{3i+1}/S_{3i+3} \cong D_3$. Moreover, S_{3i+2} has at least two maximal sublattices $\pi S_{3(i-1)+2}$ and $S_{3(i+1)+1}$, with $S_{3i+2}/\pi S_{3(i-1)+2} \cong D_1$ and $S_{3i+2}/S_{3(i+1)+1} \cong D_3$. To see this, note that $S_{3(i+1)+1}/\pi S_{3i+1}$ is the unique submodule of $S_{3i+1}/\pi S_{3i+1}$ isomorphic to D_1. Since $S_{3i+1}/S_{3i+2} \cong D_2$, we have

$S_{3i+2}/\pi S_{3i+1} \cong D_1 \oplus D_3$, implying $S_{3(i+1)+1}/\pi S_{3i+1} \subseteq S_{3i+2}/\pi S_{3i+1}$, $S_{3(i+1)+1} \subseteq S_{3i+2}$ and $S_{3i+2}/S_{3(i+1)+1} \cong D_3$. Assuming that $\pi S_{3(i-1)+2} \not\subseteq S_{3i+2}$ and using that $S_{3(i-1)+2}/S_{3i+1} \cong D_3$ by induction, we again obtain the contradiction

$$D_1 \oplus D_2 \oplus D_3 \cong S_{3i+1}/\pi S_{3i+1} \cong S_{3i+2}/\pi S_{3i+1} \oplus \pi S_{3(i-1)+2}/\pi S_{3i+1} \cong D_1 \oplus D_3 \oplus D_3 \,.$$

Hence $\pi S_{3(i-1)+2} \subseteq S_{3i+2}$ and

$$S_{3i+2}/\pi S_{3(i-1)+2} \cong (S_{3i+2}/\pi S_{3i+1})/(\pi S_{3(i-2)+1}/\pi S_{3i+1}) \cong (D_1 \oplus D_3)/D_3 \cong D_1 \,.$$

This proves assertion (c), and assertion (d) concerning the maximal sublattices of S_{3i+3} is proved analogously.

To show (e), recall that $\mathrm{Soc}(S_1/\pi S_1) \cong D_1$ and $D_1 \cong D_1^*$, by our hypotheses. Since $S_{3t+1} \cong S_1^*$, we also have $S_{3t+1}/\pi S_{3t+1} \cong (S_1/\pi S_1)^*$, so that $S_{3t+1}/\pi S_{3t+1}$ has head isomorphic to D_1; in particular, S_{3t+1} has a unique maximal sublattice. By our hypotheses, we further know that $\pi S_{3(t-1)+1} \subseteq S_{3t+1} \subseteq S_{3(t-1)+1}$ and $\dim_k(S_{3t+1}/\pi S_{3(t-1)+1}) = \dim_k(D_1)$. Hence $S_{3t+1}/\pi S_{3(t-1)+1} \cong D_1$, by Hypotheses 3.1(a), and $\pi S_{3(t-1)+1}$ is indeed the unique maximal sublattice of S_{3t+1}.

To complete the proof of the theorem, it remains to settle (f) and (g). To do so we shall apply [13, Proposition 2.3]. We first note that neither of the lattices $S_1, \ldots, S_{3t+1}$ is contained in πS_1. Namely, by Hypotheses 3.1(d) and assertions (a)-(d) above, for $j \in \{1, \ldots, 3t+1\}$, every composition factor of S_1/S_j is isomorphic to D_2 or D_3, while $S_1/\pi S_1$ has a composition factor isomorphic to D_1. Hence, by [13, Proposition 2.3], the R-forms $S_1, \ldots, S_{3t+1}$ are pairwise non-isomorphic RG-lattices. By construction, (7) is part of the submodule lattice of S_1.

Now suppose that both S_2 and S_3 have a unique maximal sublattice, which then has to be equal to S_4. We again argue by induction on i to show that each of S_{3i+2} and S_{3i+3} has precisely two maximal sublattices, for all $i \in \{1, \ldots, t-1\}$. So let $i = 1$, and assume that S_5 has a maximal sublattice T with $S_7 \neq T \neq \pi S_2$. Then we must have $S_5/\pi S_5 \cong D_1 \oplus D_2 \oplus D_3$, and $S_5/T \cong D_2$. Since $\pi S_2/\pi S_5 \cong (\pi S_2/\pi^2 S_2)/(\pi S_5/\pi^2 S_2)$ and $\pi S_5/\pi^2 S_2 \cong S_5/\pi S_2 \cong D_1$, we conclude that $\pi S_2/\pi S_5 \cong D_2 \oplus D_3$ is isomorphic to a factor module of $\pi S_2/\pi^2 S_2 \cong S_2/\pi S_2$. But $S_2/\pi S_2$ has a simple head isomorphic to D_3, hence does not have a factor module isomorphic to $D_2 \oplus D_3$.

This proves the assertions concerning S_5, and the lattice S_6 is treated analogously.

Now let $i > 1$. Assume that we have a maximal sublattice T of S_{3i+2} with $\pi S_{3(i-1)+2} \neq T \neq S_{3(i+1)+1}$. Then, as in the case where $i = 1$, we get $S_{3i+2}/\pi S_{3i+2} \cong$

$D_1 \oplus D_2 \oplus D_3$ and $S_{3i+2}/T \cong D_2$. We have $\pi S_{3i+2} \subseteq \pi S_{3i+1} \subseteq \pi S_{3(i-1)+2}$, thus

$$\pi S_{3(i-1)+2}/\pi S_{3i+2} \cong (S_{3(i-1)+2}/\pi^2 S_{3(i-1)+2})/(\pi S_{3i+2}/\pi^2 S_{3(i-1)+2})$$

and

$$(S_{3i+2}/\pi S_{3i+2})/(\pi S_{3(i-1)+2}/\pi S_{3i+2}) \cong S_{3i+2}/\pi S_{3(i-1)+2} \cong D_1\,.$$

Note that since we are assuming $S_{3i+2}/\pi S_{3i+2}$ to be semisimple, this implies that $\pi S_{3(i-1)+2}/\pi S_{3i+2} \cong D_2 \oplus D_3$. But we have just seen that $\pi S_{3(i-1)+2}/\pi S_{3i+2}$ is isomorphic to a factor module of $\pi S_{3(i-1)+2}/\pi^2 S_{3(i-1)+2} \cong S_{3(i-1)+2}/\pi S_{3(i-1)+2}$, which, by induction, has a head isomorphic to $D_1 \oplus D_3$. Since $D_1 \ncong D_2 \ncong D_3$, we obtain a contradiction.

This proves the assertions concerning S_{3i+2}, and the lattice S_{3i+3} is treated analogously. Consequently, we have now verified that (7) is the lattice of full-rank sublattices of S_1. Moreover, $S_1, \ldots, S_{3t+1}$ are precisely those sublattices of S_1 of full rank that are not contained in πS_1, which are then representatives of the isomorphism classes of R-forms of V, by [13, Proposition 2.3]. This settles (f).

Lastly suppose that D_1, D_2 and D_3 are absolutely simple. Then Λ is a graduated R-order in A, by Theorem 2.5. By (e), the lattice S_{3t+1} must be the unique sublattice of S_1 not contained in πS_1 that is a projective cover of D_1, when viewed as simple Λ-module as in 2.6(b). Consequently, S_1 is an injective Λ-lattice, and [14, Satz (I.23)(ii)], [15, Remark (II.4)] forces $m_{1j} = 0$, for $j \in \{1, 2, 3\}$.

Hence, by 2.6(b), with respect to the chosen ordering on D_1, D_2, D_3, we must have

$$M_L = M_{S_1} = \begin{pmatrix} 0 & 0 & 0 \\ t & 0 & a \\ t & b & 0 \end{pmatrix},$$

for some $a, b \in \mathbb{N}$.

Consider the uniquely determined sublattices P_2 and P_3 of S_1 not contained in πS_1 such that $P_2/\operatorname{Rad}(P_2) \cong D_2$ and $P_3/\operatorname{Rad}(P_3) \cong D_3$. That is, P_2 is a projective cover of D_2 and P_3 is a projective cover of D_3, when viewed as Λ-modules. Then, by 2.6(b), we deduce that S_1/P_2 has only composition factors isomorphic to D_3, and the number of these is b. Similarly, S_1/P_3 has only composition factors isomorphic to D_2, and the number of these is a. Since S_2 and S_3 are the only maximal sublattices of S_1, this forces $P_2 \subseteq S_3$ and $P_3 \subseteq S_2$. Therefore, we have $a = b = 1$ if and only if $P_2 = S_3$ and $P_3 = S_2$. This in turn is equivalent to S_2 and S_3 having a unique maximal sublattice, which then has to be the common sublattice S_4. $\square$

Remark 3.3. *Keep the notation of Theorem 3.2. Suppose that the simple kG-modules D_1, D_2, D_3 are absolutely simple and suppose also that $a = b = 1$ in part (g). Then the RG-lattices $S_1, \ldots, S_{3t+1}$ are representatives of the R-forms of V, and $\Lambda = \varepsilon RG\varepsilon$ is a graduated R-order of $\varepsilon KG\varepsilon$. Recall from 2.6(c) that the lattices $S_1, \ldots, S_{3t+1}$ bijectively correspond to the triples $(m_1, m_2, m_3) \in \mathbb{N}_0^3$ satisfying the following conditions:*

$$
\begin{aligned}
&t + m_1 \geqslant m_2 \geqslant m_1\\
&t + m_1 \geqslant m_3 \geqslant m_1\\
&1 + m_3 \geqslant m_2\\
&1 + m_2 \geqslant m_3 \,.
\end{aligned}
$$

Since at least one of m_1, m_2, m_3 has to be 0, this gives $m_1 = 0$ and

$$(m_1, m_2, m_3) \in \{(0, j, j), (0, j + 1, j), (0, j, j + 1), (0, t, t) : j \in \{0, \ldots, t - 1\}\} \,.$$

Moreover, by 2.6(c) and (7), the concrete correspondence between S_j and the triple (m_1, m_2, m_3) is given as follows:

j	(m_1, m_2, m_3)
$3j + 1$	$(0, j, j)$
$3j + 2$	$(0, j + 1, j)$
$3j + 3$	$(0, j, j + 1)$
$3t + 1$	$(0, t, t)$

where $j \in \{0, \ldots, t - 1\}$.

3.2. Application I: symmetric groups. Our first application of Theorem 3.2 will be concerned with the symmetric group $\mathfrak{S}_n$ of degree $n \geqslant 0$. We begin by setting up some notation that will be chosen in accordance with [5]. For details on the representations of symmetric groups and the well-known properties of these used below, we refer the reader to [9].

Notation 3.4. Suppose that R is a principal ideal domain with field of fractions K of characteristic 0. Moreover, let (π) be a maximal ideal in R such that the residue field $k := R/(\pi)$ has characteristic $p > 0$. The isomorphism classes of (absolutely) simple $K\mathfrak{S}_n$-modules are labelled by the partitions of n: for every partition λ of n, there is a simple $K\mathfrak{S}_n$-module S_K^λ, called the *Specht $K\mathfrak{S}_n$-module labelled by λ*, which carries a distinguished R-form S_R^λ, called the *Specht $R\mathfrak{S}_n$-lattice* labelled by λ. The $k\mathfrak{S}_n$-module $S_R^\lambda/\pi S_R^\lambda$ shall be denoted by S_k^λ. It is well known and easily deduced from the explicit construction of Specht modules in [8, Sections 4 and 8] that $S_R^\lambda \cong R \otimes_{\mathbb{Z}} S_{\mathbb{Z}}^\lambda$, for every partition λ of n.

The isomorphism classes of (absolutely) simple $k\mathfrak{S}_n$-modules are labelled by the p-regular partitions of n, that is, partitions λ of n each of whose parts occurs with multiplicity at most $p-1$. The simple $k\mathfrak{S}_n$-module labelled by a p-regular partition μ is usually denoted by D_k^μ. Recall also that D_k^μ is isomorphic to the head of S_k^μ.

Every simple $K\mathfrak{S}_n$-module as well as every simple $k\mathfrak{S}_n$-module is self-dual.

In [5], we considered the case where $K \in \{\mathbb{Q}, \mathbb{Q}_p\}$, $R \in \{\mathbb{Z}, \mathbb{Z}_p\}$ and $k = \mathbb{F}_p$, for some prime number p. We studied the R-forms of the Specht $K\mathfrak{S}_n$-modules labelled by hook partitions $(n-r, 1^r)$, for $r \in \{0, \ldots, n-1\}$. In [5, Theorem 6.1] we determined a set of representatives of the isomorphism classes of $\mathbb{Z}_p$-forms of the Specht $\mathbb{Q}_p\mathfrak{S}_n$-module $S_{\mathbb{Q}_p}^{(n-r,1^r)}$, for $p > 2$ and $r \in \{0, \ldots, n-1\}$; see also [14, Satz (III.8)]. The main ingredient in the proof of our result were the results of Plesken [13] and Craig [3] on the case $r = 1$. The case where $p = 2$ turned out to be much more difficult. In [5, Theorem 7.10, Theorem 7.16] we settled the case $p = 2 = r$, for $n \not\equiv 0 \pmod 4$. In Theorem 3.7 below we shall now treat the case $p = 2 = r$ and $n \equiv 0 \pmod 4$, thereby verifying [5, Conjecture 7.18(a)]. For a summary of the known relationship between Specht modules labelled by hook partitions and exterior powers, we refer to [5, Section 4; 5.5].

In what follows, for every prime number p and every $n \in \mathbb{N}$, we denote by $\nu_p(n)$ the p-adic valuation of n, that is, $\nu_p(n) = \max\{l \in \mathbb{N}_0 : p^l \mid n\}$.

3.5. Specht modules labelled by hook partitions. (a) Suppose now that $p = 2$ and $n \geqslant 4$. Let $t := \nu_2(n)$. For $R \in \{\mathbb{Z}_2, \mathbb{Q}_2, \mathbb{F}_2\}$, one has $R\mathfrak{S}_n$-isomorphisms $S_R^{(n-2,1^2)} \cong \bigwedge^2(S_R^{(n-1,1)})$; see [5, 5.5]. As in [5], we shall often view these isomorphisms as equalities, for convenience.

