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STAR OPERATIONS ON KUNZ DOMAINS

Dario Spirito

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Abstract. We study star operations on Kunz domains, a class of analytically irreducible, residually rational domains associated to pseudo-symmetric numerical semigroups, and we use them to refute a conjecture of Houston, Mimouni and Park. We also find an estimate for the number of star operations in a particular case, and a precise counting in a sub-case.

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1. Introduction

Let D be an integral domain with quotient field K, and let $\mathcal{F}(D)$ be the set of *fractional ideals* of D, i.e., the set of D-submodules I of K such that $xI \subseteq D$ for some $x \in K \setminus \{0\}$.

A *star operation* on D is a map $\star : \mathcal{F}(D) \longrightarrow \mathcal{F}(D)$, $I \mapsto I^\star$, such that, for every $I, J \in \mathcal{F}(D)$ and every $x \in K$:

- $I \subseteq I^\star$;
- if $I \subseteq J$, then $I^\star \subseteq J^\star$;
- $(I^\star)^\star = I^\star$;
- $x \cdot I^\star = (xI)^\star$;
- $D = D^\star$.

A fractional ideal I is $\star$-*closed* if $I = I^\star$.

The easiest example of a non-trivial star operation is the v-*operation* $v : I \mapsto (D : (D : I))$, where if $I, J \in \mathcal{F}(D)$ we define $(I : J) := \{x \in K \mid xJ \subseteq I\}$. An ideal that is v-closed is said to be *divisorial*; if I is divisorial and $\star$ is any other star operation then $I = I^\star$. We denote by d the identity, which is obviously a star operation.

Recently, the cardinality of the set $\mathrm{Star}(D)$ of the star operations on D has been studied, especially in the case of Noetherian [4,8] and Prüfer domains [3,5]. In particular, Houston, Mimouni and Park started studying the relationship between the cardinality of $\mathrm{Star}(D)$ and the cardinality of $\mathrm{Star}(T)$, where T is an overring of D (an *overring* of D is a ring comprised between D and K) [6,7]: they called a domain *star regular* if $|\mathrm{Star}(D)| \geq |\mathrm{Star}(T)|$ for every overring of T. While even simple domains may fail to be star regular (for example, there are domains with just one star operation having an overring with infinitely many star operations [6, Example 1.3]), they conjectured that every one-dimensional local Noetherian domain D such that $1 < |\mathrm{Star}(D)| < \infty$ is star regular, and proved it when the residue field of D is infinite [6, Corollary 1.18].

In this context, a rich source of examples are *semigroup rings*, that is, subrings of the power series ring $K[[X]]$ (where K is a field, usually finite) of the form $K[[S]] := K[[X^S]] := \{\sum_i a_i X^i \mid a_i = 0 \text{ for all } i \notin S\}$, where S is a numerical semigroup (i.e., a submonoid $S \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite). Star operations can also be defined on numerical semigroups [14], and there is a link between star operations on S and star operations on $K[[S]]$: for example, every star operation on S induces a star operation on $K[[S]]$, and $|\mathrm{Star}(S)| = 1$ if and only if $|\mathrm{Star}(K[[S]])| = 1$ [14, Theorem 5.3], with the latter result corresponding to the equivalence between S being symmetric and $K[[S]]$ being Gorenstein [2,10]. A detailed study of star operations on some numerical semigroup rings was carried out in [15].

In this paper, we study star operations on *Kunz domains*, which are, roughly speaking, a generalization of rings in the form $K[[S]]$ where S is a pseudo-symmetric semigroup (see the beginning of the next section for the definitions). We show that, if R is a Kunz domain whose residue field is finite and the length of $\overline{R}/R$ is at least 4 (where $\overline{R}$ is the integral closure of R) then R is a counterexample to Houston-Mimouni-Park's conjecture; that is, R satisfies $1 < |\mathrm{Star}(R)| < \infty$ but there is an overring T of R with more star operations than R. In Section 3, we also study more deeply one specific class of domains, linking the cardinality of $\mathrm{Star}(R)$ with the set of vector subspaces of a vector space over the residue field of R, and calculate the cardinality of $\mathrm{Star}(R)$ when the value semigroup of R is $\langle 4, 5, 7\rangle$.

We refer to [13] for information about numerical semigroups, and to [1] for the passage from numerical semigroups to one-dimensional local domains.

2. Kunz domains

A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ such that $0 \in S$, that is closed by addition and such that $\mathbb{N} \setminus S$ is finite. If S is a numerical semigroup, we let $g := g(S) := \sup(\mathbb{Z} \setminus S)$ be the *genus* of S and $\mu := \mu(S) := \min(S \setminus \{0\})$ be the *multiplicity* of S. If $a_1, \ldots, a_n$ are coprime integers, we denote by $\langle a_1, \ldots, a_n\rangle$ the numerical semigroup generated by $a_1, \ldots, a_n$, i.e., $\langle a_1, \ldots, a_n \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$.

Let (V, M_V) be a discrete valuation ring with associated valuation $\mathbf{v}$. We shall consider local subrings (R, M_R) of V with the following properties:

- R and V have the same quotient field;
- the integral closure of R is V;
- R is Noetherian;
- the conductor ideal $(R : V)$ is nonzero;
- the inclusion $R \hookrightarrow V$ induces an isomorphism of residue fields $R/M_R \longrightarrow V/M_V$.

Equivalently, R is an analytically irreducible, residually rational one-dimensional Noetherian local domain having integral closure V. For every such R, the set $\mathbf{v}(R) := \{\mathbf{v}(r) \mid r \in R\}$ is a numerical semigroup. We state explicitly a property which we will be using many times.

Proposition 2.1 ([12, Corollary to Proposition 1]). *Let R be as above, and let $I \subseteq J$ be R-submodules of the quotient field of R. Then,*

$$\ell_R(J/I) = |\mathbf{v}(J) \setminus \mathbf{v}(I)|,$$

where ℓ_R is the length of an R-module.

