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PERFECT NUMBERS WITH IDENTICAL DIGITS IN NEGATIVE BASE

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Abstract. We study perfect numbers which are repdigits in a given negative base. It is shown that in each negative base there are at most finitely many perfect repdigits, and that the set of all such numbers can effectively be computed. As an illustration we explicitly determine these numbers in bases -2 and -10.

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1. Introduction and results

For a positive integer N we denote by $\sigma(N)$ the sum-of-divisors function of N. Already the ancient Greeks were interested in this function, in particular, they studied so-called perfect numbers, i.e., solutions of the equation

$$\sigma(N) = 2N\,.$$

There are many open problems concerning the function σ (e.g., see [11]), and we do not even know whether there exist infinitely many perfect numbers. Recently, R. B. Nelsen [9] proved that every even perfect number ends in 6 or 28.

Let b be an integer different from $-1, 0, 1$. Every positive integer[1] N can uniquely be written in the form

$$N = d_{n-1}b^{n-1} + d_{n-2}b^{n-2} + \cdots + d_1 b + d_0$$

with digits $d_0, d_1, \ldots, d_{n-1} \in \{0, 1, \ldots, |b| - 1\}\,, d_{n-1} \neq 0$ (cf. [4, Section 3]), and n is the length of the representation of N in base b. We call N a repdigit in base b if all its digits are the same.

In this short note we are interested in those positive integers which are repdigits in base b and perfect. P. Pollack [10] proved that in each integer base > 1 there are at most finitely many perfect repdigits and that the set of all such numbers is

[1]For the sake of completeness it should be mentioned that in case $b < -1$ every non-zero integer can uniquely be represented in the form above (e.g., see Grünwald [5]).

effectively computable. Moreover, he showed that 6 is the only perfect repdigit in base 10. He exploited results on exponential Diophantine equations and some of the ideas implicit in F. Luca's demonstration [7,8] that there are no perfect Fibonacci or Lucas numbers. K. A. Broughan and Q. Zhou [2] computed perfect repdigits to all positive bases up to 10. In most cases their method reduces to using modular constraints or solving several particular exponential type Diophantine equations. Recently, K. A. Broughan, S. G. Sanchez and F. Luca [1] presented an algorithm to compute all perfect repdigits in positive base. Among others, they extended the computations from [2] to all bases $2 \le b \le 333$.

Adapting the arguments of P. Pollack we prove the following finiteness results for repdigits in negative base.

Theorem 1.1. *Let $b < -1$.*

(i) *There exist finitely many perfect repdigits in base b, and the set of all such numbers is effectively computable.*

(ii) *If N is an even perfect repdigit in base b then there exists a prime p such that $2^p - 1$ is prime and*

$$6 \le N = 2^{p-1}(2^p - 1) \le |b| - 1\,.$$

Theorem 1.2. (i) *There is no perfect repdigit in base -2.*

(ii) *The only perfect repdigit in base -10 is 6.*

2. Proofs

In this section we essentially prove analogues of results of [10] which are exploited in the proof of our main results above. Similarly as in [10] we use the following notions. The letters n, m, k denote positive integers, p, ℓ are reserved for primes, and $v_p(n)$ is the exponent of the highest power of p dividing n. We denote by $p^+(n)$ ($p^-(n)$, resp.) the largest (smallest, resp.) prime divisor of n. The symbol $\square$ stands for a generic element in $(\mathbb{Q}^\times)^2$. Thus, if x, y are nonzero rational numbers, then $x = y\square$ means that the quotient x/y is a square of a rational number.

Our first main ingredient is a well-known fundamental result of Euler (e.g., see [13, § 3.3]).

Lemma 2.1. [Euler] *Let N be a perfect number.*

(i) *If N is even then there exists a prime p such that $2^p - 1$ is prime and*

$$N = 2^{p-1}(2^p - 1)\,.$$

(ii) *If N is odd then there exist a natural number s and a prime p with $p \equiv 1 \pmod 4$ such that*

$$N = ps^2 .$$

Our second main ingredient is the particular Lucas sequence

$$U_n := U_n(b) := \frac{b^n - 1}{b - 1} .$$

Here we always assume $b \in \mathbb{Z} \setminus \{-1, 0, 1\}$, and we denote by d a positive integer less than $|b|$. In particular, we exploit the fact that the length of a repdigit in negative base is odd.

Lemma 2.2. *Let $b < -1$. The natural number N is a repdigit in base b if and only if there is a digit d and an odd n such that*

$$N = d\, U_n(b) .$$

In this case the length of N in the representation in base b is n.