(b) Suppose that $p \mid n$. For $j \in \{0, \ldots, \nu_2(n)\}$, let M_{2^j} be the $\mathbb{Z}_2\mathfrak{S}_n$-sublattice of $S_{\mathbb{Z}_2}^{(n-1,1)}$ in [5, Theorem 8.1]. Then

$$S_{\mathbb{Z}_2}^{(n-1,1)} = M_1 \supseteq M_2 \supseteq M_4 \supseteq \cdots \supseteq M_{2^t} \cong (S_{\mathbb{Z}_2}^{(n-1,1)})^*,$$

and $M_1, \ldots, M_{2^t}$ are representatives of the isomorphism classes of $\mathbb{Z}_2$-forms of $S_{\mathbb{Q}_2}^{(n-1,1)}$. Moreover, for $j \in \{1, \ldots, t\}$, one has $2M_{2^{j-1}} \subseteq M_{2^j} \subseteq M_{2^{j-1}}$, $[M_{2^{j-1}} : M_{2^j}] = 2^{n-2}$ and $M_{2^{j-1}}/M_{2^j} \cong D_{\mathbb{F}_2}^{(n-1,1)}$ as $\mathbb{F}_2\mathfrak{S}_n$-modules. If $j \in \{1, \ldots, t-1\}$, then $M_{2^j}/2M_{2^j} \cong D_{\mathbb{F}_2}^{(n-1,1)} \oplus \mathbb{F}_2$.

By [5, Theorem 4.5], the $\mathbb{Z}_2\mathfrak{S}_n$-lattices $\bigwedge^2(M_1), \ldots, \bigwedge^2(M_{2^t})$ are pairwise non-isomorphic $\mathbb{Z}_2$-forms of $S_{\mathbb{Q}_2}^{(n-2,1^2)}$. Lastly, recall that $S_{\mathbb{Z}_2}^{(n-1,1)}$ has rank $n-1$, and $D_{\mathbb{F}_2}^{(n-1,1)}$ has dimension $n-2$.

Lemma 3.6. *Let $n > 4$ be such that $n \equiv 0 \pmod 4$, and let $t := \nu_2(n)$. Let V be the absolutely simple $\mathbb{Q}_2\mathfrak{S}_n$-module $S_{\mathbb{Q}_2}^{(n-2,1^2)}$. Moreover, let $D_1 := D_{\mathbb{F}_2}^{(n-1,1)}$, $D_2 := D_{\mathbb{F}_2}^{(n)} \cong \mathbb{F}_2$, and $D_3 := D_{\mathbb{F}_2}^{(n-2,2)}$. With the notation as in 3.5(b), one has the following:*

(a) $\bigwedge^2(M_{2^i}) \subseteq 2^i \bigwedge^2(M_1) = 2^i S_{\mathbb{Z}_2}^{(n-2,1^2)}$, *for $i \in \{0, \ldots, t\}$;*

(b) *for $i \in \{0, \ldots, t\}$, let $S_{3i+1} := \bigwedge^2(M_{2^i})/2^i$. Then $S_1, S_4, S_7, \ldots, S_{3t+1}$ are $\mathbb{Z}_2$-forms of V satisfying Hypotheses 3.1.*

Proof. As in the proof of [5, Proposition 6.5], we have $4\bigwedge^2(M_{2^i}) \subseteq \bigwedge^2(M_{2^{i+1}}) \subseteq 2\bigwedge^2(M_{2^i})$, for $i \in \{0, \ldots, t-1\}$. Thus, by induction on i, we deduce that the modules $S_1, S_4, \ldots, S_{3t+1}$ are $\mathbb{Z}_2$-forms of V such that $2S_{3i+1} \subseteq S_{3(i+1)+1} \subseteq S_{3i+1}$, for $i \in \{0, \ldots, t-1\}$.

Let $d_i := \dim_{\mathbb{F}_2}(D_i)$, for $i \in \{1, 2, 3\}$. We have $S_1/pS_1 \cong S_{\mathbb{F}_2}^{(n-2,1^2)}$, and it follows from [11, Theorem 1.1] that $\mathrm{Rad}(S_{\mathbb{F}_2}^{(n-2,1^2)}) = \mathrm{Soc}(S_{\mathbb{F}_2}^{(n-2,1^2)}) \cong D_1$ and $\mathrm{Hd}(S_{\mathbb{F}_2}^{(n-2,1^2)}) \cong D_2 \oplus D_3$. Furthermore, $d_1 = n-2$ and $d_2 = 1$. Thus $d_3 = \dim_{\mathbb{F}_2}(S_{\mathbb{F}_2}^{(n-2,1^2)}) - d_1 - d_2 = \binom{n-1}{2} - 1 - (n-2)$. From this one immediately deduces that Hypotheses 3.1(a) and (c) are satisfied. In particular, there is a unique maximal sublattice S_2 of S_1 with $S_1/S_2 \cong D_2$, and a unique maximal sublattice S_3 of S_1 with $S_1/S_3 \cong D_3$. So Hypothesis 3.1(d) is satisfied.

Since every simple $\mathbb{F}_2\mathfrak{S}_n$-module is self-dual, also Hypothesis 3.1(b) is satisfied. It remains to verify that Hypotheses 3.1(e) are satisfied. To this end, let $i \in \{0, \ldots, t-1\}$. Then

$$
\begin{aligned}
[S_{3i+1} : S_{3(i+1)+1}] &= \left[\bigwedge^2(M_{2^i})/2^i : \bigwedge^2(M_{2^{i+1}})/2^{i+1}\right] = \left[\bigwedge^2(M_{2^i}) : \bigwedge^2(M_{2^{i+1}})/2\right] \\
&= \frac{[\bigwedge^2(M_{2^i}) : \bigwedge^2(M_{2^{i+1}})]}{2^{\binom{n-1}{2}}} = 2^{(n-2)\binom{n-2}{1} - \binom{n-1}{2}} \\
&= 2^{\binom{n-2}{2}} = 2^{\binom{n-1}{2} - (n-2)} = 2^{d_2+d_3},
\end{aligned}
$$

by [5, Lemma 4.4, Proposition 2.5] and 3.5(b). Thus $S_{3i+1}/S_{3(i+1)+1}$ has $\mathbb{F}_2$-dimension $d_2 + d_3$, and $S_{3(i+1)+1}/2S_{3i+1}$ has $\mathbb{F}_2$-dimension $\binom{n-1}{2} - d_2 - d_3 = d_1$.

Next suppose that $i \in \{1, \ldots, t-1\}$. We need to show that $S_{3i+1}/2S_{3i+1}$ is semisimple as $\mathbb{F}_2\mathfrak{S}_n$-module. By 3.5, we know that $M_{2^i}/2M_{2^i} \cong D_1 \oplus D_2$, as $\mathbb{F}_2$-modules. Moreover, using [5, 4.1(d)] and [4, (12.2)], we also know that

$$
\begin{aligned}
S_{3i+1}/2S_{3i+1} &\cong \bigwedge^2(M_{2^i})/2\bigwedge^2(M_{2^i}) \cong \bigwedge^2(M_{2^i}/2M_{2^i}) \\
&\cong \bigwedge^2(D_2) \oplus (D_1 \otimes D_2) \oplus \bigwedge^2(D_1) \cong D_1 \oplus \bigwedge^2(D_1)
\end{aligned}
$$

as $\mathbb{F}_2\mathfrak{S}_n$-modules. Consequently, $\bigwedge^2(D_1)$ must have composition factors D_2 and D_3. On the other hand, recall that D_1 is isomorphic to the head of $S_{\mathbb{F}_2}^{(n-1,1)}$. The resulting $\mathbb{F}_2\mathfrak{S}_n$-epimorphism $S_{\mathbb{F}_2}^{(n-1,1)} \to D_1$ gives rise to an $\mathbb{F}_2\mathfrak{S}_n$-epimorphism $S_{\mathbb{F}_2}^{(n-2,1^2)} \cong \bigwedge^2(S_{\mathbb{F}_2}^{(n-1,1)}) \to \bigwedge^2(D_1)$. But we have already seen above that the only factor module of $S_{\mathbb{F}_2}^{(n-2,1^2)}$ with composition factors D_2 and D_3 has to be semisimple. Hence $\bigwedge^2(D_1) \cong D_2 \oplus D_3$, and we have verified Hypothesis 3.1(e)(ii).

Lastly, we have $M_{2^t} \cong (S_{\mathbb{Z}_2}^{(n-1,1)})^*$, and thus, by [5, 4.1(c)], also

$$S_{3t+1} \cong \bigwedge^2(M_{2^t}) \cong \left(\bigwedge^2(S_{\mathbb{Z}_2}^{(n-1,1)})\right)^* \cong (S_{\mathbb{Z}_2}^{(n-2,1^2)})^* \cong S_1^*$$

as $\mathbb{Z}_2\mathfrak{S}_n$-lattices.

Therefore, $S_1, S_4, \ldots, S_{3t+1}$ indeed satisfy Hypotheses 3.1, and the proof of the lemma is complete. □

In consequence of Lemma 3.6 we can now apply Theorem 3.2 to obtain

Theorem 3.7. *Let $n > 4$ be such that $n \equiv 0 \pmod 4$, let $t := \nu_2(n)$, and keep the notation from Lemma 3.6. Then the lattice of full-rank sublattices of the $\mathbb{Z}_2\mathfrak{S}_n$-lattice $S_{\mathbb{F}_2}^{(n-2,1^2)}$ is given by (7). The lattices $S_1, S_2, \ldots, S_{3t+1}$ are representatives of the isomorphism classes of $\mathbb{Z}_2$-forms of the simple $\mathbb{Q}_2\mathfrak{S}_n$-module $S_{\mathbb{Q}_2}^{(n-2,1^2)}$.*

Proof. By Lemma 3.6 and Theorem 3.2, we conclude that (7) is part of the lattice of full-rank sublattices of $S_1 = S_{\mathbb{Z}_2}^{(n-2,1^2)}$ and that $S_1, S_2, \ldots, S_{3t+1}$ are pairwise non-isomorphic $\mathbb{Z}_2$-forms of $S_{\mathbb{Q}_2}^{(n-2,1^2)}$. Let ε be the block idempotent of $\mathbb{Q}_2\mathfrak{S}_n$ corresponding to $S_{\mathbb{Q}_2}^{(n-2,1^2)}$, and let Λ the the graduated $\mathbb{Z}_2$-order $\varepsilon\mathbb{Z}_2\mathfrak{S}_n\varepsilon$ in $\varepsilon\mathbb{Q}_2\mathfrak{S}_n\varepsilon$. By Theorem 3.2, we also know that the exponent matrix of Λ with respect to S_1 equals

$$M := (m_{ij}) := \begin{pmatrix} 0 & 0 & 0 \\ t & 0 & a \\ t & b & 0 \end{pmatrix},$$

for some $a, b \in \mathbb{N}$. In order to complete the proof of the theorem, it suffices to show that $a = b = 1$ or, equivalently, that each of the lattices S_2 and S_3 has a unique maximal sublattice, namely the common maximal sublattice S_4. We examine S_2. By construction, S_2 is the unique maximal sublattice of S_1 such that $S_1/S_2 \cong D_2 \cong \mathbb{F}_2$. Therefore, S_2 has to be the $\mathbb{Z}_2\mathfrak{S}_n$-sublattice of S_1 constructed in [5, Lemma 7.5]; in particular, S_2 does not have a maximal sublattice T such that $S_2/T \cong D_2$, by [5, Lemma 7.7(a)]. Let $L \subseteq S_1$ be the unique sublattice of S_1 such that $L \not\subseteq 2S_1$ and $L/\operatorname{Rad}(L) \cong D_3$. Then, due to the structure of M, we must have

$S_1 \neq L \not\subseteq S_3$. Hence $L \subseteq S_2$, since S_2 and S_3 are the only maximal sublattices of S_1. If $L \neq S_2$, then $a > 1$ and so there would be a maximal sublattice T of S_2 such that $S_2/T \cong D_2$, a contradiction. Therefore, $L = S_2$ and $a = 1$.

Since $S_{\mathbb{Q}_2}^{(n-2,1^2)}$ is a self-dual $\mathbb{Q}_2\mathfrak{S}_n$-module and D_1, D_2, D_3 are self-dual $\mathbb{F}_2\mathfrak{S}_n$-modules, (6) gives

$$0 + a + t = m_{12} + m_{23} + m_{31} = m_{21} + m_{32} + m_{13} = t + b + 0\,,$$

that is, $b = a = 1$ as well. So, by Theorem 3.2(f), S_4 is also the unique maximal sublattice of S_3, and the assertion of the theorem follows. □

Remark 3.8. *For completeness, we also comment on the simple $\mathbb{Q}_2\mathfrak{S}_4$-module $S_{\mathbb{Q}_2}^{(2,1^2)}$. It is well known that, for every $r \in \{0, \dots, n-1\}$ and every prime p, one has $S_{\mathbb{Q}_p}^{(r+1,1^{n-r-1})} \cong S_{\mathbb{Z}_p}^{(n-r,1^r)} \otimes \mathrm{sgn}_{\mathbb{Q}_p}$, where $\mathrm{sgn}_{\mathbb{Q}_p}$ denotes the one-dimensional sign module of $\mathbb{Q}_p\mathfrak{S}_n$; see* [9, Theorem 6.7].

Thus, in particular, if M_1, M_2, M_3 denote the $\mathbb{Z}_2\mathfrak{S}_4$-sublattices of $S_{\mathbb{Z}_2}^{(3,1)}$ mentioned in 3.5, then $M_1 \otimes \mathrm{sgn}_{\mathbb{Z}_2}$, $M_2 \otimes \mathrm{sgn}_{\mathbb{Z}_2}$ and $M_3 \otimes \mathrm{sgn}_{\mathbb{Z}_2}$ are representatives of the isomorphism classes of $\mathbb{Z}_2$-forms of $S_{\mathbb{Q}_2}^{(2,1^2)}$.

Together with [5, Theorem 1.1, Corollary 3.4] we now also have the following immediate corollary:

Corollary 3.9. *Let $n \geqslant 4$, and let $V \in \{S_{\mathbb{Q}}^{(n-2,1^2)}, S_{\mathbb{Q}}^{(3,1^{n-3})}\}$. Let $h(V)$ be the number of isomorphism classes of $\mathbb{Z}$-forms of V, and let $d(n)$ be the number of divisors of n in $\mathbb{N}$. Then one has*

$$h(V) = \begin{cases} 3d(n) & \text{if } 2 \nmid n\,, \\ 2d(n) & \text{if } n \equiv 2 \pmod 4\,, \\ 3 & \text{if } n = 4\,, \\ \frac{(3\nu_2(n)+1)d(n)}{\nu_2(n)+1} & \text{if } n \equiv 0 \pmod 4,\ n > 4\,. \end{cases}$$

3.3. Application II: projective special linear groups. Our second application of Theorem 3.2 will involve the Steinberg module of the projective special linear group $\mathrm{PSL}_2(q)$ over suitable local fields of characteristic 0. We begin by setting up the necessary notation.