We say that a numerical semigroup S is *pseudo-symmetric* if $g = g(S)$ is even and, for every $a \in \mathbb{N}$, $a \neq g/2$, either $a \in S$ or $g - a \in S$. Following [1] (and using the characterization in [1, Proposition II.1.12]), we give the following definition.

Definition 2.2. A ring R satisfying the previous conditions is a *Kunz domain* if $\mathbf{v}(R)$ is a pseudo-symmetric semigroup.

From now on, we suppose that R is a Kunz domain, and we set $g := g(\mathbf{v}(R))$ and $\tau := g/2$. The hypotheses on R guarantee that, if $x \in V$ is such that $\mathbf{v}(x) > g$, then $x \in R$ [10, Theorem, p.749].

Our first stage is constructing an overring T of R which we will use in the counterexample.

Lemma 2.3. *Let $y \in V$ be an element of valuation g, and let $T := R[y]$. Then:*

(a) *T contains all elements of valuation g;*
(b) *$\mathbf{v}(T) = \mathbf{v}(R) \cup \{g\}$;*
(c) *$\ell_R(T/R) = 1$;*
(d) *$T = R + yR$.*

Proof. Let $y' \in V$ be another element of valuation g. Then, $\mathbf{v}(y/y') = 0$, and thus $c := y/y'$ is a unit of V. Hence, there is a $c' \in R$ such that the images of c and c' in the residue field of V coincide; in particular, $c = c' + m$ for some $m \in M_V$. Hence,

$$y' = cy = (c' + m)y = c'y + my.$$

Since $c' \in R$, we have $c'y \in R[y]$; furthermore, $\mathbf{v}(my) = \mathbf{v}(m) + \mathbf{v}(y) > \mathbf{v}(y) = g$, and thus $my \in R$. Hence, $y' \in R[y]$, and thus $R[y]$ contains all elements of valuation g.

The fact that $\mathbf{v}(T) = \mathbf{v}(R) \cup \{g\}$ is trivial; hence, $\ell_R(T/R) = |\mathbf{v}(T) \setminus \mathbf{v}(R)| = 1$. The last point follows from the fact that $R+yR$ is an R-module, from $R \subsetneq R+yR \subseteq T$ and from $\ell_R(T/R) = 1$. □

In particular, the previous proposition shows that T is independent from the element y chosen. From now on, T will always denote this ring.

We denote by $\mathcal{F}_0(R)$ the set of R-fractional ideals I such that $R \subseteq I \subseteq V$. If I is any fractional ideal over R, and $\alpha \in I$ is an element of minimal valuation, then $\alpha^{-1} I \in \mathcal{F}_0(R)$; hence, the action of any star operation is uniquely determined by its action on $\mathcal{F}_0(R)$. Furthermore, $V^\star = V$ for all $\star \in \mathrm{Star}(R)$ (since $(R : (R : V)) = V$) and thus $I^\star \in \mathcal{F}_0(R)$ for all $I \in \mathcal{F}_0(R)$, i.e., $\star$ restricts to a map from $\mathcal{F}_0(R)$ to itself.

To analyze star operations, we want to subdivide them according to whether they close T or not. One case is very simple.

Proposition 2.4. *If $\star \in \mathrm{Star}(R)$ is such that $T \neq T^\star$, then $\star = v$.*

Proof. Suppose $\star \neq v$: then, there is a fractional ideal $I \in \mathcal{F}_0(R)$ that is $\star$-closed but not divisorial. By [1, Lemma II.1.22], $\mathbf{v}(I)$ is not divisorial (in $\mathbf{v}(R)$) and thus by [1, Proposition I.1.16] there is a positive integer $n \in \mathbf{v}(I)$ such that $n+\tau \notin \mathbf{v}(I)$.

Let $x \in I$ be an element of valuation n, and consider the ideal $J := x^{-1} I \cap V$: being the intersection of two $\star$-closed ideals, it is itself $\star$-closed. Since $\mathbf{v}(x) > 0$, every element of valuation g belongs to J; on the other hand, by the choice of n, no element of valuation τ can belong to J.

Consider now the ideal $L := (R : M_R)$: then, L is divisorial (since M_R is divisorial) and, using [1, Proposition II.1.16(1)],

$$\mathbf{v}(L) = (\mathbf{v}(R) - \mathbf{v}(M_R)) = \mathbf{v}(R) \cup \{\tau, g\}.$$

We claim that $T = J \cap L$: indeed, clearly $J \cap L$ contains R, and if y has valuation g then $y \in J \cap L$ by construction; thus $T = R + yR \subseteq J \cap L$. On the other hand, $\mathbf{v}(J \cap L) \subseteq \mathbf{v}(J) \cap \mathbf{v}(L) = \mathbf{v}(R) \cup \{g\}$, and thus $J \cap L \subseteq T$.

Hence, $T = J \cap L$; since J and L are both $\star$-closed, so is T. Therefore, if $T \neq T^\star$ then $\star$ must be the divisorial closure, as claimed. $\square$

Suppose now that $T = T^\star$. Then, $\star$ restricts to a star operation $\star_1 := \star|_{\mathcal{F}(T)}$, and the amount of information we lose in the passage from $\star$ to $\star_1$ depends on the R-fractional ideals that are not ideals over T. We can determine them explicitly.

Recall that the *canonical ideal* of a ring R is a (fractional) ideal ω such that $(\omega : (\omega : I)) = I$ for every fractional ideal I. Not every integral domain has a canonical ideal; however, Kunz domains (or, more generally, Noetherian one-dimensional local domains whose completion is reduced [11, Korollar 2.12], so in particular domains satisfying the five properties at the beginning of this section) have a canonical ideal; furthermore, if S is a Kunz domain, then an ideal $\omega \in \mathcal{F}_0(R)$ is canonical if and only if $\mathbf{v}(\omega) = S \cup \{x \in \mathbb{N} \mid g(S) - x \notin S\} = S \cup \{\tau\}$ [9, Satz 5].