Proof. We observe that

$$N = d \sum_{j=0}^{n-1} b^j = d\, U_n$$

is positive if and only if n is odd (e.g., see [4, Proposition 3.1]). □

We now proceed as follows. First, we show the second part of Theorem 1.1. Then we list several auxiliary lemmas; the proof of the last lemma is based on Theorem 1.1 (ii). These lemmas are applied in the final part where we complete the proof of Theorem 1.1 and establish Theorem 1.2.

2.1. Proof of the second part of Theorem 1.1. We essentially proceed analogously as in [10, Lemma 7]. By Lemma 2.2 we have $N = dU_n$ with n odd, and Euler's Theorem (see Lemma 2.1) yields a prime p such that $q := 2^p - 1$ is prime and

$$N = dU_n = 2^{p-1} q .$$

If $n = 1$ then we have $6 \le N = d \le |b| - 1$, since 6 is the smallest perfect number.

For the sake of contradiction we now assume $n > 1$. Then U_n must be odd, because otherwise

$$2 \mid \underbrace{b^{n-1} + \cdots + b + 1}_{n \text{ summands}}$$

which implies b odd and n even which is impossible. Thus we can write

$$U_n = q \ge 3 \text{ and } d = 2^{p-1} .$$

If $n = 3$ we have

$$b^2 + b + 1 = 2^p - 1 = 2d - 1 < -2b - 1,$$

yielding

$$b^2 + 3b + 2 < 0,$$

which has no solution in $\mathbb{Z}$.

Finally, the assumption $n \geq 5$ implies

$$b^2 < U_5 \leq U_n < 2^p\,,$$

and therefore

$$|b| < 2^{p/2} \leq 2^{p-1} = d$$

which is excluded. □

2.2. Auxiliary results on the sequence $U_n(b)$. First we collect some divisibility properties which will be needed in the sequel.

Lemma 2.3. *If ℓ is a prime such that $b \equiv 1 \pmod{\ell}$ then we have*

$$v_\ell(U_n(b)) = v_\ell(n)\,.$$

Proof. If $v_\ell(n) = 0$ then our claim follows from

$$U_n = \sum_{j=0}^{n-1} b^j \equiv n \pmod{\ell}\,.$$

For $v_\ell(n) > 0$ our assertion is clear by [13, Lemma 6.21]. □

Lemma 2.4. *Let $n \in \mathbb{N}$, p, ℓ be primes and e be the order of b modulo ℓ.*

(i) *If ℓ divides $b^n - 1$ then e divides n.*
(ii) *If $e = p$ is a prime then $p < \ell$ and $\ell \equiv 1 \pmod{p}$.*
(iii) *Suppose $\ell \mid U_p(b)$. Then we have $e \in \{1, p\}$. Moreover, if $e = 1$ then $\ell = p \leq |b| + 1$, and if $p > |b| + 1$ then $e = p$.*

Proof. (i) In view of

$$\ell t = b^n - 1$$

with some integer t we have $b^n \equiv 1 \pmod{\ell}$, which yields our claim.
(ii) We have $\ell > 2$ and

$$p \mid \#(\mathbb{Z}/\ell\mathbb{Z})^\times = \ell - 1$$

yielding $p < \ell$ and $\ell \equiv 1 \pmod{p}$.
(iii) Our prerequisites and (i) yield $e \mid p$, hence $e \in \{1, p\}$. If $e = 1$ we have $\ell \mid b - 1$. We know from Lemma 2.3 that $\ell \mid p$, hence

$$p = \ell \le |b-1| \le |b| + 1\,.$$

Finally, we observe that $p > |b| + 1$ implies $e \ne 1$. □

Lemma 2.5. *Let $p = p^+(n)$.*

(i) *If the prime ℓ divides*

$$\gcd\Bigl(U_p(b),\, \frac{U_n(b)}{U_p(b)}\Bigr)$$

then we have $b \equiv 1 \pmod{\ell}$.