3.10. Two-fold transitive permutation lattices. (a) Let R be a principal ideal domain with field of fractions K of characteristic 0. Moreover, let $r \in \mathbb{N}$ with $r > 2$ and let G be a finite group acting two-transitively on a set $\Omega := \{\omega_1, \dots, \omega_r\}$. Let further M_R be the corresponding permutation RG-lattice with R-basis Ω, and let

$$L_R := {}_R\langle \omega_2 - \omega_1, \ldots, \omega_r - \omega_1 \rangle\,,$$

which is an RG-sublattice of M_R of rank $r-1$. Then the KG-module $V_K := KL_R$ is absolutely simple, and $KM_R \cong V_K \oplus K$; see [7, Satz V.20.2]. Since KM_R and K are a self-dual KG-modules, so is V_K. Note that we also have $M_R \cong R \otimes_{\mathbb{Z}} M_{\mathbb{Z}}$ and $L_R \cong R \otimes_{\mathbb{Z}} L_{\mathbb{Z}}$.

(b) Now suppose that K is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. Let R be the valuation ring of K with respect to the extension of the p-adic valuation, and let $\mathfrak{m} = (\pi)$ be the maximal ideal in R.

Suppose that $p \mid r$, and let $t \in \mathbb{N}$ be such that $(r) = \mathfrak{m}^t = (\pi^t)$. For $j \in \{0, \ldots, t\}$, the map $\iota_j : L_R/\pi^j L_R \to M_R/\pi^j M_R\,, x + \pi^j L_R \mapsto x + \pi^j M_R$ is easily checked to be an injective homomorphism of RG-modules and $R/(\pi^j)[G]$-modules. Since $\pi^j \mid r$, the element $\sum_{i=1}^r \omega_i + \pi^j M_R$ spans a trivial submodule of $M_R/\pi^j M_R$, and so $\sum_{i=2}^r (\omega_i - \omega_1) + \pi^j L_R$ spans a trivial submodule of $L_R/\pi^j L_R$. The preimage of the latter under the canonical surjective RG-homomorphism $L_R \to L_R/\pi^j L_R$ is

$$M_{\pi^j} := {}_R\langle \sum_{i=2}^r (\omega_i - \omega_1) \rangle + \pi^j L_R \subseteq L_R\,; \tag{8}$$

in particular, $M_{\pi^0}, M_\pi, \ldots, M_{\pi^t}$ are RG-sublattices of L_R. As well, M_{π^j} has R-basis $(\sum_{i=2}^r (\omega_i - \omega_1), \pi^j(\omega_3 - \omega_1), \ldots, \pi^j(\omega_r - \omega_1))$, for $j \in \{0, \ldots, t\}$. In particular, $M_{\pi^0}, M_\pi, \ldots, M_{\pi^t}$ are R-forms of the KG-module V_K not contained in πL_R. Thus, by [13, Proposition 2.3], they are pairwise non-isomorphic as RG-lattices.

Now suppose, in addition, that the factor module D of $L_R/\pi L_R$ modulo the trivial submodule D_1 mentioned above is a non-trivial, simple kG-module. By Lemma 3.11 below, the kG-module $L_R/\pi L_R$ then has to be indecomposable. Hence, by [15, Theorem (VI.1)] and [13, Proposition 2.3], one deduces that there are precisely $t+1$ isomorphism classes of R-forms of V_K, and hence $M_{\pi^0}, \ldots, M_{\pi^t}$ are representatives of these.

In fact, in this case, the kG-modules $M_{\pi^j}/\pi M_{\pi^j}$, for $j \in \{1, \ldots, t-1\}$ are all semisimple, hence isomorphic to $D \oplus D_1$. Moreover, for $j \in \{0, \ldots, t-1\}$, one has $M_{\pi^j}/M_{\pi^{j+1}} \cong D$ as kG-modules. This can, for instance, be deduced from Plesken's results in [13, Theorem 3.22] and [12, Satz (I.6)], which we shall also recall in 4.7.

Note that, in this case, the permutation kG-module $k\Omega \cong M_R/\pi M_R$ has composition factors D (with multiplicity 1) and k (with multiplicity 2). Since $k\Omega$ and k are self-dual, so is D.

The following properties of the RG-lattices introduced above are certainly well known. They will be important in the proofs of Lemma 3.17 and Proposition 3.18 below, so we include their proofs here.

Lemma 3.11. *In the notation of 3.10(a), let R be a principal ideal domain with field of fractions K, let $\mathfrak{m} := (\pi)$ be a maximal ideal in R, let $k := R/(\pi)$ be the corresponding residue field. Then the kG-module $L_R/\pi L_R$ does not have a trivial factor module.*

Proof. For $x \in \pi L_R$, let $\bar{x} := x + \pi L_R \in L_R/\pi L_R$. Assume that $L_R/\pi L_R$ has a trivial factor module, and let $\varphi : L_R/\pi L_R \to k$ be a non-trivial kG-homomorphism. Since the elements $\overline{\omega_i - \omega_1}$, for $i \in \{2, \ldots, r\}$, form a k-basis of $L_R/\pi L_R$, there is some $j \in \{2, \ldots, r\}$ with $\varphi(\overline{\omega_j - \omega_1}) \neq 0$. For $g \in G$, we have $\varphi(g(\overline{\omega_j - \omega_1})) = g\varphi(\overline{\omega_j - \omega_1}) = \varphi(\overline{\omega_j - \omega_1})$. Since $r > 2$, there is some $i \in \{2, \ldots, r\}$ such that $i \neq j$. Since G acts two-transitively on Ω, there exist $g, h \in G$ such that $g(\omega_j, \omega_1) = (\omega_j, \omega_i)$ and $h(\omega_j, \omega_1) = (\omega_i, \omega_1)$. We get

$$\begin{aligned}0 \neq g\varphi(\overline{\omega_j - \omega_1}) &= \varphi(\overline{\omega_j - \omega_i}) = \varphi(\overline{\omega_j - \omega_1}) - \varphi(\overline{\omega_i - \omega_1}) \\ &= \varphi(\overline{\omega_j - \omega_1}) - h\varphi(\overline{\omega_j - \omega_1}) = 0\,,\end{aligned}$$

a contradiction. □

Lemma 3.12. *In the notation of 3.10(b), one has an RG-isomorphism $M_{\pi^t} \cong M_1^*$.*

Proof. The permutation KG-module $M_K = KM_R$ carries its natural non-degenerate symmetric G-invariant bilinear form β such that $\beta(\omega_i, \omega_j) = \delta_{ij}$, for $i, j \in \{1, \ldots, r\}$. Via restriction, β induces a symmetric G-invariant bilinear form on V_K, and one easily checks that this is still non-degenerate. Let $N \subseteq V_K$ be any R-form of V_K, let $(b_1, \ldots, b_{r-1})$ be an R-basis of N, and consider $N^\# := \{v \in V_K : \beta(v, N) \subseteq R\}$. Then $N^\#$ is also an R-form of V_K. Moreover, $N \subseteq N^\#$ and, since R is local with finite residue field k, one has

$$[N^\# : N] = |R/\det((\beta(b_i, b_j))_{1,\leqslant i,j \leqslant r-1}|\,;$$

see [5, Proposition 2.5]. Also note that the map $N^\# \to N^*\,, x \mapsto (y \mapsto \beta(x, y))$ defines an RG-isomorphism.

Next observe that we have $\pi^t M_{\pi^t} \subseteq M_{\pi^t} \subseteq M_1 \subseteq \pi^t M_{\pi^t}^\#$, by the choice of t. So, in order to complete the proof of the lemma, it suffices to show that $[M_1 : \pi^t M_{\pi^t}] = [M_{\pi^t}^\# : M_{\pi^t}]$. For then we get $[M_1 : \pi^t M_{\pi^t}] = [\pi^t M_{\pi^t}^\# : \pi^t M_{\pi^t}]$, and hence $M_1 = \pi^t M_{\pi^t}^\# \cong M_{\pi^t}^\# \cong M_{\pi^t}^*$ as RG-lattices.

So consider the R-basis $(b_1, \ldots, b_{r-1}) := (\omega_2 - \omega_1, \ldots, \omega_r - \omega_1)$ of M_1 and the R-basis $(c_1 \ldots, c_{r-1}) := (\sum_{i=2}^{r}(\omega_i - \omega_1), \pi^t(\omega_3 - \omega_1), \ldots, \pi^t(\omega_r - \omega_1))$ of $\pi^t M_{\pi^t}$. Then one has

$$\det((\beta(c_i, c_i)_{1 \leqslant i,j \leqslant r-1}) = (\pi^t)^{2(r-2)} \cdot \det((\beta(b_i, b_j)_{1 \leqslant i,j \leqslant r-1}) = (\pi^t)^{2(r-2)} \cdot r\,,$$

hence $[M_{\pi^t}^{\#} : M_{\pi^t}] = |R/((\pi^t)^{2(r-2)} \cdot r)| = |R/r^{2r-3}R|$. On the other hand, using [5, Proposition 2.5], we also see that $[M_1 : \pi^t M_{\pi^t}] = |R/((\pi^t)^{2(r-2)} \cdot \pi^t)| = |R/r^{2r-3}R|$.

This completes the proof of the lemma. □

3.13. (Projective) general linear and (projective) special linear groups. Keep the notation as in 3.10.

(a) Let $n \in \mathbb{N}$ with $n > 1$, let p be a prime, and let q be a power of a prime. The general linear group $\mathrm{GL}_n(q)$ acts two-transitively on the set of one-dimensional subspaces of $\mathbb{F}_q^n$; as above, we simply denote this set by $\Omega = \{\omega_1, \ldots, \omega_r\}$, where $r = 1 + q + \cdots + q^{n-1}$. Suppose that $p \nmid q$. In the notation of 3.10 above, the $k[\mathrm{GL}_n(q)]$-module $L_R/\pi L_R$ is absolutely simple if $p \nmid r$; see [10, p.16, p.47, Theorem 20.3] So, in this case L_R is up to isomorphism the unique R-form of V_K; this follows from [4, Proposition (16.16)], see also [5, Proposition 2.12]. If $p \mid r$, then L_R/pL_R has precisely two absolutely simple composition factors D and D_1 satisfying the properties of 3.10(b); see, for instance, [10, (11.12)(iii), Theorem 16.3, Theorem 20.7]. The same is true when replacing $\mathrm{GL}_n(q)$ by the projective general linear group $\mathrm{PGL}_n(q)$. So, in these cases, Plesken's result [15, Theorem (VI.1)] determines the R-forms of V_K listed in (8).

(b) One has $\mathrm{SL}_n(q) \trianglelefteq \mathrm{GL}_n(q)$, and one can also regard $\mathrm{PSL}_n(q)$ as a normal subgroup of $\mathrm{PGL}_n(q)$ in the obvious way. Via restriction, the $K[\mathrm{GL}_n(q)]$-module V_K in (a) becomes a $K[\mathrm{SL}_n(q)]$-module (and also a $K[\mathrm{PSL}_n(q)]$-module), which is still absolutely simple, since also $\mathrm{SL}_n(q)$ acts two-transitively on Ω.

(c) In the following, we shall focus on the case $n = 2$. Then V_K is the Steinberg module of the (projective) general and (projective) special linear groups under consideration; see [1, Chapter 9]. Moreover, we shall from now on suppose that K is a finite unramified extension of $\mathbb{Q}_p$ and R is the valuation ring in K with respect to the extension of the p-adic valuation. In this case we may take $\pi := p$. The residue field k is isomorphic to $\mathbb{F}_{p^f}$, where f is the degree of K over $\mathbb{Q}_p$. We shall use the results of Plesken in [15, Theorem (VI.1); Chapter VII] and Theorem 3.2 to obtain representatives of the isomorphism classes of R-forms of $\mathrm{Res}_{\mathrm{PSL}_2(q)}^{\mathrm{PGL}_2(q)}(V_K)$.

Remark 3.14. (a) *Keep the notation from 3.13(c). Suppose first that p is odd and that q is power of a prime different from p. Let $G := \mathrm{PGL}_2(q)$ and $H := \mathrm{PSL}_2(q)$. If p does not divide the order of H, then the kH-module $\mathrm{Res}^G_H(L_R/pL_R)$ is absolutely simple. Thus, in this case, $\mathrm{Res}^G_H(L_R)$ is up to isomorphism the unique R-form of $\mathrm{Res}^G_H(V_K)$; see* [4, Proposition (16.16)], [5, Proposition 2.12].

Hence, we may suppose that $p \mid |H|$. Recall that $|H|$ is a divisor of $(q+1)q(q-1)$. If $p \mid (q-1)$, then, by [1, Section 9.4.2]*, the kH-module $\mathrm{Res}^G_H(L_R/pL_R)$ is absolutely simple. So, also in this case, $\mathrm{Res}^G_H(L_R)$ is up to isomorphism the unique R-form of $\mathrm{Res}^G_H(V_K)$. If $p \mid (q+1)$, then it follows from* [1, Section 9.4.3] *that the kH-module $\mathrm{Res}^G_H(L_R/pL_R)$ has a trivial submodule with (absolutely) simple quotient. So, we are in the situation of 3.10(b), and representatives of the isomorphism classes of R-forms of V_K are given by the RH-lattices in (8), for $t = \nu_p(q+1)$.*

(b) *Now let $p = 2$, and let q be odd. If $q \equiv \pm 3 \pmod 8$ and if the degree of K over $\mathbb{Q}_2$ is odd, then the kH-module $\mathrm{Res}^G_H(L_R/2L_R)$ has two composition factors; see* [15, p. 110]*. In consequence of 3.13(a) and Lemma 3.11, we are then again in the situation of 3.10(b), and representatives of the isomorphism classes of R-forms of $\mathrm{Res}^G_H(V_K)$ are given by the lattices in (8), for $t = \nu_2(q+1)$.*

If $q \equiv \pm 1 \pmod 8$, or if $q \equiv \pm 3 \pmod 8$ and the degree of K over $\mathbb{Q}_2$ is even, then $\mathrm{Res}^G_H(L_R/2L_R)$ has three (absolutely) simple composition factors; see [15, p. 110], [1, Section 9.4.4]*. These cases will be dealt with in the following.*

Remark 3.15. *As for the case of equal characteristic, that is, in the case where $p \mid q$, note that the kH-module $\mathrm{Res}^G_H(L_R/pL_R)$ is projective and absolutely simple; see* [1, Lemma 10.2.4]*. So, also in this case, $\mathrm{Res}^G_H(L_R)$ is up to isomorphism the unique R-form of $\mathrm{Res}^G_H(V_K)$.*

Hypotheses 3.16. For the remainder of this subsection, we suppose that q is an odd prime power and $p = 2$. We set $G := \mathrm{PGL}_2(q)$, and let $H := \mathrm{PSL}_2(q)$. Let (K, R, k) be as in 3.13(c). If $q \equiv \pm 1 \pmod 8$, then we may take K to be any finite unramified extension of $\mathbb{Q}_2$. If $q \equiv \pm 3 \pmod 8$, then let K be a finite unramified extension of $\mathbb{Q}_2$ of even degree; in particular, k then contains the field with four elements. Lastly, let V_K be the absolutely simple KG-module with R-form L_R as defined in 3.13(a).