Lemma 2.5. *Let $I \in \mathcal{F}_0(R)$, $I \neq R$. Then, the following are equivalent.*

(i) $\mathbf{v}(I) = \mathbf{v}(R) \cup \{\tau\}$;
(ii) *I does not contain any element of valuation g;*
(iii) $IT \neq I$;
(iv) *I is a canonical ideal of R.*

Furthermore, in this case, there is a unit u of R such that $R \subseteq uI \subseteq (R : M_R)$ and $(uI)^v = (R : M_R)$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Since $R \subseteq I$, there is an element x of I of valuation 0; hence, IT contains an element of valuation g, and thus $IT \neq I$.

(iii) $\Rightarrow$ (i) Suppose there is an $x \in I$ such that $\mathbf{v}(x) \notin \mathbf{v}(R) \cup \{\tau\}$. Since $\mathbf{v}(R)$ is pseudo-symmetric, there is an $y \in R$ such that $\mathbf{v}(y) = g - \mathbf{v}(x)$; hence, I contains an element (explicitly, xy) of valuation g and, by the proof of Lemma 2.3, it follows that it contains every element of valuation g.

Fix now an element $y \in V$ of valuation g. Since $IT \neq I$, there are $i \in I$, $t \in T$ such that $it \notin I$. By Lemma 2.3(d), there are $r, r' \in R$ such that $t = r + yr'$; hence, $it = i(r + yr') = ir + iyr'$. Both ir and iyr' are in I, the former since it belongs to $IR = I$ and the latter because its valuation is at least g. However, this contradicts $it \notin I$; therefore, $\mathbf{v}(I) \subseteq \mathbf{v}(R) \cup \{\tau\}$.

If $\mathbf{v}(I) = \mathbf{v}(R)$, then we must have $I = R$, against our hypothesis; therefore, $\mathbf{v}(I) = \mathbf{v}(R) \cup \{\tau\}$.

(i) $\Leftrightarrow$ (iv) follows from [9, Satz 5] and the fact that R is a Kunz domain.

For the last claim, we first note that $(R : M_R)$ is divisorial (since M_R is divisorial). By [1, Proposition II.1.16], $\mathbf{v}((R : M_R)) = S \cup \{\tau, g\}$; if $x \in (R : M_R)$ has valuation τ, then $I' := R + xR$ is a canonical ideal (since $\mathbf{v}(I') = S \cup \{\tau\}$) and is contained between R and $(R : M_R)$. Since R is local, and I is a canonical ideal too, there is a unit u of R such that $I' = uI$ [11, Satz 2.8(b)]; the claim follows. $\square$

Proposition 2.6. *The map*

$$\begin{aligned}\Psi\colon \mathrm{Star}(R) \setminus \{d, v\} &\longrightarrow \mathrm{Star}(T)\\ \star &\longmapsto \star|_{\mathcal{F}(T)}\end{aligned}$$

is well-defined and injective.

Proof. By Proposition 2.4, if $\star \neq v$ then $T = T^\star$, and thus $\star|_{\mathcal{F}(T)}$ is a star operation on T; hence, Ψ is well-defined. We claim that it is injective: suppose $\star_1 \neq \star_2$. Then, there is an $I \in \mathcal{F}_0(R)$ such that $I^{\star_1} \neq I^{\star_2}$. If I is a T-module then $\Psi(\star_1) \neq \Psi(\star_2)$; suppose I is not a T-module.

By Lemma 2.5, I can only be R or a canonical ideal of R. In the former case, since $\star_1$ and $\star_2$ are star operations, $R^{\star_1} = R = R^{\star_2}$, a contradiction. In the latter case, by multiplying by a unit we can suppose that $I \subseteq (R : M_R)$. Then, $\ell((R : M_R)/I) = 1$, and thus $I^{\star_i}$ can only be I or $(R : M_R)$; suppose now that $I^\star = I$ for some $\star \in \mathrm{Star}(R)$. By definition of the canonical ideal, $J = (I : (I : J))$ for every ideal J; since $(I : L)$ is always $\star$-closed if I is $\star$-closed, it follows that $\star$ must be the identity. Since $\star_1, \star_2 \neq d$, we must have $I^{\star_1} = (R : M_R) = I^{\star_2}$, against the assumptions. Thus, Ψ is injective. $\square$

An immediate corollary of the previous proposition is that $|\mathrm{Star}(R)| \leq |\mathrm{Star}(T)| + 2$. Our counterexample thus involves finding star operations of T that do not belong to the image of Ψ; to do so, we restrict to the case $\ell_R(V/R) \geq 4$ or, equivalently, $|\mathbb{N} \setminus \mathbf{v}(R)| \geq 4$. This excludes exactly two pseudo-symmetric numerical semigroups, namely $\langle 3, 4, 5\rangle$ and $\langle 3, 5, 7\rangle$.

Lemma 2.7. *Let S be a pseudo-symmetric numerical semigroup, let $g := \max(\mathbb{N} \setminus S)$ and let $S' := S \cup \{g\}$. If $|\mathbb{N} \setminus S| \geq 4$, then there are $a, b \in (S' - M_{S'}) \setminus S'$, $a \neq b$, such that $2a, 2b, a + b \in S'$.*

Proof. Let μ be the multiplicity of S. We claim that $a := \tau$ and $b := g - \mu$ are the two elements we are looking for.