(ii) *If $b < -1$ and $p \ge |b| + 2$ then we have*

$$\gcd\Bigl(U_p(b),\, \frac{U_n(b)}{U_p(b)}\Bigr) = 1\,.$$

Proof. (i) Let e be the order of b modulo ℓ, thus $e \in \{1, p\}$ by Lemma 2.4. Assume $e = p$, hence $p < \ell$. On the other hand, we have

$$\frac{U_n}{U_p} = \sum_{j=0}^{\frac{n}{p}-1} (b^p)^j \equiv \frac{n}{p} \pmod{\ell}\,,$$

and since ℓ divides the quotient on the left hand side we conclude

$$\ell \mid \frac{n}{p} \mid n\,,$$

thus $\ell \mid n$ yielding $\ell \le p$ by our choice of p: Contradiction.
(ii) Suppose that there exists some prime ℓ which divides

$$\gcd\Bigl(U_p,\, \frac{U_n}{U_p}\Bigr)\,.$$

Then $b \equiv 1 \pmod{\ell}$ by (i), and $\ell \mid p$ by Lemma 2.3, hence

$$p = \ell \mid b - 1\,,$$

which yields the absurdity

$$|b| + 2 \le p \le |b-1| \le |b| + 1.$$

□

Lemma 2.6. *If $k\,|\,n$ and*

$$g := \gcd\Big(\frac{b^{n/k}-1}{b-1}, \sum_{j=0}^{k-1} b^{jn/k}\Big),$$

then $g \mid k$ and

$$b^{n/k} \equiv 1 \pmod{g}.$$

Proof. With $m := n/k$ we have $g \mid b^m - 1$, hence $b^m \equiv 1 \pmod{g}$ and further

$$gz = \sum_{j=0}^{k-1} (b^m)^j \equiv k \pmod{g},$$

with some integer z, which implies $g\,|\,k$. □

Similarly as in [10, Lemma 4] we consider the case that the product of two members U_n, U_m of our Lucas sequence is a square. Luckily, for a negative base this can only happen in the trivial case provided that both n and m are odd.

Lemma 2.7. *Let $b < -1$. If n, m are odd and $U_n(b)\,U_m(b) = \square$ then $n = m$.*

Proof. With $k := \gcd(n, m)$ we infer

$$\gcd(U_n(b),\, U_m(b)) = U_k(b)$$

from [3, Lemma 1]. Thus there exist $s, t \in \mathbb{N}$ such that

$$\frac{(b^k)^{n/k}-1}{b^k-1} = \frac{U_n}{U_k} = \frac{U_n U_k}{U_k^2} = s^2$$

and

$$\frac{(b^k)^{m/k}-1}{b^k-1} = \frac{U_m}{U_k} = t^2\,. \tag{2.1}$$

For the sake of contradiction we assume $n < m$.

First, suppose $n/k \in \{1, 2\}$. Since n is odd this implies $n = k$, hence $n\,|\,m$ and therefore $m/k \notin \{1, 2\}$. But then (2.1) and [6] (see [10, Lemma 3]) yield

$$(b^k, \frac{m}{k}, t) \in \{(\pm 3, 5, 11),\ (\pm 7, 4, 20)\},$$

which implies

$$(b^k, \frac{m}{k}, t) \in \{(\pm 3, 5, 11)\},$$

because m is odd. Therefore, we have $b = -3$, $n = k = 1$, $m = 5$ and $t = 11$ yielding the absurdity

$$61 = \frac{(-3)^5 - 1}{-3-1} = \frac{b^3-1}{b-1} = 11^2\,.$$

Second, suppose $n/k \notin \{1,2\}$. Again we deduce

$$(b^k, \frac{n}{k}, s) \in \{(\pm 3, 5, 11),\ (\pm 7, 4, 20)\}\,,$$

and the proof is completed similarly as above. The details are left to the reader. $\square$

To complete the proof of Theorem 1.1 we need the following important consequence of [14, Theorem 9.6]. Recall that for a given finite set S of primes the integer z is an S-number if every prime which divides z is contained in S.

Lemma 2.8. *Let S be a finite set of primes. Then the set*

$$\{n \in \mathbb{N} \,:\, U_n(b) = a\square \text{ for some } S\text{-number } a\}$$

is finite, and all such n do not exceed an effectively computable constant which depends only on S and b.

Based on Lemma 2.8 the proof of the next lemma follows the same lines as the proof of its analogue [10, Lemma 8]. For the convenience of the reader we include the details here.

Lemma 2.9. *Let $M \in \mathbb{N}$, $b < -1$, r a prime and $N = dU_n(b) = r\square$ a repdigit in base b such that $p^-(n) \le M$. Then N is bounded by an effectively computable constant which depends only on b and M.*

Proof. Certainly, it suffices to show that n is effectively bounded. Observe that n is odd, hence $p := p^-(n) > 2$. Set $m := n/p$.