Lemma 3.17. *Let q be an odd prime power, and let $t := \nu_2(q+1)$. With Hypotheses 3.16 one has the following:*

(a) *The kH-module $\mathrm{Res}^G_H(L/2L)$ has precisely three composition factors D_1, D_2 and D_3, all of which are absolutely simple and pairwise non-isomorphic.*

More precisely, $D_1 \cong k$, *and* $\mathrm{Res}^G_H(D) \cong D_2 \oplus D_3$; *in particular* $\dim_k(D_2) = (q-1)/2 = \dim_k(D_3)$.

(b) *For* $i \in \{0, \ldots, t\}$, *let* $M_{2^i} \subseteq L_R$ *be the* RG*-lattice in (8), and let* $S_{3i+1} := \mathrm{Res}^G_H(M_{2^i})$. *Then* $S_1, S_4, \ldots, S_{3t+1}$ *are* R*-forms of* $\mathrm{Res}^G_H(V_K)$ *satisfying Hypotheses 3.1.*

Proof. Assertion (a) is well known; see, for instance, [1, Section 9.4.4] and [15, p.110]. Note that D_2 is G-conjugate to D_3. As for assertion (b), note first that $S_1, S_4, \ldots, S_{3t+1}$ are of course R-forms of $\mathrm{Res}^G_H(V_K)$ satisfying Hypothesis 3.1(e)(i). Recall from 3.10(b) and 3.13(a) that $M_{2^i}/2M_{2^i} \cong D_1 \oplus D$, for $i \in \{1, \ldots, t-1\}$, and that $M_{2^i}/M_{2^{i+1}} \cong D$, for $i \in \{0, \ldots, t-1\}$, as kG-modules. Hence $S_1, S_4, \ldots, S_{3t+1}$ satisfy Hypothesis 3.1(e)(ii). By Lemma 3.12, also Hypothesis 3.1(e)(iii) is satisfied.

Since the simple kG-module D is self-dual and since $\mathrm{Res}^G_H(D) \cong D_2 \oplus D_3$, Hypotheses 3.1(b) are satisfied. Setting $d_i := \dim_k(D_i)$, for $i \in \{1, 2, 3\}$, we have $d_2 = d_3 = (q-1)/2 \neq 1 = d_1$ and $d_1 \neq d_2 + d_3$.

It remains to verify the assertion on the submodule structure of the kH-module $\mathrm{Res}^G_H(\bar{L}) := \mathrm{Res}^G_H(L_R/2L_R)$. We know that $\bar{L}$ has a trivial submodule U with factor module isomorphic to $\mathrm{Res}^G_H(D) \cong D_2 \oplus D_3$; in particular, $\mathrm{Rad}(\mathrm{Res}^G_H(\bar{L})) \subseteq U$. By Lemma 3.11, we also know that $\mathrm{Res}^G_H(\bar{L})$ does not have a trivial factor module; in particular, $\mathrm{Res}^G_H(\bar{L})$ cannot be semisimple, implying $\mathrm{Rad}(\mathrm{Res}^G_H(\bar{L})) = U$. Therefore, $\mathrm{Res}^G_H(\bar{L})$ has precisely two maximal submodules, U_2 and U_3, where U_2 has composition factors D_2 and D_1, and U_3 has composition factors D_1 and D_3. So U_2 has to be G-conjugate to U_3. Since $\mathrm{Res}^G_H(\bar{L})$ is not semisimple, both U_2 and U_3 are indecomposable, and the common trivial submodule of U_2 and U_3 is the unique simple submodule of $\mathrm{Res}^G_H(\bar{L})$. Thus $\mathrm{Soc}(\mathrm{Res}^G_H(\bar{L})) = \mathrm{Rad}(\mathrm{Res}^G_H(\bar{L})) \cong k \cong D_1$. This completes the proof of the lemma. □

Proposition 3.18. *Let* q *be an odd prime power, and let* $t := \nu_2(q+1)$. *In the notation of Lemma 3.17, the* RH*-lattice* L_R *has submodule lattice (7). The lattices* $S_1, S_2, \ldots, S_{3t+1}$ *are representatives of the isomorphism classes of* R*-forms of the absolutely simple* KH*-module* $\mathrm{Res}^G_H(V_K)$.

Proof. Let ε be the block idempotent of KH corresponding to the absolutely simple module $\mathrm{Res}^G_H(V_K)$. Consider the graduated R-order $\Lambda := \varepsilon RH\varepsilon$ in $\varepsilon KH\varepsilon$. By Lemma 3.17 and Theorem 3.2, we know that (7) is part of the lattice of full-rank sublattices of $L_R = S_1$. Moreover, $S_1, S_2, \ldots, S_{3t+1}$ are pairwise non-isomorphic R-forms of the KH-module V_K. On the other hand, by [15, Chapter VII], there is a KH-lattice $L' \subseteq V_K$ that is an R-form of V_K and the exponent matrix of Λ with respect to L' is

$$M_{L'} = \begin{pmatrix} 0 & t & t \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

From this and 2.6(c) one easily deduces that the isomorphism classes of R-forms of the KH-module V_K are in bijection with the set

$$\{(0,0,0),(j,0,0),(j,0,1),(j,1,0) : j \in \{1,\ldots,t\}\},$$

which has cardinality $3t+1$. Hence, (7) has to be the complete lattice of full-rank sublattices of $L_R = S_1$, and $S_1, S_2, \ldots, S_{3t+1}$ are representatives of the isomorphism classes of R-forms of the KH-module V_K. □

Question 3.19. To conclude this section, suppose again that G is any finite group acting two-transitively on a finite set $\Omega = \{\omega_1, \ldots, \omega_r\}$ with $r > 2$. In the notation of 3.10, we know that the absolutely simple KG-module V_K admits at least $t+1$ pairwise non-isomorphic R-forms, representatives of which are given by the RG-lattices $L_R = M_{\pi^0}, \ldots, M_{\pi^t}$ in (8). We also know, by Lemma 3.12, that M_{π^t} is isomorphic to the dual lattice $M_1^* = L_R^*$. Consider the kG-module $L_k := L_R/\pi L_R$ and its trivial submodule D_1 as in 3.10. Suppose that L_k/D_1 is the direct sum of two non-trivial pairwise non-isomorphic simple kG-modules D_2 and D_3. For $i \in \{0, \ldots, t\}$, set $S_{3i+1} := M_{\pi^i}$.

Do these RG-lattices then satisfy Hypotheses 3.1? If so, is S_4 the only maximal RG-sublattice of each of the maximal sublattices S_2 and S_3 of S_1? If this was the case, then Theorem 3.2 would be applicable to determine representatives of the isomorphism classes of R-forms of V_K, generalizing Proposition 3.18.

4. Zeta functions

In this section we briefly review the notion of zeta functions of modules. We follow Solomon [16], who introduced these objects to study enumerative problems in integral representation theory. After introducing the general zeta function, we focus on the case over local principal ideal domains and determine zeta functions of various types of lattices, including the ones from Section 3.

4.1. Local and global zeta functions.

Notation 4.1. Let R be a unitary ring, and let M be a left R-module such that, for all $n \in \mathbb{N}$, the number a_n of R-sublattices of M with index n is finite. One defines the *zeta function of* M by

$$\zeta_R(M,s) := \sum_{n=1}^{\infty} a_n n^{-s}, \text{ where } s \in \mathbb{C}.$$

We view this as a formal Dirichlet series and ignore questions of convergence. Note that if M is a free $\mathbb{Z}$-module of finite rank r, then, by comparing $\zeta_R(M,s)$ with $\zeta_{\mathbb{Z}}(M,s)$, one sees that $\zeta_R(M,s)$ converges absolutely for $\mathrm{Re}(s) > r$; see also [2].

4.2. Local zeta functions. (a) Let R be a local principal ideal domain with maximal ideal $\mathfrak{m} = (\pi)$, field of fractions K and finite residue field $k = R/\mathfrak{m}$ of cardinality q. Assume that Λ is an R-order and M a Λ-lattice. Since k is finite, the number of Λ-sublattices of M with bounded index is finite. Moreover, as the indices of such Λ-sublattices of M must be powers of q, there exists some $Z(M) \in \mathbb{Z}[[X]]$ such that $Z(M)(q^{-s}) = \zeta_\Lambda(M,s)$. By defining $(M:N) := X^i$, for $i \in \mathbb{N}_0$ and every R-submodule of N of M with $[M:N] = q^i$, this can be rewritten as

$$Z(M) = \sum_{N \subseteq M} (M:N),$$

where the sum is taken over all Λ-sublattices N of M such that $[M:N] < \infty$.

(b) Assume that, up to isomorphism, there are only finitely many Λ-sublattices $M = M_1, M_2, \ldots, M_r$ of M with finite index, for some $r \in \mathbb{N}$. For $j \in \{1, \ldots, r\}$, one defines

$$Z(M, M_j) = \sum_{\substack{N \subseteq M \\ [M:N] < \infty \\ N \cong M_j}} (M:N)\,.$$

Analogously, one defines $Z(M_i, M_j)$, for all $i, j \in \{1, \ldots, r\}$. Having fixed an ordering on $M_1, \ldots, M_r$, the matrix $(Z(M_i, M_j))_{1 \leqslant i,j \leqslant r} \in \mathbb{Z}[[X]]^{r \times r}$ is uniquely determined by M, and we shall from now on denote it by $\mathbf{B}_M$. Note that $Z(M_i)$ is the sum of the entries of the ith row of $\mathbf{B}_M$, for $i \in \{1, \ldots, r\}$.

(c) We denote by $\max(M)$ the set of maximal Λ-sublattices of M, and by $\mathrm{Rad}(M) = \bigcap_{N \in \max(M)} N$ again the Jacobson radical of M. Furthermore, we consider the following sets of Λ-sublattices of M: $\Phi(M) := \{N : \mathrm{Rad}(M) \subseteq N \subseteq M\}$ and $\Phi(M, L) := \{N \in \Phi(M) : N \cong L\}$. Finally, for $L \in \Phi(M)$, let $\mu(M, L) := \sum_J (-1)^{|J|}$, where the sum runs over all subsets $J \subseteq \max(M)$ with $\bigcap_{N \in J} N = L$. The matrix $(A_{ij})_{1 \leqslant i,j \leqslant r} \in \mathbb{Z}[X]^{r \times r}$ defined by

$$A_{ij} = \sum_{L \in \Phi(M_i, M_j)} \mu(M_i, L)(M_i : L)$$

is uniquely determined by M, and we denote it by $\mathbf{A}_M$. By [16, Lemma 3], the matrix $\mathbf{A}_M$ is the inverse of $\mathbf{B}_M$.

4.3. Global zeta functions. Assume that M is a $\mathbb{Z}$-form of a $\mathbb{Q}G$-module V, for which we want to determine the zeta function $\zeta_{\mathbb{Z}G}(M,s)$. Then, for every prime p, the p-adic completion $M_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} M$ is a $\mathbb{Z}_p$-form of the $\mathbb{Q}_pG$-module $V_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} V$, giving rise to a local zeta function $\zeta_{\mathbb{Z}_pG}(M_p,s)$. By 4.2, we know that

$$\zeta_{\mathbb{Z}_pG}(M_p,s) = Z(M_p)(p^{-s})\,.$$

Let $\mathbb{P} \subset \mathbb{N}$ be the set of all prime numbers. By [16], one has

$$\zeta_{\mathbb{Z}G}(M,s) = \prod_{p\in\mathbb{P}} \zeta_{\mathbb{Z}_pG}(M_p,s)\,.$$

In [16] it is also shown that there exists a complex function

$$\zeta_V(s) = \prod_{p\in\mathbb{P}} \zeta_{V,p}(s)\,,$$

depending only on V, such that $\zeta_{V,p}(s) = \zeta_{\mathbb{Z}_pG}(M_p,s)$, for all primes p not dividing the group order $|G|$. In particular, if P is a finite set of prime numbers containing all prime divisors of $|G|$, then

$$\zeta_{\mathbb{Z}G}(M,s) = \zeta_V(s) \prod_{p\in P} \frac{\zeta_{\mathbb{Z}_pG}(M_p,s)}{\zeta_{V,p}(s)}\,.$$

Thus, when determining the global zeta function $\zeta_{\mathbb{Z}G}(M,s)$, it is sufficient to determine $\zeta_V(s)$ as well as the local zeta functions $\zeta_{\mathbb{Z}_pG}(M_p,s)$, for all prime divisors of $|G|$. The task of determining $\zeta_V(s)$ is straightforward, once the structure of the blocks of $\mathbb{Q}G$ containing the indecomposable direct summands of V are known; see [16, (1.2)]. If V is absolutely simple of dimension d, then $\zeta_V(s) = \zeta_{\mathbb{Q}}(ds)$, where $\zeta_{\mathbb{Q}}$ is the Riemann zeta function. In particular, in this case, we have $\zeta_{V,p}(s) = (1-p^{-ds})^{-1}$, for all $p \in \mathbb{P}$.