Since $a + M_S \subseteq S$ and $a + g > g$ (and so $a + g \in M_S$) we have $a \in (S' - M_{S'})$. Furthermore, since $|\mathbb{N} \setminus S| \geq 4$, we have $g > \mu$, and thus $b + m \geq g$ for all $m \in M_{S'}$.

By the previous point, $a+m, b+m \in S' \cup \{a,b\}$ for every $m \in M_{S'}$. Since $a = \tau$, we have $2a = g \in S'$.

If $g > 2\mu$, then $a > \mu$, and so $a + b \geq g$, which implies $a + b \in S'$; moreover, also $b > \mu$, and thus $2b = b + b \geq g - \mu + \mu = g$, so that $2b \in S'$.

If $g < 2\mu$, then g must be equal to $2\mu - 2$ or to $\mu - 1$; the latter case is impossible since $|\mathbb{N} \setminus S| \geq 4$. Hence, $b = \mu - 2$ and $a = \mu - 1$. Then, $2b = 2\mu - 4$ and $a+b = 2\mu - 3$; again since $|\mathbb{N} \setminus S| \geq 4$, we must have $\mu > 3$, and thus $2b > a + b \geq \mu$. Furthermore, in this case $S' = \{0, \mu, \ldots\}$, and so $a + b, 2b \in S'$, as claimed. □

Proposition 2.8. *Let K be the residue field of R, and suppose that $\ell_R(V/R) \geq 4$. There are at least $|K| + 1$ star operations on T that do not close $(R : M_R)$.*

Proof. We first note that $(R : M_R)$ is a T-module. Indeed, let $x \in (R : M_R)$ and $t \in T$: then, $t = r + ay$, with $r, a \in R$ and $\mathbf{v}(y) = g$, and so $xt = xr + axy$. Both xr and axy belong to $(R : M_R)$, the former because $(R : M_R)$ is an R-module and the latter since its valuation is at least g: hence, $xt \in (R : M_R)$. Thus, it makes sense to ask if a star operation on T closes $(R : M_R)$.

Furthermore, $T \subsetneq (R : M_R) \subseteq (T : M_T)$: the first containment follows from the previous reasoning (and the fact that $(R : M_R)$ contains an element of valuation τ while T does not). To see the second containment, we note that $M_T = M_R + \{x \in V \mid \mathbf{v}(x) = g\}$; thus, if $x \in (R : M_R)$ and $y \in M_T$, we can write $y = y_1 + y_2$ (with $y_1 \in M_R$ and $\mathbf{v}(y_2) = g$) and so $xy = x(y_1 + y_2) = xy_1 + xy_2$. Now $xy_1 \in R \subseteq T$, while $\mathbf{v}(xy_2) \geq g$ since $\mathbf{v}(x) \geq 0$, and thus both xy_1 and xy_2 belong to T. It follows that $x \in (T : M_T)$, i.e., $(R : M_R) \subseteq (T : M_T)$. Therefore, $(R : M_R)^{v_T} = (T : M_T)$, where v_T is the v-operation on T.

Let $S' := \mathbf{v}(T)$: by Lemma 2.7, we can find $a, b \in (S' - M_{S'}) \setminus S'$ such that $2a, 2b, a + b \in S'$. Choose $x, y \in (T : M_T)$ such that $\mathbf{v}(x) = a$ and $\mathbf{v}(y) = b$ (and, without loss of generality, suppose $y \notin (R : M_R)$): they exist since $\mathbf{v}((T : M_T)) = (S' - M_{S'})$ [1, Proposition II.1.16].

Let $\{\alpha_1, \ldots, \alpha_q\}$ be a complete set of representatives of R/M_R (or, equivalently, of T/M_T). Then, $x + \alpha_i y \in (T : M_T)$ for each i, and by the choice of $\mathbf{v}(x)$ and $\mathbf{v}(y)$ the module $T_i = T + (x + \alpha_i y)T$ is a ring, equal to $T[x + \alpha_i y]$. Define $\star_i$ as the star operation

$$I \mapsto I^{v_T} \cap IT_i.$$

We claim that $\star_i$ closes T_i but not T_j for $j \neq i$.

Indeed, clearly $T_i^{\star_i} = T_i$. If $j \neq i$, then T_iT_j contains both $x + \alpha_i y$ and $x + \alpha_j y$, and thus it contains their difference $(\alpha_i - \alpha_j)y$. Since α_i and α_j are units corresponding to different residues, it follows that $\alpha_i - \alpha_j$ is a unit of R, and thus of T; hence, $y \in T_iT_j$. By construction, $y \in (T : M_T)$: thus, $y \in T_i^{\star_j}$. On the other hand, $y \notin T_i$, and thus $T_i^{\star_j} \neq T_i$.

Thus, $\{\star_1, \ldots, \star_q\}$ are $q = |K|$ different star operations. Furthermore, none of them closes $(R : M_R)$, since

$$(R : M_R)^{\star_i} = (T : M_T) \cap (R : M_R)T[x + a_i y]$$

contains y, while $y \notin (R : M_R)$.

To conclude the proof, it is enough to note that none of the $\star_i$ are the divisorial closure (since they close one of the T_i, none of which is divisorial); thus, adding v_T to the $\star_i$, we have $q + 1$ star operations that do not close $(R : M_R)$. □

We are now ready to show that R is the desired counterexample.

Theorem 2.9. *Let R be a Kunz domain with finite residue field, and suppose that $\ell_R(V/R) \geq 4$. Then, $1 < |\mathrm{Star}(R)| < \infty$, but R is not star regular.*

Proof. Since K is a finite field and R is not Gorenstein, by [4, Theorem 2.5] $1 < |\mathrm{Star}(R)| < \infty$, and the same for T.