First, suppose $r \,|\, d$. Then we have $U_n = rd\square$. Since rd is supported on the primes dividing $|b|\,!$ our assertion is clear by Lemma 2.8.

Second, suppose $r \nmid d$. For some $s \in \mathbb{N}$ we have

$$rs^2 = dU_n = d \cdot \frac{b^m - 1}{b - 1} \cdot \sum_{j=0}^{p-1} (b^m)^j\,,$$

For

$$g := \gcd\Big(\frac{b^m - 1}{b - 1},\ \sum_{j=0}^{p-1} (b^m)^j\Big),$$

Lemma 2.6 yields

$$b^m \equiv 1 \pmod g \qquad \text{and} \qquad g \,|\, p\,,$$

and thus there exist square-free $u, v \in \mathbb{N}$ supported on the primes dividing dpr such that

$$\frac{b^m - 1}{b - 1} = u\square \qquad \text{and} \qquad \sum_{j=0}^{p-1} (b^m)^j = v\square\,. \tag{2.2}$$

Thus we have $duv = r\square$, and in particular $v_r(duv)$ is odd. Therefore $r \mid uv$ since $r \nmid d$. Clearly, $r \mid u$ or $r \mid v$, but not both because otherwise $v_r(duv) = 2$.

Let S be the set of primes dividing $|b|\,!\,M!$, and assume $r \nmid u$. Hence $r \mid v$, and then $u \mid dp$ by the above. In view of $d < |b|$ and $p \leq M$ we conclude that u is supported on S. Applying Lemma 2.8 to S and the first equality in (2.2) we see that n/p is bounded by an effectively computable constant depending only on b and M. Keeping in mind $p \leq M$ also n is bounded.

Now, assume $r \mid u$. So $r \nmid v$, and then v is supported on the set S of primes introduced above. Writing $w := b^{n/p}$ the second equality in (2.2) yields

$$1 + w + w^2 + \cdots + w^{p-1} = vk^2 \tag{2.3}$$

with some integer k.

Consider the case $p = 3$ and write $w = b^\delta x^2$ with $\delta \in \{0, 1\}$ and $x \in \mathbb{N}$. In both cases an application of [10, Lemma 6] yields our assertion.

Finally, consider $p > 3$. Then the roots of the polynomial on the left hand side of (2.3) are pairwise distinct, thus again by [10, Lemma 6] w is bounded by an effectively computable constant depending only on p and v. Since $p \leq M$ and $v \mid \prod_{\ell \in S} \ell$ we conclude that n is also bounded. $\square$

We exploit this result to establish the analogue of [10, Lemma 9].

Lemma 2.10. *Let $b < -1$, $n \in \mathbb{N}$ composite, r a prime and $U_n(b) = r\square$. Then n is bounded by an effectively computable constant depending only on b.*

Proof. Since $U_n(b)$ is positive n is odd. Set $p := p^+(n)$ and observe that in view of Lemma 2.9 we may suppose

$$p^+(n) \geq \max\{7,\ |b| + 2\}\ .$$

Using Lemmas 2.5 and 2.7 the proof of [10, Lemma 9] can easily be adapted. $\square$

The next inequality is extracted from the proof of [10, Lemma 10]. It will be needed to establish Lemma 2.12 and Theorem 1.2.

Lemma 2.11. *Let $b < -1$ and $p > |b| + 1$. Then we have*

$$\frac{\sigma(U_p(b))}{U_p(b)} < \Big(1 + \frac{2}{p}\Big)\Big(1 + \frac{2}{p} \cdot \log\big(\frac{3p}{2}\log(p-2)\big)\Big)\,.$$

Proof. Setting $m := U_p$ and $S := \{\ell \mid m\ :\ \ell \equiv 1 \pmod p\}$ and applying Lemma 2.4 and well-known facts we have

$$\frac{\sigma(U_p)}{U_p} = \sum_{d|m} \frac{1}{d} = \prod_{\ell \mid m} \sum_{j=0}^{\infty} \frac{1}{\ell^j} = \prod_{\ell \in S} \sum_{j=0}^{\infty} \frac{1}{\ell^j}\,.$$

Exploiting some calculus we deduce

$$\frac{\sigma(m)}{m} < \exp\Big(\sum_{\ell\in S}\frac{1}{\ell-1}\Big) \le \exp\Big(\sum_{j=1}^{\omega(m)}\frac{1}{jp}\Big) \le \exp\Big(\frac{1}{p}(1+\log\omega(m))\Big) = \exp\Big(\frac{1}{p}\Big)\exp\Big(\frac{\log\omega(m)}{p}\Big),$$

where we denote by $\omega(m)$ the number of prime divisors of m. Clearly, we have

$$2^{\omega(m)} \le m < |b|^p,$$

hence

$$\omega(m) < \frac{p}{\log 2}\log|b| < \frac{3p}{2}\log(p-2),$$

and then our claim is easily verified. □

Now we show the analogue of [10, Lemma 10]. Observe that the result in particular applies to the case $d = 1$, i.e., to repunits.