Lastly, note that if V is simple and if $p \in \mathbb{P}$ is such that the $\mathbb{F}_pG$-module $M/pM \cong M_p/pM_p$ is also simple, then $\zeta_{\mathbb{Z}_pG}(M_p,s) = \zeta_{V,p}(s)$. Namely, in this case, pM_p is the unique maximal sublattice of M_p and $\{p^iM_p : i \in \mathbb{N}_0\}$ is the set of all $\mathbb{Z}_pG$-sublattices of M_p. Hence

$$\zeta_{\mathbb{Z}_pG}(M_p,s) = \sum_{i=0}^{\infty} (p^{id})^{-s} = \frac{1}{1-p^{-ds}} = \zeta_{V,p}(s)\,.$$

4.2. Uniserial reductions. Throughout this subsection, let R be a local principal ideal domain with maximal ideal $\mathfrak{m} = (\pi)$, field of fractions K and finite residue field $k = R/\mathfrak{m}$ of cardinality q. Assume further that G is a finite group and M is an R-form of an absolutely simple KG-module V such that the lattice of RG-sublattices of M of full R-rank is totally ordered. This happens, for instance, if the reduction modulo $\mathfrak{m}$ of every R-form of V is a uniserial kG-module; see [5, Proposition 3.7].

Denote by $\pi M = M_{r+1} \subseteq M_r \subseteq \cdots \subseteq M_1 = M$ a chain of R-forms of V, such that, for $i \in \{1, \ldots, r\}$, the module M_{i+1} is a maximal RG-sublattice of M_i. By [13, Proposition 2.3], we know that $M_1, \ldots, M_r$ form a set of representatives of the R-forms of V.

Our next aim is to determine, for each $i \in \{1, \ldots, r\}$, the zeta function $\zeta_{RG}(M_i, s)$, by determining $\mathbf{B}_M$ and $Z(M_i) \in \mathbb{Z}[[X]]$. For $i \in \{1, \ldots, r\}$, we denote by d_i the k-dimension of M_i/M_{i+1}, and we set $d := d_1 + \cdots + d_r$.

Lemma 4.4. *With the above notation, the matrix $\mathbf{A}_M = (A_{ij}) \in \mathbb{Z}[X]^{r \times r}$ is given by*

$$A_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -X^{d_i} & \text{if } i \neq r,\ j = i+1, \\ -X^{d_r} & \text{if } i = r,\ j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\mathbf{A}_M = \begin{pmatrix} 1 & -X^{d_1} & 0 & 0 & \ldots & 0 \\ 0 & 1 & -X^{d_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -X^{d_{r-1}} \\ -X^{d_r} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. Since the lattice of full-rank RG-sublattices of M is totally ordered, we have $\max(M_i) = \{M_{i+1}\}$, for $i \in \{1, \ldots, r-1\}$, and $\max(M_r) = \{\pi M_1\}$. Thus $\Phi(M_i) = \{M_i, M_{i+1}\}$, for $i \in \{1, \ldots, r-1\}$, and $\Phi(M_r) = \{M_r, \pi M_1\}$. In particular,

$$\Phi(M_i, M_j) = \begin{cases} \{M_j\} & \text{if } i \neq r, j \in \{i, i+1\}, \\ \{M_j\} & \text{if } i = r, j \in \{1, r\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

For $i, j \in \{1, \ldots, r\}$ and $L \in \Phi(M_i, M_j)$, we have $\mu(M_i, L) = 1$ if $L = M_i$, and $\mu(M_i, L) = -1$ otherwise. Thus the claim follows. $\square$

Proposition 4.5. *The matrix* $\mathbf{B} = (B_{ij})_{1 \leqslant i,j \leqslant r}$ *defined by*

$$B_{ij} = \frac{1}{1-X^d} \begin{cases} X^{d_i+\cdots+d_{j-1}} & \text{if } j \geqslant i, \\ X^d B_{ji}^{-1} & \text{if } j < i \end{cases}$$

satisfies $\mathbf{B} = \mathbf{B}_M$.

Proof. It is sufficient to show that $\mathbf{B}\mathbf{A}_M$ is the identity matrix. To this end, let A_j be the jth column of $\mathbf{A}_M$ and B_i the ith row of $\mathbf{B}$. Then

$$\begin{aligned}
&(1-X^d) \cdot B_j \cdot A_1 \\
&= B_{j1} - X^{d_r} B_{jr} \\
&= \begin{cases} 1 - X^{d_r} B_{1r} = 1 - X^d & \text{if } j = 1, \\ X^d B_{1j}^{-1} - X^{d_r} B_{jr} = X^{d_j+\cdots+d_r} - X^{d_j+\cdots+d_r} & \text{if } 1 < j < r \\ B_{r1} - X^{d_r} B_{rr} = X^{d_r} - X^{d_r} & \text{if } j = r \end{cases} \\
&= \begin{cases} 1 - X^d & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Now let $i > 1$ and consider $(1-X^d)B_j \cdot A_i = B_{j,i-1} \cdot (-X^{d_{i-1}}) + B_{ji}$. If $j < i-1$, then

$$(1-X^d)B_j \cdot A_i = X^{\sum_{k=j}^{i-2} d_k}(-X^{d_{i-1}}) + X^{\sum_{k=j}^{i-1} d_k} = 0\,;$$

if $j = i-1$, then

$$(1-X^d)B_j \cdot A_i = -X^{d_{i-1}} + B_{i-1,i} = -X^{d_{i-1}} + X^{d_{i-1}} = 0\,;$$

and if $j > i$, then

$$(1-X^d)B_j \cdot A_i = -X^{d_{i-1}} X^d B_{i-1,j}^{-1} + X^d B_{i,j}^{-1} = 0\,,$$

since $B_{i,j}/B_{i-1,j} = X^{d_{i-1}}$. Finally, if $j = i$, then

$$(1-X^d)B_i \cdot A_i = B_{i,i-1}(-X^{d_{i-1}}) + 1 = X^d B_{i-1,i}^{-1}(-X^{d_{i-1}}) = 1 - X^d.$$

□

By summing up the entries of $\mathbf{B}_M$ row-wise, we obtain:

Corollary 4.6. *For* $i \in \{1, \ldots, r\}$, *one has*

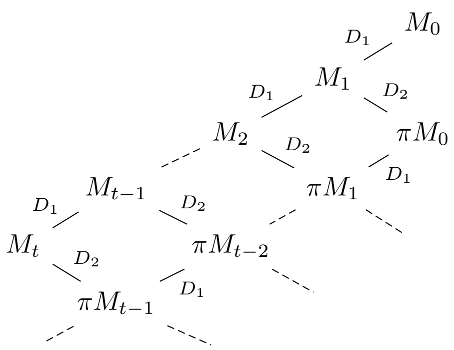
$$Z(M_i) = \frac{1}{1-X^d} \left(\sum_{j=1}^{i-1} X^{d-d_j-\cdots-d_{i-1}} + \sum_{j=i}^{r} X^{d_i+\cdots+d_{j-1}} \right).$$

4.3. Modular reductions with two non-isomorphic composition factors. In this section, let R be a local principal ideal domain with maximal ideal $\mathfrak{m} = (\pi)$, field of fractions K, and finite residue field $k = R/\mathfrak{m}$ of cardinality q. Let V be an absolutely simple KG-module of dimension d such that the reduction modulo $\mathfrak{m}$ of any R-form of V has two non-isomorphic composition factors D_1 and D_2. Assume further that the Jordan–Zassenhaus theorem holds for R-forms of V, that is, up to isomorphism there are only finitely many R-forms of V. By [12, Satz (I.6)], there exists an R-form M of V such that $M/\pi M$ is indecomposable. We fix such an R-form M of V, for the remainder of this subsection. We shall suppose that the head of $M/\pi M$ is isomorphic to D_1. In fact, [12, Satz (I.6)] is stated in the case where $R = \mathbb{Z}$ and $\mathfrak{m}$ is any maximal ideal in $\mathbb{Z}$. The proof, however, generalizes literally to our situation. Alternatively, see also [13, Theorem 3.22].

4.7. The submodule lattice of M. Now let $t+1$ be the number of isomorphism classes of R-forms of V. In [13, Theorem 3.22] (see also [12, Satz (I.6)] for the case $R = \mathbb{Z}$) Plesken has shown that there exist R-forms $M_0 = M, M_1, \ldots, M_t$ of V such that

(a) M_1 is the unique maximal RG-sublattice of M_0,
(b) for $i \in \{1, \ldots, t-1\}$, $M_{i+1} \neq \pi M_{i-1}$ are the only maximal RG-sublattices of M_i,
(c) πM_{t-1} is the unique maximal RG-sublattice of M_s.

Moreover, $M_0, \ldots, M_t$ are representatives of the isomorphism classes of R-forms of V. The lattice of full-rank RG-sublattices of M therefore looks as in diagram (9) below.



(9)

We set $\dim_k(M_0/M_1) = \dim_k(D_1) =: d_1$ and $\dim_k(M_1/\pi M_0) = \dim_k(D_2) =: d_2$, so that $d = d_1 + d_2$.

Lemma 4.8. *The matrix* $\mathbf{A}_M = (A_{ij}) \in \mathbb{Z}[X]^{(t+1)\times(t+1)}$ *satisfies*

$$A_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i = j = t+1, \\ 1 + X^d & \text{if } i = j,\ i \notin \{1, t+1\}, \\ -X^{d_1} & \text{if } j = i+1, \\ -X^{d_2} & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$A = \begin{pmatrix} 1 & -X^{d_1} & 0 & 0 & \dots & 0 \\ -X^{d_2} & 1+X^d & -X^{d_1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -X^{d_2} & 1+X^d & -X^{d_1} \\ 0 & 0 & 0 & 0 & -X^{d_2} & 1 \end{pmatrix}.$$

Proof. From 4.7 and diagram (9) we can read off the following properties. First of all we have $\max(M_0) = \{M_1\}$, $\max(M_t) = \{\pi M_{t-1}\}$, and $\max(M_i) = \{M_i, \pi M_{i-1}\}$, for $i \in \{1, \dots, t-1\}$. In particular, $\Phi(M_0) = \{M_0, M_1\}$, $\Phi(M_t) = \{M_t, \pi M_{t-1}\}$ and $\Phi(M_i) = \{M_i, M_{i+1}, \pi M_{i-1}, \pi M_i\}$, for $i \in \{1, \dots, t-1\}$. Since $M_0, \dots, M_t$ are representatives of the isomorphism classes of R-forms of V, we conclude that $\Phi(M_0, M_j) = \{M_j\}$, for $j \in \{0, 1\}$, and $\Phi(M_0, M_j) = \emptyset$ otherwise. Moreover, for $i \in \{1, \dots, t\}$, we have

$$\Phi(M_i, M_j) = \begin{cases} \{M_i, \pi M_i\} & \text{if } i = j,\ i \neq t, \\ \{M_i\} & \text{if } i = j,\ i = t, \\ \{M_{i+1}\} & \text{if } j = i+1, \\ \{\pi M_{i-1}\} & \text{if } j = i-1. \end{cases}$$

From this the assertion of the lemma follows. $\square$

Proposition 4.9. *The matrix* $\mathbf{B} = (B_{ij})_{1 \leqslant i,j \leqslant t+1}$ *defined by*

$$B_{ij} = \frac{1}{1 - X^d} \begin{cases} X^{(j-i)d_1} & \text{if } j \geqslant i, \\ X^{(i-j)d_2} & \text{if } i \geqslant j. \end{cases}$$

satisfies $\mathbf{B}_M = \mathbf{B}$. *Thus*

$$\mathbf{B}_M = \frac{1}{1-X^d}\begin{pmatrix} 1 & X^{d_1} & X^{2d_1} & X^{3d_1} & X^{4d_1} & \cdots & X^{td_1} \\ X^{d_2} & 1 & X^{d_1} & X^{2d_1} & X^{3d_1} & \cdots & X^{(t-1)d_1} \\ X^{2d_2} & X & 1 & X^{d_1} & X^{2d_1} & \cdots & X^{(t-2)d_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X^{(t-2)d_2} & X^{(t-3)d_2} & \cdots & X & 1 & X^{d_1} & X^{2d_1} \\ X^{(t-1)d_2} & X^{(t-2)d_2} & X^{(t-3)d_2} & \cdots & X & 1 & X^{d_1} \\ X^{td_2} & X^{(t-1)d_2} & X^{(t-2)d_2} & \cdots & X^{2d_2} & X^{d_2} & 1 \end{pmatrix}.$$

Proof. As usual we prove the assertion by showing that the matrix $\mathbf{B}$ is the inverse of $\mathbf{A}_M$, which is a straightforward calculation. Let A_i be the ith row of $\mathbf{A}_M$ and B_j the jth column of $\mathbf{B}$. We need to show that $A_iB_j = \delta_{ij}$, for $i, j \in \{1, \ldots, t+1\}$. For instance, for $i, j \in \{2, \ldots, t-1\}$, we have

$$\begin{aligned} A_i \cdot B_j &= A_{i,i-1}B_{j,i-1} + A_{i,i}B_{j,i} + A_{i,i+1}B_{j,i+1} \\ &= -X^{d_2}B_{j,i-1} + (1 + X^{d_1+d_2})B_{j,i} - X^{d_1}B_{j,i+1}. \end{aligned}$$

Now $A_iB_j = \delta_{ij}$ follows by observing that

$$(B_{j,i-1}, B_{j,i}, B_{j,i+1}) = \begin{cases} (X^{(j-i+1)d_1}, X^{(j-i)d_1)}, X^{(j-i-1)d_1}) & \text{if } i < j, \\ (X^{d_1}, 1, X^{d_2}) & \text{if } i = j, \\ (X^{(i-j-1)d_2}, X^{(i-j)d_2}, X^{(i-j+1)d_2}) & \text{if } i > j. \end{cases}$$

The remaining cases are treated analogously. □

We have now established the following:

Corollary 4.10. *With the notation as in 4.7, for* $i \in \{0, \ldots, t\}$*, one has*

$$Z(M_i) = \frac{1}{1-X^d}\left(\sum_{j=0}^{i} X^{jd_2} + \sum_{j=1}^{t-i} X^{jd_1}\right).$$

4.4. Modular reductions with three non-isomorphic composition factors. Throughout this subsection, let R be a local principal ideal domain with maximal ideal $\mathfrak{m} = (\pi)$, field of fractions K, and finite residue field $k = R/\mathfrak{m}$ of cardinality q. Assume further that G is a finite group and $S = S_1$ is an R-form of an absolutely simple KG-module V satisfying Hypotheses 3.1. Let $S_1, S_2, \ldots, S_{3t+1}$ be RG-lattices as in Theorem 3.2, and suppose that $S_1, S_2, \ldots, S_{3t+1}$ are representatives of the isomorphism classes of R-forms of V.

Our next aim in this subsection is to determine $Z(S)$.