By Proposition 2.6, we have $|\mathrm{Star}(R)| \leq 2 + |\Psi(\mathrm{Star}(R))|$; by Proposition 2.8, we have $|\Psi(\mathrm{Star}(R))| \leq |\mathrm{Star}(T)| - |K| - 1$. Hence,

$$\begin{aligned}|\mathrm{Star}(R)| &\leq 2 + |\mathrm{Star}(T)| - |K| - 1 = \\ &= |\mathrm{Star}(T)| - |K| + 1 < |\mathrm{Star}(T)|\end{aligned}$$

since $|K| \geq 2$. The claim is proved. □

3. The case $\mathbf{v}(R) = \langle n, n+1, \ldots, 2n-3, 2n-1\rangle$

In this section, we specialize to the case of Kunz domains R such that $\mathbf{v}(R) = \langle n, n+1, \ldots, 2n-1, 2n-3\rangle = \{0, n, n+1, \ldots, 2n-1, 2n-3, \ldots\}$, where $n \geq 4$ is an integer. It is not hard to see that this semigroup is pseudo-symmetric, with $g = 2n - 2$ and $\tau = n - 1$.

We note that this semigroup is pseudo-symmetric also if $n = 3$, for which the number of star operations has been calculated in [8, Proposition 2.10]: in this case, we have $|\mathrm{Star}(R)| = 4$.

By Lemma 2.5, the only $I \in \mathcal{F}_0(R)$ such that $IT \neq I$ are R and the canonical ideals. From now on, we denote by $\mathcal{G}$ the set $\{I \in \mathcal{F}_0(R) \mid IT = I\}$; we want to parametrize $\mathcal{G}$ by subspaces of a vector space.

Lemma 3.1. *Let K be the residue field of R. Then, there is an order-preserving bijection between $\mathcal{G}$ and the set of vector subspaces of K^{n-1}.*

Proof. Every $I \in \mathcal{G}$ contains T. The quotient of R-modules $\pi : V \mapsto V/T$ induces a map

$$\widetilde{\pi} \colon \mathcal{G} \longrightarrow \mathcal{P}(V/T)$$
$$I \longmapsto \pi(I),$$

where $\mathcal{P}(V/T)$ denotes the power set of V/T. It is obvious that $\widetilde{\pi}$ is injective.

The map π induces on V/T a structure of K-vector space of dimension $n-1$. If $I \in \mathcal{G}$, then its image along $\widetilde{\pi}$ will be a vector subspace; conversely, if W is a vector subspace of V/T then $\pi^{-1}(W)$ will be an ideal in $\mathcal{G}$. The claim is proved. $\square$

For an arbitrary domain D and a fractional ideal I of D, the star operation *generated* by I is the map [14, Section 5]

$$\star_I : J \mapsto (I:(I:J)) \cap J^v = J^v \cap \bigcap_{\gamma \in (I:J)\setminus\{0\}} \gamma^{-1} I;$$

this star operation has the property that, if I is $\star$-closed for some $\star \in \mathrm{Star}(D)$ and J is $\star_I$-closed, then J is also $\star$-closed. If $\Delta \subseteq \mathcal{F}(D)$, we define $\star_\Delta$ as the map

$$\star_\Delta : J \mapsto \bigcap_{I \in \Delta} J^{\star_I}.$$

In the present case, we can characterize when an ideal is $\star_\Delta$-closed.

Proposition 3.2. *Let $I, J \in \mathcal{G}$ and let $\Delta \subseteq \mathcal{G}$ be a set of nondivisorial ideals.*

(a) *I is divisorial if and only if $n-1 \in \mathbf{v}(I)$;*
(b) *$I^v = I \cup \{x \mid \mathbf{v}(x) \geq n-1\}$;*
(c) *if I, J are nondivisorial, then $I = I^{\star_J}$ if and only if $I \subseteq \gamma^{-1}J$ for some γ of valuation 0;*
(d) *if I is nondivisorial, then I is $\star_\Delta$-closed if and only if $I \subseteq \gamma^{-1}J$ for some $J \in \Delta$ and some γ of valuation 0.*

Proof. (a) If I is divisorial, then (since $I \neq R$) we must have $(R : M_R) \subseteq I$; in particular, $n-1 \in \mathbf{v}(I)$.

Suppose $n-1 \in \mathbf{v}(I)$; since I contains every element of valuation at least n (being $IT = I$), it contains also all elements of valuation $n-1$. Let x be such that $\mathbf{v}(x) = n-1$: then, $\mathbf{v}(x+r) \geq n-1$ for every $r \in V$, and thus $x + I \subseteq I$. Hence, I is divisorial by [1, Proposition II.1.23].

(b) Let $L := I \cup \{x \mid \mathbf{v}(x) \geq n-1\}$. If $n-1 \in \mathbf{v}(I)$, then $L = I$ and $I^v = L$ by the previous point. If $n-1 \notin \mathbf{v}(I)$, then (since I contains any element of valuation at least n), L is a fractional ideal of R such that $\mathbf{v}(L) = \mathbf{v}(I) \cup \{n-1\}$; hence, it is divisorial and $\ell(L/I) = 1$. It follows that $L = I^v$, as claimed.

(c) Suppose $I \subseteq \gamma^{-1}J$, where $\mathbf{v}(\gamma) = 0$. Since J is not divisorial, $n-1 \notin \mathbf{v}(J) = \mathbf{v}(\gamma^{-1}J)$; hence, using the previous point, $I = I^v \cap \gamma^{-1}J$ is closed by $\star_J$.

Conversely, suppose $I = I^{\star_J}$. Since I is nondivisorial, there must be $\gamma \in (I : J)$, $\gamma \neq 0$ such that $I \subseteq \gamma^{-1}J$ and $I^v \nsubseteq \gamma^{-1}J$. If $\mathbf{v}(\gamma) > 0$, then $\gamma^{-1}J$ contains the elements of valuation $n-1$; it follows that $I^v \subseteq \gamma^{-1}J$ and thus that $I^v \subseteq I^{\star_J}$, against $I = I^{\star_J}$. Hence, $\mathbf{v}(\gamma) = 0$, as claimed.