Lemma 2.12. *Let $b < -1$, $d = \square$ and $dU_n(b)$ be perfect. Then n is bounded by an effectively computable constant depending only on b.*

Proof. In view of Theorem 1.1 (ii) which we have proved above we may suppose that dU_n is odd, hence d is odd and Euler's Theorem yields a prime r such that

$$dU_n = r\square\,.$$

Using Lemma 2.10 we may suppose that $p := n$ is prime. Then [10, Lemma 2] implies that d is not perfect which yields

$$\left|\frac{\sigma(d)}{d} - 2\right| \ge \frac{1}{d} > \frac{1}{|b|}.$$

Further, we must have

$$\sigma(d) < 2d$$

because otherwise $\sigma(d) > 2d$ yielding

$$2dU_p = \sigma(dU_p) \ge \sigma(d)U_p \ge (1+2d)U_p \ge 1 + 2dU_p$$

which is absurd. Therefore we have

$$2 - \frac{\sigma(d)}{d} = \left|\frac{\sigma(d)}{d} - 2\right| > \frac{1}{|b|}$$

yielding

$$\frac{\sigma(d)}{d} < 2 - \frac{1}{|b|}.$$

Now it suffices to show

$$p < 48^2 b^2\,.$$

For the sake of contradiction we suppose the contrary and observe

$$\frac{6\log p}{p} < \frac{1}{8|b|}\,.$$

From Lemma 2.11 we infer

$$\frac{\sigma(U_p)}{U_p} < \Big(1+\frac{2}{p}\Big)\Big(1+\frac{6\log p}{p}\Big) \le \Big(1+\frac{6\log p}{p}\Big)^2 < \Big(1+\frac{8}{|b|}\Big)^2 < 1+\frac{1}{2|b|}\,,$$

and this leads to the absurdity

$$2 = \frac{\sigma(dU_p)}{dU_p} \le \frac{\sigma(d)}{d}\cdot\frac{\sigma(U_p)}{U_p} < \Big(2-\frac{1}{|b|}\Big)\Big(1+\frac{1}{2|b|}\Big) < 2\,.$$

$\square$

2.3. Completion of the proofs of Theorems 1.1 and 1.2. Now we are in a position to complete the proof of our main results.

Proof of Theorem 1.1 (i). Let dU_n be perfect. In view of Theorem 1.1 (ii) we may assume that dU_n is odd, thus by Euler's Theorem

$$dU_n = r\Box$$

with some prime r. In view of Lemma 2.12 we may suppose that d is not a square. Thus there exists a prime ℓ with $v_\ell(d)$ odd, in particular $\ell \le d < |b|$.

If $\ell = r$ then $U_n = d\ell\Box$, and we are done by Lemma 2.8 where S is the set of prime divisors of $|b|\,!$. Finally, let $\ell \ne r$, hence $\ell \,|\, U_n$. Lemma 2.5 tells us that the order e of b modulo ℓ divides n, and clearly we have $e < \ell$. Therefore, $p^-(n) < |b|$: Indeed, if $e = 1$ we have $\ell \,|\, n$ by Lemma 2.3, and if $e > 1$ then we know $e \,|\, n$ and $e < \ell < |b|$. Thus our assertion drops out from Lemma 2.9. $\square$

Proof of Theorem 1.2. (i) In view of Theorem 1.1 it suffices to assume that $U_n := U_n(-2)$ is an odd perfect repdigit in base -2. Clearly, n is odd and at least 3. Euler's Theorem yields a prime r with $r \equiv 1 \pmod 4$ and an integer s such that

$$rs^2 = U_n = (-2)^{n-1} + \cdots + (-2) + 1,$$

hence $rs^2 \equiv 3 \pmod 4$ which is impossible.