Lemma 4.11. *With the above notation, let*

$$\mathbf{A}_1 = \begin{pmatrix} 1 & -X^{d_2} & -X^{d_3} & X^{d_2+d_3} \\ 0 & 1 & 0 & -X^{d_3} \\ 0 & 0 & 1 & -X^{d_2} \end{pmatrix} \in \mathbb{Z}[X]^{3\times 4}.$$

Then

$$\mathbf{A}_S = \left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{0}_{3\times 3(t-1)} \\ \hline \multicolumn{2}{c}{*} \end{array} \right].$$

Proof. From Theorem 3.2 and diagram (7) we know that $\max(S_1) = \{S_2, S_3\}$, $\max(S_2) = \max(S_3) = \{S_4\}$, and $\mathrm{Rad}(S_i) = S_4$, for $i \in \{1,2,3\}$. Thus $\Phi(S_1) = \{S_1, S_2, S_3, S_4\}$ and $\Phi(S_i) = \{S_i, S_4\}$, for $i \in \{2,3\}$. As $S_1, \ldots, S_{3t+1}$ are representatives of the isomorphism classes of R-forms of V, we obtain $\Phi(S_1, S_j) = \{S_j\}$ for $j \in \{1,2,3,4\}$, and $\Phi(S_1, S_j) = \emptyset$ otherwise. As well, for $i \in \{2,3\}$, we have $\Phi(S_i, S_j) = \{S_j\}$ for $j \in \{i, 4\}$, and $\Phi(S_i, S_j) = \emptyset$ otherwise. From this the assertion of the lemma follows. □

Lemma 4.12. *Let $i \in \{1, \ldots, t-1\}$, and let $\mathbf{A}' = (A_{lj})_{3i+1\leqslant l\leqslant 3i+3, 1\leqslant j\leqslant 3t+1}$ be the submatrix of $\mathbf{A}_S$ consisting of rows $3i+1, 3i+2, 3i+3$. Then*

$$\mathbf{A}' = \left[\begin{array}{c|c|c} \mathbf{0}_{3\times 3(i-1)} & \mathbf{A}_2 & \mathbf{0}_{3\times 3(t-i)-3} \end{array} \right],$$

where

$$\mathbf{A}_2 = \begin{pmatrix} -X^{d_1} & X^{d_1+d_2} & X^{d_1+d_3} & 1-X^{d} & -X^{d_2} & -X^{d_3} & X^{d_2+d_3} \\ 0 & -X^{d_1} & 0 & X^{d_1+d_3} & 1 & 0 & -X^{d_3} \\ 0 & 0 & -X^{d_1} & X^{d_1+d_2} & 0 & 1 & -X^{d_2} \end{pmatrix} \in \mathbb{Z}[X]^{3\times 7}.$$

Proof. Let $i \in \{1, \ldots, t-1\}$. We use again Theorem 3.2 and diagram (7) to conclude that $\max(S_{3i+1}) = \{S_{3i+2}, S_{3i+3}, \pi S_{3(i-1)+1}\}$, $\max(S_{3i+2}) = \{S_{3(i+1)+1}, \pi S_{3(i-1)+2}\}$ and $\max(S_{3i+3}) = \{S_{3(i+1)+1}, \pi S_{3(i-1)+3}\}$. In particular, we obtain $\mathrm{Rad}(S_{3i+1}) = \mathrm{Rad}(S_{3i+2}) = \mathrm{Rad}(S_{3i+3}) = \pi S_{3i+1}$ and

$$\begin{aligned} \Phi(S_{3i+1}) &= \{S_{3i+1}, S_{3i+2}, S_{3i+3}, S_{3(i+1)+1}, \\ &\qquad \pi S_{3(i-1)+1}, \pi S_{3(i-1)+2}, \pi S_{3(i-1)+3}, \pi S_{3i+1}\}, \\ \Phi(S_{3i+2}) &= \{S_{3(i+1)+1}, \pi S_{3(i-1)+2}, \pi S_{3i+1}\}, \\ \Phi(S_{3i+3}) &= \{S_{3(i+1)+1}, \pi S_{3(i-1)+3}, \pi S_{3i+1}\}. \end{aligned}$$

Thus

$$\Phi(S_{3i+1}, S_j) = \begin{cases} \{S_j\} & \text{if } j \in \{3i+2, 3i+3\}, \\ \{\pi S_j\} & \text{if } j \in \{3(i-1)+1, 3(i-1)+2, 3(i-1)+3\}, \\ \{S_{3i+1}\pi S_{3i+1}\} & \text{if } j = 3i+1, \\ \emptyset & \text{otherwise,} \end{cases}$$

$\Phi(S_{3i+2}, S_j) = \Phi(S_{3i+3}, S_j) = \{S_j\}$ for $j = 3(i+1)+1$, $\Phi(S_{3i+2}, S_j) = \{\pi S_j\}$ for $j \in \{3(i-1)+2, 3i+1\}$ and $\Phi(S_{3i+2}, S_j) = \emptyset$ otherwise. Analogously, $\Phi(S_{3i+3}, S_j) = \{\pi S_j\}$ for $j \in \{3(i-1)+3, 3i+1\}$, and $\Phi(S_{3i+3}, S_j) = \emptyset$ otherwise. $\square$

We have now determined all the rows of $\mathbf{A}_S$, except for the last one.

Lemma 4.13. *The last row of $\mathbf{A}_S$ is equal to $\left[\begin{array}{c|c} \mathbf{0}_{1\times(3t-6)} & \mathbf{A}_3 \end{array}\right]$, where*

$$\mathbf{A}_3 = \begin{pmatrix} 0 & 0 & 0 & -X^{d_1} & 0 & 0 & 1 \end{pmatrix} \in \mathbb{Z}[X]^{1\times 7}.$$

Proof. We have $\max(S_{3t+1}) = \{\pi S_{3(t-1)+1}\}$ and $\mathrm{Rad}(S_{3t+1}) = \pi S_{3(t-1)+1}$. Hence $\Phi(S_{3t+1}) = \{S_{3t+1}, \pi S_{3(t-1)+1}\}$, and it follows that $\Phi(S_{3t+1}, S_j) = \{S_j\}$ for $j = 3t+1$, $\Phi(S_{3t+1}, S_j) = \{\pi S_j\}$ for $j = 3(t-1)+1$, and $\Phi(S_{3t+1}, S_j) = \emptyset$ otherwise. $\square$

To summarize, we have now established

Proposition 4.14. *The matrix $\mathbf{A}_S$ is given as follows:*

$$\left[\begin{array}{c} \begin{array}{|c|c|} \hline \mathbf{A}_1 & \mathbf{0}_{3\times(3t-3)} \\ \hline \end{array} \\ \begin{array}{|c|c|} \mathbf{A}_2 & \mathbf{0}_{3\times(3t-6)} \\ \hline \end{array} \\ \begin{array}{|c|c|c|} \mathbf{0}_{3\times 3} & \mathbf{A}_2 & \mathbf{0}_{3\times(3t-9)} \\ \hline \end{array} \\ \begin{array}{|c|c|c|} \mathbf{0}_{3\times 6} & \mathbf{A}_2 & \mathbf{0}_{3\times(3t-12)} \\ \hline \end{array} \\ \vdots \quad \ddots \quad \ddots \quad \ddots \quad \ddots \quad \vdots \\ \begin{array}{|c|c|c|} \hline \mathbf{0}_{3\times 3t-9} & \mathbf{A}_2 & \mathbf{0}_{3\times 3} \\ \hline \end{array} \\ \begin{array}{|c|c|} \mathbf{0}_{3\times 3t-6} & \mathbf{A}_2 \\ \hline \end{array} \\ \begin{array}{|c|c|} \mathbf{0}_{1\times 3t-6} & \mathbf{A}_3 \\ \hline \end{array} \end{array}\right]$$

4.15. The matrix B. We shall now define a matrix $\mathbf{B} = (B_{ij}) \in \mathbb{Q}(X)^{(3t+1)\times(3t+1)}$, and shall successively prove that this is precisely the matrix $\mathbf{B}_S$. For $i, j \in \{1, \ldots, 3t+1\}$, we write $B_{ij} = b_{ij} \cdot \frac{1}{1-X^d}$, where b_{ij} is given as follows:

(a) If $m \equiv 1 \pmod 3$, then, for $i \geqslant 0$, we set $b_{m+3i+1,m} = X^{(i+1)d_1+d_3}$, $b_{m+3i+2,m} = X^{(i+1)d_1+d_2}$, and $b_{m+3i+3,m} = X^{(i+1)d_1}$.

(b) If $m \equiv 2 \pmod 3$, then, for $i \geqslant 0$, we set $b_{m+3i+1,m} = X^{(i+1)d_1+2d_2}$, $b_{m+3i+2,m} = X^{(i+1)d_1+d_2}$, and $b_{m+3i+3,m} = X^{(i+1)d_1}$.

(c) If $m \equiv 0 \pmod 3$, then, for $i \geqslant 0$, we set $b_{m+3i+1,m} = X^{(i+1)d_1+d_3}$, $b_{m+3i+2,m} = X^{(i+2)d_1+2d_3}$, and $b_{m+3i+3,m} = X^{(i+1)d_1}$.

(d) If $m \equiv 1 \pmod 3$, then, for $i \geqslant 0$, we set $b_{m,m+3i+1} = X^{(i+1)d_2+id_3}$, $b_{m,m+3i+2} = X^{id_2+(i+1)d_3}$, and $b_{m,m+3i+3} = X^{(i+1)(d_2+d_3)}$.

(e) If $m \equiv 2 \pmod 3$, then we set $b_{m,m+1} = X^{d_1+2d_3}$, $b_{m,m+3i+1} = X^{(i-1)d_2+(i+1)d_3}$ for $i \geqslant 1$, $b_{m,m+3i+2} = X^{id_2+(i+1)d_3}$ for $i \geqslant 0$, and $b_{m,m+3i+3} = X^{(i+1)(d_2+d_3)}$ for $i \geqslant 0$.

(f) If $m \equiv 0 \pmod 3$, then, for $i \geqslant 0$, we set $b_{m,m+3i+1} = X^{(i+1)d_2+id_3}$, $b_{m,m+3i+2} = X^{(i+2)d_2+id_3}$, and $b_{m,m+3i+3} = X^{(i+1)(d_2+d_3)}$.

Moreover, we set $b_{ii} := 1$, for $i \geqslant 1$.

We aim to show that $\mathbf{B}$ is the inverse of $\mathbf{A}$. To do this, we partition $(1-X^d)\cdot\mathbf{B}$ into blocks: First we define $\mathbf{B}_1 = (b_{ij})_{1\leqslant i,j\leqslant 7} \in \mathbb{Z}[X]^{7\times 7}$, that is,

$$\mathbf{B}_1 = \begin{pmatrix} 1 & X^{d_2} & X^{d_3} & X^{d_2+d_3} & X^{2d_2+d_3} & X^{d_2+2d_3} & X^{2d_2+2d_3} \\ X^{d_1+d_3} & 1 & X^{d_1+2d_3} & X^{d_3} & X^{d_2+d_3} & X^{2d_3} & X^{d_2+2d_3} \\ X^{d_1+d_2} & X^{d_1+2d_2} & 1 & X^{d_2} & X^{2d_2} & X^{d_2+d_3} & X^{2d_2+d_3} \\ X^{d_1} & X^{d_1+d_2} & X^{d_1+d_3} & 1 & X^{d_2} & X^{d_3} & X^{d_2+d_3} \\ X^{2d_2+d_3} & X^{d_1} & X^{2d_1+2d_3} & X^{d_1+d_3} & 1 & X^{d_1+2d_3} & X^{d_3} \\ X^{2d_1+d_2} & X^{2d_1+2d_2} & X^{d_1} & X^{d_1+d_2} & X^{d_1+2d_2} & 1 & X^{d_2} \\ X^{2d_1} & X^{2d_1+d_2} & X^{2d_1+d_3} & X^{d_1} & X^{d_1+d_2} & X^{d_1+d_3} & 1 \end{pmatrix}.$$

Next we set $\mathbf{B}_3 = (b_{ij})_{(3t+1)-7\leqslant i\leqslant 3t+1, 1\leqslant j\leqslant 3t+1} \in \mathbb{Z}[X]^{7\times(3t+1-7)}$ and $\mathbf{B}_2 = (b_{ij})_{1\leqslant i\leqslant 7, 8\leqslant j\leqslant 3t+1} \in \mathbb{Z}[X]^{7\times(3t+1-7)}$. Thus we have

$$\mathbf{B} = \frac{1}{1-X^d}\left(\begin{array}{c|c} \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \multicolumn{2}{c}{*} \\ \hline \mathbf{B}_3 & \mathbf{B}_1 \end{array}\right).$$

Let $D = (1-X^d)$.

Lemma 4.16. *With the above notation, the following holds:*

(a) *For $l \in \{1,\dots,t-2\}$, one has $\mathbf{B}_1 = (b_{ij})_{\substack{3l+1\leqslant i\leqslant 3l+7,\\ 3l+1\leqslant j\leqslant 3l+7}}$.*

(b) *For $l \in \{1,\dots,t-2\}$, there exists some $c_l' \in \mathbb{Q}(X)$ such that*

$$(b_{ij})_{\substack{3l+1\leqslant i\leqslant 3l+7,\\ 1\leqslant j\leqslant 3l}} = c_l' \cdot ((\mathbf{B}_3)_{ij})_{\substack{1\leqslant i\leqslant 7,\\ 1\leqslant j\leqslant 3l}}.$$

(c) *For $l \in \{1,\dots,t-2\}$, there exists some $c_l'' \in \mathbb{Q}(X)$ such that*

$$(b_{ij})_{\substack{3l+1\leqslant i\leqslant 3l+7,\\ 3l+8\leqslant j\leqslant 3t+1}} = c_l'' \cdot ((\mathbf{B}_2)_{ij})_{\substack{1\leqslant i\leqslant 7,\\ 3l+8\leqslant j\leqslant 3t+1}}.$$

Proof. Let $l \in \{1,\dots,t-2\}$, and let $i,j \in \{1,\dots,7\}$. Then 4.15 immediately gives $b_{i+3l,j+3l} = b_{ij}$, proving (a).

Next let $m \in \{1,\dots,3t-2\}$ and $r \in \{1,\dots,m\}$. Then, by 4.15(a)–(c), we have $b_{m,r} = X^{-d_1} \cdot b_{m+3,r}$. This, in particular, implies (b).