(d) If $I \subseteq \gamma^{-1}J$ for some $J \in \Delta$ and some γ such that $\mathbf{v}(\gamma) = 0$, then $I^{\star_\Delta} \subseteq I^{\star_J} = I$, and thus I is $\star_\Delta$-closed.

Conversely, suppose $I = I^{\star_\Delta}$. For every $J \in \Delta$, the ideal $I^{\star_J}$ is contained in $I^v = I \cup \{x \mid \mathbf{v}(x) \geq n-1\}$; since $\ell(I^v/I) = 1$, it follows that $I^{\star_J}$ is either I or I^v. Since $I = I^{\star_\Delta}$, it must be $I^{\star_J} = I$ for some J; by the previous point, $I \subseteq \gamma^{-1}J$ for some γ, as claimed. $\square$

An important consequence of the previous proposition is the following: suppose that Δ is a set of nondivisorial ideals in $\mathcal{F}_0(R)$ such that, when $I \neq J$ are in Δ, then $I \nsubseteq \gamma^{-1}J$ for all γ having valuation 0. Then, for every subset $\Lambda \subseteq \Delta$, the set of ideals of Δ that are $\star_\Lambda$-closed is exactly Λ; in particular, each nonempty subset of Δ generates a different star operation.

We will use this observation to estimate the cardinality of $\mathrm{Star}(R)$ when the residue field is finite.

Proposition 3.3. *Let R be a Kunz domain such that $\mathbf{v}(R) = \langle n, n+1, \ldots, 2n-3, 2n-1\rangle$, and suppose that the residue field of R has cardinality $q < \infty$. Then,*

$$|\mathrm{Star}(R)| \geq 2^{\frac{q^{n-2}-1}{q-1}} \geq 2^{q^{n-3}}.$$

Proof. Let $L := \{x \in V \mid \mathbf{v}(x) \geq n\}$; then, $A := V/L$ is a K-algebra. Let e_1 be an element of valuation 1, and let $e_i := e_1^i$; then, $\{1 = e_0, e_1, \ldots, e_{n-1}\}$ projects to a K-basis of A, which for simplicity we still denote by $\{e_0, \ldots, e_{n-1}\}$. The vector subspace spanned by e_0 is exactly the field K.

Since V and L are stable by multiplication by every element of valuation 0, asking if $\gamma I \subseteq J$ for some $I, J \in \mathcal{F}_0(R)$ and some γ is equivalent to asking if there is a $\overline{\gamma} \in A$ of "valuation" 0 such that $\overline{\gamma}\overline{I} \subseteq \overline{J}$, where $\overline{I}$ and $\overline{J}$ are the images of I and J, respectively, in A. Hence, instead of working with ideals in $\mathcal{F}_0(R)$ we can work with vector subspaces of A containing e_0.

Furthermore, if V is a vector subspace of A and γ has valuation 0, then γV has the same dimension of V; thus, if V and W have the same dimension, $\gamma V \subseteq W$ if and only if $\gamma V = W$. Let $\sim$ denote the equivalence relation such that $V \sim W$ if and only if $\gamma V = W$ for some γ of valuation 0.

Let X be the set of 2-dimensional subspaces of A that contain e_0 but not e_{n-1}. Then, the preimage of every element of X is an element of $\mathcal{F}_0(R)$ that does not contain any element of valuation $n-1$, and thus it is nondivisorial by Proposition 3.2(b).

An element of X is in the form $\langle e_0, \lambda_1 e_1 + \cdots + \lambda_{n-1}e_{n-1}\rangle$, where at least one among $\lambda_1, \ldots, \lambda_{n-2}$ is not 0; since $\langle e_0, f\rangle = \langle e_0, \lambda f\rangle$ for all $\lambda \in K$, $\lambda \neq 0$, there are exactly $(q^{n-1}-q)/(q-1)$ such subspaces.

Let $V \in X$, say $V = \langle e_0, f\rangle$, and consider the equivalence class Δ of V with respect to $\sim$. Then, $W \in \Delta$ if and only if $\gamma W = V$ for some γ; since $1 \in W$, it follows that such a γ must belong to V. Since γ has valuation 0, it must be in the form $\lambda_0 e_0 + \lambda_1 f$ with $\lambda_0 \neq 0$; furthermore, if $\gamma' = \lambda\gamma$ then $\gamma^{-1}V = \gamma'^{-1}W$. Hence, the cardinality of Δ is at most $\frac{q^2-q}{q-1} = q$.

Therefore, X contains elements belonging to at least

$$\frac{1}{q}\frac{q^{n-1}-q}{q-1} = \frac{q^{n-2}-1}{q-1} \geq q^{n-3}$$

equivalence classes; let X' be a set of representatives of such classes, and let Y be the preimage of X' in the power set of $\mathcal{F}_0(R)$. Then, every subset of Y generates a different star operation (with the empty set corresponding to the v-operation); it follows that

$$|\mathrm{Star}(R)| \geq 2^{\frac{q^{n-2}-1}{q-1}} \geq 2^{q^{n-3}},$$

as claimed. □

For $n = 4$, we can even calculate $|\mathrm{Star}(R)|$.

Proposition 3.4. *Let R be a Kunz domain such that $\mathbf{v}(R) = \langle 4, 5, 7\rangle$, and suppose that the residue field of R has cardinality $q < \infty$. Then, $|\mathrm{Star}(R)| = 2^{2q} + 3$.*

Proof. Consider the same setup of the previous proof. We start by claiming that two vector subspaces W_1, W_2 of A of dimension 3 that contain e_0 but not e_3 are equivalent under $\sim$.