(ii) Let N be a perfect repdigit in base -10. If N is even we are done by Theorem 1.1. Therefore, we suppose $N = dU_n(-10)$ is odd. Then both d and n are odd, and we have $n \ge 3$. From [10, Lemma 2] we infer that there exist a prime r and a positive integer s such that $r \equiv 1 \pmod 4$ and $N = rs^2$. Therefore $N \equiv 1 \pmod 4$. Let $p := p^+(n)$, thus $p > 2$ and further $d \in \{3,7\}$: Indeed, the assumption $d \in \{1,5,9\}$ implies $d \equiv 1 \pmod 4$ and then the impossibility $N \equiv 3 \pmod 4$.

Let us start with $d = 3$. Then we have

$$3 \cdot \frac{10^p + 1}{11} \cdot \frac{10^n + 1}{10^p + 1} = 3U_p \frac{U_n}{U_p} = 3U_n = rs^2 \,. \tag{2.4}$$

From Lemma 2.5 we deduce that 3 is the only prime which can divide

$$\gcd\Big(3 \cdot \frac{10^p + 1}{11}, \frac{10^n + 1}{10^p + 1}\Big) .$$

Therefore (2.4) yields that one of

$$\frac{10^p + 1}{11}, \; 3 \cdot \frac{10^p + 1}{11}, \; \frac{10^n + 1}{10^p + 1}, \; 3 \cdot \frac{10^n + 1}{10^p + 1}$$

is a square. If the first is a square we have $2^p + 1 \equiv 3 \pmod 4$ yielding the absurdity

$$1 \equiv 3 \pmod 4 .$$

If the second or the fourth is a square we have $3 \equiv k^2 \pmod 5$, which is impossible. Thus we find

$$\frac{U_n}{U_p} = \frac{10^n + 1}{10^p + 1} = \square ,$$

hence $U_n U_p = \square$, and then $n = p$ by Lemma 2.7. Checking $N = 3U_p$ numerically we see $p \geq 29$. Now Lemma 2.11 shows

$$\frac{\sigma(U_p)}{U_p} < \frac{3}{2}$$

and then the contradiction

$$\frac{\sigma(N)}{N} = \frac{\sigma(3U_p)}{3U_p} \leq \frac{4}{3} \frac{\sigma(U_p)}{U_p} < \frac{4}{3} \cdot \frac{3}{2} = 2 .$$

Finally, let $d = 7$. Then we have

$$7 \cdot \frac{10^p + 1}{11} \cdot \frac{10^n + 1}{10^p + 1} = 7U_p \frac{U_n}{U_p} = 7U_n = N = rs^2 \tag{2.5}$$

with $r \neq 7$. Put

$$g := \gcd\Big(\frac{10^p + 1}{11}, \frac{10^n + 1}{10^p + 1}\Big)$$

First, suppose $g = 1$. From (2.5) we infer that one of

$$\frac{10^p + 1}{11}, \; 7 \cdot \frac{10^p + 1}{11}, \; \frac{10^n + 1}{10^p + 1}, \; 7 \cdot \frac{10^n + 1}{10^p + 1}$$

is a square, and we already know that the first cannot be a square. The second and the fourth cannot be squares because then we would have the impossibility $2 \equiv k^2 \pmod 5$. Similarly as above we then have

$$\frac{U_n}{U_p} = \frac{10^n + 1}{10^p + 1} = \square ,$$

hence $U_nU_p = \square$, and then $n = p$ by Lemma 2.7. Checking $N = 7U_p$ numerically we see $p \geq 23$. Lemma 2.11 shows

$$\frac{\sigma(U_p)}{U_p} < 1.52\,,$$

which implies the contradiction

$$\frac{\sigma(N)}{N} = \frac{\sigma(7U_p)}{7U_p} \leq \frac{8}{7}\frac{\sigma(U_p)}{U_p} < 2\,.$$

Second, suppose $g > 1$. From Lemma 2.5 we deduce that 11 is the only prime which can divide g. Since $-10 \equiv 1 \pmod{11}$ Lemma 2.4 yields $p = \ell = 11$. Then (2.5) reads

$$rs^2 = 7U_n = 7 \cdot \frac{10^{11}+1}{11} \cdot k = 7^2 \cdot 13 \cdot 19\,m\,.$$

In view of $19 \equiv 3 \pmod 4$ we observe $r \neq 19$, and thus $19^2 \mid U_n$. We check that the order of -10 modulo 19^2 equals $171 = 3^2 \cdot 19$, hence $19 \mid n$ yielding the contradiction

$$19 \leq p^+(n) = p = 11\,.$$

This completes the proof. $\square$

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