If $r \in \{m+4,\dots,3t-1\}$ and $r > m+4$ in the case where $m \equiv 2 \pmod 3$, then 4.15(d)–(f) gives $b_{m,r} = X^{d_2+d_3} \cdot b_{m+3,r}$, which implies (c). □

Lemma 4.17. *With the above notation, one has the following:*

(a) $\mathbf{A}_2 \cdot \mathbf{B}_1 = [\ \mathbf{0}_{3\times 3} \mid D\cdot \mathbf{1}_3 \mid \mathbf{0}_{3\times 1}\ ]$,
(b) $\mathbf{A}_2 \cdot \mathbf{B}_2 = \mathbf{0}$,
(c) $\mathbf{A}_2 \cdot \mathbf{B}_3 = \mathbf{0}$,
(d) $[\ \mathbf{A}_1 \mid \mathbf{0}_{3\times 3}\ ] \cdot \mathbf{B}_1 = [\ D\cdot \mathbf{1}_3 \mid \mathbf{0}_{3\times 4}\ ]$,
(e) $[\ \mathbf{0}_{1\times(3t-6)} \mid \mathbf{A}_3\ ] \cdot \mathbf{B} = [\ \mathbf{0}_{1\times 3t} \mid D\cdot \mathbf{1}_1\ ]$,
(f) $[\ \mathbf{A}_1 \mid \mathbf{0}_{3\times 3}\ ] \cdot \mathbf{B}_2 = \mathbf{0}$.

Proof. To prove (a) and (d), we compute

$$\left(\begin{array}{c|c} \mathbf{A}_1 & \mathbf{0} \\ \hline \multicolumn{2}{c}{\mathbf{A}_2} \end{array}\right) \cdot \mathbf{B}_1 = [\ D\cdot \mathbf{1}_6 \mid \mathbf{0}_{6\times 1}\ ].$$

Next we show (b) and (f): We claim that every column of $\mathbf{B}_2$ is of the form

$$a \cdot (1, X^{-d_2}, X^{-d_3}, X^{-d_2-d_3}, X^{-2d_2-d_3}, X^{-d_2-2d_3}, X^{-2d_2-2d_3})^t\,, \tag{10}$$

for some $a \in \mathbb{Q}(X)$. Once we have this, we immediately get

$$\left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{0}_{3\times 3} \\ \hline \multicolumn{2}{c}{\mathbf{A}_2} \end{array}\right] \cdot \mathbf{B}_2 = 0\,.$$

Obviously, (10) is true for the first three columns of $\mathbf{B}_2$. So suppose that $m \in \{1,\dots,7\}$ and $r \in \{8,\dots,3t-2\}$. Then 4.15(d)-(f) shows that $b_{m,r+3} = X^{d_2+d_3} \cdot b_{m,r}$, whence (10).

To prove (c), we first claim that, for $i \in \{1,\dots,3t-6\}$, the ith column of $\mathbf{B}_3$ has the form

$$\begin{aligned} a_i(X^{-2d_1}, X^{d_3-d_1}, X^{d_2-d_1}, X^{-d_1}, X^{d_3}, X^{d_2}, 1)^t &\quad \text{if } i \equiv 1 \pmod 3,\\ a_i(X^{-2d_1}, X^{-d_2-2d_1}, X^{d_2-d_1}, X^{-d_1}, X^{-d_2-d_1}, X^{d_2}, 1)^t &\quad \text{if } i \equiv 2 \pmod 3,\\ a_i(X^{-2d_1}, X^{-d_1+d_3}, X^{-2d_1-d_3}, X^{-d_1}, X^{d_3}, X^{-d_1-d_3}, 1)^t &\quad \text{if } i \equiv 0 \pmod 3, \end{aligned}$$

for suitable $a_i \in \mathbb{Q}(X)$. Once we have this, we immediately get $\mathbf{A}_2 \cdot \mathbf{B}_3 = \mathbf{0}$. The claim is easily verified for the first three columns of $\mathbf{B}_3$. If $m \in \{3t-5, \dots, 3t+1\}$ and $r \in \{1, \dots, 3t-9\}$, then 4.15(a)–(c) gives $b_{m,r} = X^{d_1} \cdot b_{m,r+3}$. Thus the claim follows by induction.

It remains to prove (e). To do so, it is sufficient to compute

$$\mathbf{A}_3 \cdot \left[\begin{array}{c|c} \mathbf{B}_3 & \mathbf{B}_1 \end{array}\right].$$

Denoting by $\mathbf{B}_{3,i}$ the ith column of $\mathbf{B}_3$, we have

$$\mathbf{A}_3 \cdot \mathbf{B}_{3,i} = (1 - X^d(a_i - a_i)) = 0\,.$$

On the other hand, a quick calculation shows

$$\mathbf{A}_3 \cdot \mathbf{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1-X^d \end{pmatrix},$$

which finishes the proof. □

We now obtain

Theorem 4.18. *The matrix* $\mathbf{B}$ *defined in 4.15 is the inverse of* $\mathbf{A}_S$*, that is,* $\mathbf{B} = \mathbf{B}_S$.

Proof. We show that $\mathbf{A}_S \cdot \mathbf{B} = \mathbf{1}_{3t+1}$. To this end, we multiply $\mathbf{B}$ with the rows of $\mathbf{A}$ successively from the left. First note that, by Lemma 4.17, we have

$$\left[\begin{array}{c|c|c} \mathbf{A}_1 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3t-6} \\ \hline \multicolumn{2}{c|}{\mathbf{A}_2} & \mathbf{0}_{3\times 3t-6} \end{array}\right] \cdot \mathbf{B} = \left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{0}_{3\times 3} \\ \hline \multicolumn{2}{c}{\mathbf{A}_2} \end{array}\right] \cdot [\ \mathbf{B}_1 \mid \mathbf{B}_2\ ] = [\ \mathbf{1}_6 \mid \mathbf{0}_{6\times 3t-5}\ ].$$

Next consider

$$[\ \mathbf{0}_{3\times 3} \mid \mathbf{A}_2 \mid \mathbf{0}_{3t-9\times 3t-9}\ ] \cdot \mathbf{B} = \mathbf{A}_2 \cdot (B_{ij})_{4\leqslant i\leqslant 10, 1\leqslant j\leqslant 3t+1}\,.$$

Let us write $(B_{ij})_{4\leqslant i\leqslant 10,1\leqslant j\leqslant 3t+1} = [\,\mathbf{B}' \mid \mathbf{B}_1 \mid \mathbf{B}''\,]$ with $\mathbf{B}' = (B_{ij})_{4\leqslant i\leqslant 10,1\leqslant j\leqslant 3}$ and $\mathbf{B}'' = (B_{ij})_{4\leqslant i\leqslant 10,11\leqslant j\leqslant 3t+1}$. By Lemma 4.16, we know $\mathbf{B}' = c'((\mathbf{B}_3)_{ij})_{1\leqslant i\leqslant 7,1\leqslant j\leqslant 3}$ and $\mathbf{B}'' = c''((\mathbf{B}_2)_{ij})_{1\leqslant i\leqslant 7,11\leqslant j\leqslant 3t+1}$, for some $c', c'' \in \mathbb{Q}(X)$. Together with Lemma 4.17 this shows that

$$\begin{aligned}[\ \mathbf{0}_{3\times 3} \mid \mathbf{A}_2 \mid \mathbf{0}_{3\times 3t-9}\ ] \cdot \mathbf{B} = \mathbf{A}_2 \cdot [\ \mathbf{B}' \mid \mathbf{B}_1 \mid \mathbf{B}''\ ] &= [\mathbf{A}_2 \cdot \mathbf{B}' \mid \mathbf{A}_2 \cdot \mathbf{B}_1 \mid \mathbf{A}_2 \cdot \mathbf{B}''\ ] \\ &= [\ \mathbf{0}_{3\times 6} \mid \mathbf{1}_3 \mid \mathbf{0}_{3\times 3t-8}\ ]\,.\end{aligned}$$

We now do the previous step for an arbitrary slice of $\mathbf{A}_S$. Let $1 \leqslant l \leqslant t-1$ and consider

$$(A_{ij})_{\substack{3l+1\leqslant i\leqslant 3(l+1),\\ 1\leqslant j\leqslant 3t+1}} \cdot \mathbf{B} = [\ \mathbf{0}_{3\times 3(l-1)} \mid \mathbf{A}_2 \mid \mathbf{0}_{3\times 3(t-l-1)}\ ] \cdot \mathbf{B}$$
$$= \mathbf{A}_2 \cdot (B_{ij})_{\substack{3(l-1)+1\leqslant i\leqslant 3(l-1)+7,\\ 1\leqslant j\leqslant 3t+1}}.$$

We write

$$\mathbf{B}' = (B_{ij})_{3(l-1)+1\leqslant i\leqslant 3(l-1)+7, 1\leqslant j\leqslant 3(l-1)},$$
$$\mathbf{B}'' = (B_{ij})_{3(l-1)+1\leqslant i\leqslant 3(l-1)+7, 3(l-1)+8\leqslant j\leqslant 3t+1},$$

and note that $\mathbf{B}_1 = (B_{ij})_{3(l-1)+1\leqslant i\leqslant 3(l-1)+7, 3(l-1)+1\leqslant j\leqslant 3(l-1)+7}$. By Lemma 4.16, we know $\mathbf{B}' = c'((\mathbf{B}_3)_{ij})_{1\leqslant i\leqslant 7, 1\leqslant j\leqslant 3(l-1)}$ and $\mathbf{B}'' = c''((\mathbf{B}_2)_{ij})_{1\leqslant i\leqslant 7, 3(l-1)+8\leqslant j\leqslant 3t+1}$, for some $c', c'' \in \mathbb{Q}(X)$. Thus, using Lemma 4.17,

$$(A_{ij})_{\substack{3l+1\leqslant i\leqslant 3(l+1),\\ 1\leqslant j\leqslant 3t+1}} \cdot \mathbf{B} = [\ \mathbf{A}_2 \cdot \mathbf{B}' \mid \mathbf{A}_2 \cdot \mathbf{B}_1 \mid \mathbf{A}_2 \cdot \mathbf{B}''\ ] \cdot \mathbf{B}$$
$$= [\ \mathbf{0}_{3\times 3(l-1)} \mid \mathbf{1}_3 \mid \mathbf{0}_{3\times 3t-3(l-1)-3}\ ].$$

Finally note that $[\ \mathbf{0}_{1\times 3t-6} \mid \mathbf{A}_3\ ] \cdot \mathbf{B} = [\ \mathbf{0}_{1\times 3t} \mid D \cdot \mathbf{1}_1\ ]$ by Lemma 4.17 $\square$

As an immediate consequence of Theorem 4.18 and 4.2, we have

Corollary 4.19. *With the above notation, one has*

$$Z(S) = \frac{1}{1-X^d}\left(1 + \sum_{i=1}^{t} X^{id_2+id_3} + X^{(i-1)d_2+id_3} + X^{id_2+(i-1)d_3}\right).$$

Remark 4.20. *Of course, given $\mathbf{B}_S$, we can also read off $Z(S_2), \ldots, Z(S_{3t+1})$. Since the formulae become more complicated and we shall not need them for our applications below, we do, however, not write them down explicitly here.*

4.5. Application I: Specht lattices labelled by hook partitions. We want to use the previous results to determine the zeta functions of certain lattices over p-adic group algebras, the first of which will come from the symmetric group $\mathfrak{S}_n$ of degree $n \geqslant 4$. In the language of 3.4, our aim is to determine the zeta functions of the Specht lattices associated to the hook partitions $(n-r, 1^r)$, for $r \in \{0, \ldots, n-1\}$. For $r = 1$, the zeta functions of $S_{\mathbb{Z}}^{(2,1^{n-2})}$ and $S_{\mathbb{Z}_p}^{(2,1^{n-2})}$, where p is any prime, have been determined by the second author in [6]. Here we shall investigate $S_{\mathbb{Z}_p}^{(n-r,1^r)}$, for all $r \in \{1, \ldots, n-2\}$ and odd primes p, as well as $S_{\mathbb{Z}_2}^{(n-2,1^2)}$ and $S_{\mathbb{Z}}^{(n-2,1^2)}$. We begin with the local zeta functions at the odd primes p. Recall from 3.4 and 3.5 that $S_{\mathbb{Z}_p}^{(n-r,1^r)} \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} S_{\mathbb{Z}}^{(n-r,1^r)}$ and that $\mathrm{rk}_{\mathbb{Z}_p}(S_{\mathbb{Z}_p}^{(n-r,1^r)}) = \mathrm{rk}_{\mathbb{Z}}(S_{\mathbb{Z}}^{(n-r,1^r)}) = \binom{n-1}{r}$, for $r \in \{0, \ldots, n-1\}$.

Proposition 4.21. *Let $p \geqslant 3$ be a prime number, let $r \in \{1, \ldots, n-2\}$, and let $d := \binom{n-1}{r}$.*

(a) *If $p \nmid n$, then*

$$\zeta_{\mathbb{Z}_p\mathfrak{S}_n}(S_{\mathbb{Z}_p}^{(n-r,1^r)}, s) = \frac{1}{1-p^{-ds}}\,.$$

(b) *If $p \mid n$, then*

$$\zeta_{\mathbb{Z}_p\mathfrak{S}_n}(S_{\mathbb{Z}_p}^{(n-r,1^r)}, s) = \frac{1}{1-p^{-ds}} \sum_{i=0}^{\nu_p(n)} p^{-sia}\,,$$

where $a = \binom{n-2}{r}$.

Proof. (a) Since $p \geqslant 3$ is not dividing n, by [8, Theorem 23.7], the reduction of $S_{\mathbb{Z}_p}^{(n-r,1^r)}$ modulo p is absolutely simple. So the assertion follows from 4.3.