Indeed, any such subspace must have a basis of the form $\{e_0, e_1+\theta_1 e_3, e_2+\theta_2 e_3\}$, and different pairs (θ_1, θ_2) induce different subspaces; let $W(\theta_1, \theta_2) := \langle e_0, e_1 + \theta_1 e_3, e_2 + \theta_2 e_3\rangle$. To show that two such subspaces are equivalent, we prove that they are all equivalent to $W(0,0)$. Let $\gamma := e_0 - \theta_2 e_1 - \theta_1 e_2$: we claim that $\gamma W(\theta_1, \theta_2) = W(0,0)$. Indeed, $\gamma e_0 = \gamma \in W(0,0)$; on the other hand,

$$\gamma(e_1 + \theta_1 e_3) = e_1 + \theta_1 e_3 - \theta_2 e_2 - \theta_1 e_3 = e_1 - \theta_2 e_2 \in W(0,0),$$

and likewise

$$\gamma(e_2 + \theta_2 e_3) = e_2 + \theta_2 e_3 - \theta_2 e_3 = e_2 \in W(0,0).$$

Hence, $W(\theta_1,\theta_2)\sim W(0,0)$.

Consider now the set Δ of nondivisorial ideals in $\mathcal{F}_0(R)$. By Lemma 2.5 and Proposition 3.2, Δ is equal to the union of the set $\mathcal{C}$ of the canonical ideals and the set $\mathcal{G}$ of the $I\in\mathcal{F}_0(R)$ such that $IT=I$. By Lemma 3.1 and Proposition 3.2, the elements of the latter correspond to the subspaces of V/T containing e_0 but not e_3: hence, we can write $\mathcal{G}=\mathcal{G}_1\cup\mathcal{G}_2\cup\mathcal{G}_3$, where $\mathcal{G}_i$ contains the ideals of $\mathcal{G}$ corresponding to subspaces of dimension i.

Given $\star\in\mathrm{Star}(R)$, let $\Delta(\star):=\{I\in\Delta\mid I=I^\star\}$. We claim that $\Delta(\star)$ is one of the following:

- Δ;
- $\Delta\setminus\mathcal{C}$;
- $\Lambda\cup\{T\}$ for some $\Lambda\subseteq\mathcal{G}_2$;
- the empty set.

By Proposition 2.4, if $T\neq T^\star$ (i.e., if $T\notin\Delta(\star)$) then $\star=v$, and $\Delta(\star)=\emptyset$.

If $\Delta(\star)$ contains a canonical ideal then $\star$ is the identity, and thus $\Delta(\star)=\Delta$.

If I is $\star$-closed for some $I\in\mathcal{G}_3$, but no canonical ideal is $\star$-closed, then every element of $\mathcal{G}_3$ must be closed, since any other $I'\in\mathcal{G}_3$ is in the form γI for some γ of valuation 0 (by the first part of the proof); furthermore, every element of $\mathcal{G}_2$ is the intersection of the elements of $\mathcal{G}_3$ containing it, and thus it is $\star$-closed. It follows that $\Delta(\star)=\Delta\setminus\mathcal{C}$; in particular, there is only one such star operation.

Let $\star$ be any star operation different from the three above. Then, $\Delta(\star)$ must contain T and cannot contain any canonical ideal nor any element of $\mathcal{G}_3$. Hence, $\Delta(\star)$ must be equal to $\Lambda\cup\{T\}$ for some $\Lambda\subseteq\mathcal{G}_2$. Moreover, $\Lambda\cup\{T\}$ is equal to $\Delta(\star)$ for some $\star$ if and only if Λ is the (possibly empty) union of equivalence classes under $\sim$. It follows that $|\mathrm{Star}(R)|=2^x+3$, where x is the number of such equivalence classes.

By the proof of Proposition 3.3, the image of an element of $\mathcal{G}_2$ is in the form $\langle e_0,f\rangle$, where $f=\lambda_1e_1+\lambda_2e_2+\lambda_3e_3$ with at least one between λ_1 and λ_2 nonzero. Let $V(\lambda_1,\lambda_2,\lambda_3)$ denote the subspace $\langle e_0,f\rangle$; clearly, $V(\lambda_1,\lambda_2,\lambda_3)=V(c\lambda_1,c\lambda_2,c\lambda_3)$ for every $c\in K\setminus\{0\}$. The subspaces equivalent to V must have the form $(e_0+\theta f)^{-1}V$ for some $\theta\in K$, and, by using the basis $\{e_0,e_0+\theta f\}$ of V, we see that $(e_0+\theta f)^{-1}V(\lambda_1,\lambda_2,\lambda_3)=\langle e_0,(e_0+\theta f)^{-1}\rangle$. If $\theta=0$, then $e_0+\theta f=e_0$, and thus $(e_0+\theta f)^{-1}V(\lambda_1,\lambda_2,\lambda_3)=V(\lambda_1,\lambda_2,\lambda_3)$; suppose, from now on, that $\theta\neq 0$.

To calculate $(e_0+\theta f)^{-1}=e_0+\alpha_1e_1+\alpha_2e_2+\alpha_3e_3$, we can simply expand the product $(e_0+\theta f)(e_0+\alpha_1e_1+\alpha_2e_2+\alpha_3e_3)$, using $e_i=0$ for $i>3$; we obtain

$$\begin{cases}\alpha_1=-\theta\lambda_1\\ \alpha_2=-\theta(\lambda_1\alpha_1+\lambda_2)\\ \alpha_3=-\theta(\lambda_1\alpha_2+\lambda_2\alpha_1+\lambda_3).\end{cases}$$

Since $\theta\neq 0$, the set $\{e_0,(e_0+\theta f)^{-1}-e_0\}$ is a basis of $(e_0+\theta f)^{-1}V(\lambda_1,\lambda_2,\lambda_3)$; hence, $(e_0+\theta f)^{-1}V(\lambda_1,\lambda_2,\lambda_3)=V(\alpha_1,\alpha_2,\alpha_3)$. We distinguish two cases.

If $\lambda_1=0$, then $\lambda_2\neq 0$, and so we can suppose $\lambda_2=1$. Then, we have

$$\begin{cases}\alpha_1=0\\ \alpha_2=-\theta\\ \alpha_3=-\theta\lambda_3.\end{cases}$$

and so $(e_0+\theta f)^{-1}V(0,1,\lambda_3)=V(0,-\theta,-\theta\lambda_3)=V(0,1,\lambda_3)$ since $\theta\neq 0$. It follows that the only subspace equivalent to $V(0,1,\lambda_3)$ is $V(0,1,\lambda_3)$ itself; since we have q choices for λ_3, this case gives q different equivalence classes.

If $\lambda_1\neq 0$, we can suppose $\lambda_1=1$. Then, we get

$$\begin{cases}\alpha_1=-\theta\\ \alpha_2=-\theta(\alpha_1+\lambda_2)=-\theta(-\theta+\lambda_2)\\ \alpha_3=-\theta(-\theta(-\theta+\lambda_2)-\theta\lambda_2+\lambda_3).\end{cases}$$

Since $\theta\neq 0$, we can divide by $-\theta$, obtaining

$$(e_0+\theta f)^{-1}V(1,\lambda_2,\lambda_3)=V(1,-\theta+\lambda_2,\theta^2-2\theta\lambda_2+\lambda_3).$$

Since $-\theta+\lambda_2\neq-\theta'+\lambda_2$ if $\theta\neq\theta'$, we have $(e_0+\theta f)^{-1}V(1,\lambda_2,\lambda_3)\neq(e_0+\theta' f)^{-1}V(1,\lambda_2,\lambda_3)$ for all $\theta\neq\theta'$; thus, every equivalence class is composed by q subspaces. Since there are q^2 such subspaces, we get q additional equivalence classes.

Therefore, $\mathcal{G}_2$ is partitioned into $2q$ equivalence classes, and so $|\mathrm{Star}(R)|=2^{2q}+3$, as claimed. $\square$

Remark 3.5. (1) *The estimate obtained in Proposition 3.3 grows very quickly; for example, if q is fixed, it follows that the double logarithm of* $|\mathrm{Star}(R)|$ *grows (at least) linearly in $n=\ell(V/R)+1$. This should be compared with* [8, Theorem 3.21]*, where the authors analyzed a case where the growth of* $|\mathrm{Star}(R)|$ *was linear in $\ell(\overline{R}/R)$ (where $\overline{R}$ is the integral closure of R, which in this case is nonlocal).*

(2) *Let* $V = K[[X]]$ *be the ring of power series and consider the case* $n = 4$*. Then,* $T = K + X^4K[[X]]$*, and using Theorem 2.9 and Proposition 3.3, we have the lower bound* $|\mathrm{Star}(T)| \geq 2^{2q}+q+2$*. This estimate is not very far from the precise counting* $|\mathrm{Star}(T)| = 2^{2q+1} + 2^{q+1} + 2$ *obtained in* [15, Corollary 4.1.2].

References

[1] V. Barucci, D. E. Dobbs and M. Fontana, *Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains,* Mem. Amer. Math. Soc., 125(598) (1997), x+78 pp.

[2] H. Bass, *On the ubiquity of Gorenstein rings,* Math. Z., 82 (1963), 8-28.

[3] E. G. Houston, A. Mimouni and M. H. Park, *Integral domains which admit at most two star operations,* Comm. Algebra, 39(5) (2011), 1907-1921.

[4] E. G. Houston, A. Mimouni and M. H. Park, *Noetherian domains which admit only finitely many star operations,* J. Algebra, 366 (2012), 78-93.

[5] E. G. Houston, A. Mimouni and M. H. Park, *Integrally closed domains with only finitely many star operations,* Comm. Algebra, 42(12) (2014), 5264-5286.

[6] E. Houston, A. Mimouni and M. H. Park, *Star operations on overrings of Noetherian domains,* J. Pure Appl. Algebra, 220(2) (2016), 810-821.

[7] E. Houston, A. Mimouni and M. H. Park, *Star operations on overrings of Prüfer domains,* Comm. Algebra, 45(8) (2017), 3297-3309.

[8] E. G. Houston and M. H. Park, *A characterization of local Noetherian domains which admit only finitely many star operations: The infinite residue field case,* J. Algebra, 407 (2014), 105-134.

[9] J. Jäger, *Längenberechnung und kanonische Ideale in eindimensionalen Ringen,* Arch. Math. (Basel), 29(5) (1977), 504-512.

[10] E. Kunz, *The value-semigroup of a one-dimensional Gorenstein ring,* Proc. Amer. Math. Soc., 25 (1970), 748-751.

[11] E. Kunz, *Beispiel: Die kanonische Idealklasse eines eindimensionalen Cohen-Macaulay-Rings,* 103, Lecture Notes in Math., Vol. 238, Springer, Berlin, (1971), 17-24.

[12] T. Matsuoka, *On the degree of singularity of one-dimensional analytically irreducible Noetherian local rings,* J. Math. Kyoto Univ., 11 (1971), 485-494.

[13] J. C. Rosales and P. A. García-Sánchez, Numerical Semigroups, Developments in Mathematics, 20, Springer, New York, 2009.

[14] D. Spirito, *Star operations on numerical semigroups,* Comm. Algebra, 43(7) (2015), 2943-2963.

[15] B. White, *Star Operations and Numerical Semigroup Rings*, Ph.D. thesis, The University of New Mexico, 2014.

Dario Spirito
Dipartimento di Matematica e Fisica
Università degli Studi "Roma Tre"
Roma, Italy
email: spirito@mat.uniroma3.it