(b) By [5, Section 6] we know that there are $\nu_p(n)+1$ isomorphism classes of $\mathbb{Z}_p$-forms in $S_{\mathbb{Q}_p}^{(n-r,1^r)}$. Moreover the reduction of $S_{\mathbb{Z}_p}^{(n-r,1^r)}$ modulo p has precisely two non-isomorphic composition factors, and is indecomposable. By [8, Theorem 24.1], the head of $S_{\mathbb{Z}_p}^{(n-r,1^r)}/pS_{\mathbb{Z}_p}^{(n-r,1^r)}$ has $\mathbb{F}_p$-dimension $d - \binom{n-2}{r-1} = \binom{n-2}{r} =: a$. Thus, by Corollary 4.10, it follows that

$$Z(S_{\mathbb{Z}_p}^{(n-r,1^r)}) = \frac{1}{1-X^d} \sum_{i=0}^{\nu_p(n)} X^{ia}.$$

Now the assertion follows from 4.2. $\square$

Remark 4.22. *For $r = 2$, the next proposition yields the local zeta functions for $p = 2$. This will involve the dimension of the simple $\mathbb{F}_2\mathfrak{S}_n$-module $D_{\mathbb{F}_2}^{(n-2,2)}$, for $n > 4$, which is well known, by* [8, Theorem 24.15]*. For $n > 4$, one has*

$$\dim_{\mathbb{F}_2}(D_{\mathbb{F}_2}^{(n-2,2)}) = \begin{cases} \frac{1}{2}(n^2-5n+4) & \text{if } n \equiv 0 \pmod 4\,, \\ \frac{1}{2}(n^2-3n-2) & \text{if } n \equiv 1 \pmod 4\,, \\ \frac{1}{2}(n^2-5n+2) & \text{if } n \equiv 2 \pmod 4\,, \\ \frac{1}{2}(n^2-3n) & \text{if } n \equiv 3 \pmod 4\,. \end{cases}$$

This can also be read off from [5, Proposition 7.2]*, recalling that* $\dim_{\mathbb{F}_2}(D_{\mathbb{F}_2}^{(n-1,1)}) = n-1$ *if* $2 \nmid n$*, and* $\dim_{\mathbb{F}_2}(D_{\mathbb{F}_2}^{(n-1,1)}) = n-2$ *if* $2 \mid n$*.*

Proposition 4.23. *Let $d := \binom{n-1}{2}$, and let $d_3 := \dim_{\mathbb{F}_2}(D_{\mathbb{F}_2}^{(n-2,2)})$ if $n > 4$. Then the following hold:*

(a) *If $n \equiv 1 \pmod 4$, then*

$$\zeta_{\mathbb{Z}_2\mathfrak{S}_n}(S_{\mathbb{Z}_2}^{(n-2,1^2)}, s) = \frac{1}{1-2^{-sd}}(1 + 2^{-s} + 2^{-s(d_3+1)})\,.$$

(b) *If* $n \equiv 2 \pmod 4$, *then*

$$\zeta_{\mathbb{Z}_2\mathfrak{S}_n}(S_{\mathbb{Z}_2}^{(n-2,1^2)}, s) = \frac{1}{1-2^{-sd}}(1 + 2^{-s} + 2^{-s(d_3+1)} + 2^{-s(d_3+2)})\,.$$

(c) *If* $n \equiv 3 \pmod 4$, *then*

$$\zeta_{\mathbb{Z}_2\mathfrak{S}_n}(S_{\mathbb{Z}_2}^{(n-2,1^2)}, s) = \frac{1}{1-2^{-sd}}(1 + 2^{-s} + 2^{-sd_3})\,.$$

(d) *If* $n \equiv 0 \pmod 4$ *and* $n > 4$, *then*

$$\zeta_{\mathbb{Z}_2\mathfrak{S}_n}(S_{\mathbb{Z}_2}^{(n-2,1^2)}, s) = \frac{1}{1-2^{-sd}}\left(1 + \sum_{i=1}^{\nu_2(n)} 2^{-is-isd_3} + 2^{-s(i-1)-sid_3} + 2^{-si-s(i-1)d_3}\right).$$

(e) *If* $n = 4$, *then*

$$\zeta_{\mathbb{Z}_2\mathfrak{S}_4}(S_{\mathbb{Z}_2}^{(2,1^2)}, s) = \frac{1}{1-2^{-sd}}(1 + 2^{-s} + 2^{-2s})\,.$$

Proof. We set $M_1 = S_{\mathbb{Z}_2}^{(n-2,1^2)}$. We shall determine $Z(M_1)$, and then apply 4.2.

(a) Recall that, by [5, Section 7], the (partial) submodule lattice of M_1 is given by

$$2M_1 \subseteq M_3 \subseteq M_2 \subseteq M_1\,,$$

with $M_1/M_2 \cong \mathbb{F}_2$, $M_2/M_3 \cong D_{\mathbb{F}_2}^{(n-2,2)}$ and $M_3/2M_1 \cong \mathbb{F}_2$. Here $\mathbb{F}_2$ denotes the trivial $\mathbb{F}_2 S_n$-module. Moreover, M_1, M_2, M_3 are representatives of the isomorphism classes of $\mathbb{Z}_2$-forms of $S_{\mathbb{Q}_2}^{(n-1,1^2)}$. By [5, proof of Theorem 7.10], the reduction modulo 2 of every $\mathbb{Z}_2$-form of $S_{\mathbb{Q}_2}^{(n-2,1^2)}$ is a uniserial $\mathbb{F}_2\mathfrak{S}_n$-module. So, by [5, Proposition 3.7], we can apply Corollary 4.6, which shows that

$$Z(M_1) = \frac{1}{1-X^d}(1 + X + X^{d_3+1})\,.$$

(b) By [5, Section 7], the (partial) submodule lattice of M_1 is give by

$$2M_1 \subseteq M_4 \subseteq M_3 \subseteq M_2 \subseteq M_1\,,$$

with $M_1/M_2 \cong M_3/M_4 \cong \mathbb{F}_2$, $M_2/M_3 \cong D_{\mathbb{F}_2}^{(n-2,2)}$ and $M_4/2M_1 \cong D_{\mathbb{F}_2}^{(n-1,1)}$. Moreover, $M_1, \ldots, M_4$ are representatives of the isomorphism classes of $\mathbb{Z}_2$-forms of $S_{\mathbb{Q}_2}^{(n-1,1^2)}$ In [5, proof of Theorem 7.16] it is also shown that the reduction modulo 2 of every $\mathbb{Z}_2$-form of $S_{\mathbb{Q}_2}^{(n-2,1^2)}$ is a uniserial $\mathbb{F}_2\mathfrak{S}_n$-module. Thus, by [5, Proposition 3.7] and Corollary 4.6, we have

$$Z(M_1) = \frac{1}{1-X^d}(1 + X + X^{d_3+1} + X^{d_3+2}).$$

(c) By [5, Proposition 7.9, Theorem 7.10], there are precisely three isomorphism classes of $\mathbb{Z}_2$-forms of $S_{\mathbb{Q}_2}^{(n-2,1^2)}$. There are representatives M_1, M_2, M_3 of these isomorphism classes with the following properties: $M_1 = S_{\mathbb{Z}_2}^{(n-2,1^2)}$, M_2 and M_3 are maximal in M_1, $M_2 \cap M_3 = 2M_1$, $M_1/M_2 \cong M_3/2M_1 \cong \mathbb{F}_2$, $M_1/M_3 \cong M_2/2M_1 \cong D_{\mathbb{F}_2}^{(n-2,2)}$. Both $M_2/2M_2$ and $M_3/2M_3$ are indecomposable, with head isomorphic to $D_{\mathbb{F}_2}^{(n-2,2)}$ and $\mathbb{F}_2$, respectively, Thus, we can apply Corollary 4.10 to M_2 and the chain of sublattices $2M_3 \subseteq 2M_1 \subseteq M_2$. With respect to this ordering, the second row of the matrix $\mathbf{B}_{M_2}$ gives

$$Z(M_1) = \frac{1}{1 - X^d}(1 + X + X^{d_3}).$$

(d) This follows from Corollary 4.19 together with Lemma 3.6.

(e) Lastly, let $n = 4$ and let $M := S_{\mathbb{Z}_2}^{(2,1^2)}$. Then, by [8, Theorem 8.15], $M/2M \cong (S_{\mathbb{F}_2}^{(3,1)})^*$ is indecomposable, with two composition factors, socle isomorphic to $D_{\mathbb{F}_2}^{(3,1)}$ and trivial head. So the assertion follows from Corollary 4.10. □

Theorem 4.24. *Let $n \geqslant 4$, and let $S = S_{\mathbb{Z}}^{(n-2,1^2)}$ be the Specht $\mathbb{Z}\mathfrak{S}_n$-lattice labelled by the hook partition $(n-2, 1^2)$. Set $d = (n-1)(n-2)/2$, and $d_3 = \dim_{\mathbb{F}_2}(D_{\mathbb{F}_2}^{(n-2,2)})$ if $n > 4$. Then one has*

$$\zeta_{\mathbb{Z}\mathfrak{S}_n}(S, s) = \zeta_{\mathbb{Q}}(ds) \prod_{p|n} \varphi_p(p^{-s})\,,$$

where $\zeta_{\mathbb{Q}}$ is the Riemann zeta function, $\varphi_p(X) = \sum_{i=0}^{\nu_p(n)} X^i \in \mathbb{Z}[X]$ if $p \geqslant 3$, and

$$\varphi_2(X) = \begin{cases} 1 + X + X^{d_3+1} & \text{if } n \equiv 1 \pmod 4, \\ 1 + X + X^{d_3+1} + X^{d_3+2} & \text{if } n \equiv 2 \pmod 4, \\ 1 + X + X^{d_3} & \text{if } n \equiv 3 \pmod 4, \\ 1 + \sum_{i=1}^{\nu_2(n)} X^{i+id_3} + X^{(i-1)+id_3} + X^{i+(i-1)d_3} & \text{if } n \equiv 0 \pmod 4, \\ & \quad \text{and } n > 4, \\ 1 + X + X^2 & \text{if } n = 4\,. \end{cases}$$

Proof. Since $V = S_{\mathbb{Q}}^{(n-2,1^2)}$ is an absolutely simple $\mathbb{Q}G$-module of dimension d, we have $\zeta_V(s) = \zeta_{\mathbb{Q}}(ds)$, as mentioned in 4.3. The claim now follows from 4.3, Proposition 4.21 and Proposition 4.23. □

4.6. Application II: projective special linear groups. For the rest of this subsection, let q be a prime power, and let $H = \mathrm{PSL}_2(q)$. If R is a principal ideal domain, we denote by M_R the permutation RH-lattice associated to the action of H on the one-dimensional subspaces of $\mathbb{F}_q^2$, as defined in 3.13. Let $L_R \subseteq M_R$ be the RH-lattice investigated in 3.10 and 3.13. Our aim is to determine the zeta function

$\zeta_{\mathbb{Z}H}(L_{\mathbb{Z}}, s)$. To this end we first compute the local zeta functions $\zeta_{\mathbb{Z}_pH}(L_{\mathbb{Z}_p}, s)$, for all primes p.

If $R = \mathbb{Z}$ and $K = \mathbb{Q}$, or $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$, for some prime p, then we denote by V_K the absolutely simple KH-module with R-form L_R as defined in 3.13(b).

Lemma 4.25. *Let p be a prime.*

(a) *If $p \mid q$, or if p is odd with $p \nmid |H|$ or $p \mid (q-1)$, then*

$$\zeta_{\mathbb{Z}_pH}(L_{\mathbb{Z}_p}, s) = \frac{1}{1-p^{-qs}}.$$

(b) *If p is odd and $p \mid (q+1)$, then*

$$\zeta_{\mathbb{Z}_pH}(L_{\mathbb{Z}_p}, s) = \frac{1}{1-p^{-qs}} \sum_{i=0}^{v_p(q+1)} p^{-i(q-1)s}.$$

Proof. (a) We have already seen in Remark 3.14 that in all three cases $L_{\mathbb{Z}_p}$ is up to isomorphism the unique $\mathbb{Z}_p$-form of $V_{\mathbb{Q}_p}$. Thus the claim follows from 4.2.

(b) By Remark 3.14, we know that $L_{\mathbb{Z}_p}/pL_{\mathbb{Z}_p}$ has two non-isomorphic composition factors, is indecomposable and has a trivial submodule. Thus, by 3.10 and Corollary 4.10, it follows that

$$Z(L_{\mathbb{Z}_p}) = \frac{1}{1-X^q} \sum_{i=0}^{v_p(q+1)} X^{i(q-1)}.$$

Now the assertion follows from 4.2. ☐

Lemma 4.26. (a) *If $q \equiv \pm 3 \bmod 8$, then*

$$\zeta_{\mathbb{Z}_2H}(L_{\mathbb{Z}_2}, s) = \frac{1}{1-2^{-sq}} \sum_{i=0}^{v_2(q+1)} 2^{-si(q-1)}.$$

(b) *If $q \equiv \pm 1 \bmod 8$, then*

$$\zeta_{\mathbb{Z}_2H}(L_{\mathbb{Z}_2}, s) = \frac{1}{1-2^{-sq}} \left(1 + \sum_{i=1}^{\nu_2(q+1)} 2^{-si(q-1)} + 2^{-s(2i-1)(q-1)/2+1} \right).$$

Proof. (a) This follows from Remark 3.14 (b) and Corollary 4.10.

(b) Lemma 3.17 and Proposition 3.18 show that $L_{\mathbb{Z}_2}/2L_{\mathbb{Z}_2}$ has three composition factors of dimensions $d_1 = 1$ and $d_2 = d_3 = (q-1)/2$, respectively. Moreover, Corollary 4.19 applies, with $t = \nu_2(q+1)$. So we get

$$Z(L_{\mathbb{Z}_2}) = \frac{1}{1-X^q} \left(1 + \sum_{i=1}^{\nu_2(q+1)} X^{i(q-1)} + 2X^{(2i-1)(q-1)/2} \right).$$

The assertion of the lemma now follows from 4.2. ☐

Theorem 4.27. *The zeta function of the $\mathbb{Z}H$-lattice $L_{\mathbb{Z}}$ is given as*

$$\zeta_{\mathbb{Z}H}(L_{\mathbb{Z}}, s) = \zeta_{\mathbb{Q}}(qs)\varphi_2(q^{-s}) \prod_{\substack{p\geqslant 3\\ p\mid q+1}} \varphi_p(q^{-s})\,,$$

where $\zeta_{\mathbb{Q}}$ is the Riemann zeta function, $\varphi_p(X) = \sum_{i=0}^{\nu_p(q+1)} X^{i(q-1)}$ if $p \geqslant 3$ and $p \mid q+1$, and

$$\varphi_2(X) = \begin{cases} \sum_{i=0}^{\nu_2(q+1)} X^{i(q-1)} & \text{if } q \equiv \pm 3 \pmod 8, \\ 1 + \sum_{i=1}^{\nu_2(q+1)} (X^{q-1} + 2X^{(2i-1)(q-1)/2}) & \text{if } q \equiv \pm 1 \pmod 8. \end{cases}$$

Proof. First note that, since $V_{\mathbb{Q}}$ is an absolutely simple $\mathbb{Q}H$-module of dimension q, we have $\zeta_V(s) = \zeta_{\mathbb{Q}}(qs)$, as mentioned in 4.3. The assertion of the theorem follows from Lemma 4.25 and Lemma 4.26 together with 4.2 and 4.3. □

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