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GROUP PARTITIONS VIA COMMUTATIVITY

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Abstract. Let G be a nonabelian group, $A \subset G$ an abelian subgroup and $n \geqslant 2$ an integer. We say that G has an n-abelian partition with respect to A, if there exists a partition of G into A and n disjoint commuting subsets $A_1, A_2, \ldots, A_n$ of G, such that $|A_i| > 1$ for each $i = 1, 2, \ldots, n$. We classify all nonabelian groups, up to isomorphism, which have an n-abelian partition, for $n = 2$ and 3.

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1. Introduction and preliminaries

Let Γ be a simple graph and m, n two non-negative integers. We say that Γ is (m, n)-*partitionable* if its vertex set can be partitioned into m independent sets $I_1, \ldots, I_m$ and n cliques $C_1, \ldots, C_n$; that is

$$V_\Gamma = I_1 \uplus I_2 \uplus \cdots \uplus I_m \uplus C_1 \uplus C_2 \uplus \cdots \uplus C_n.$$

Such a partition of V_Γ is called a (m, n)-*partition* of Γ (see [3]). We shall note some special cases: $(1, 1)$-partitionable graphs are called split graphs (see [6]), $(1, 0)$-partitionable graphs are called edgeless graphs, $(0, 1)$-partitionable graphs are called complete graphs. In particular, in the case when $m = 0$ or $n = 0$, we essentially split Γ into n cliques,

$$V_\Gamma = C_1 \uplus C_2 \uplus \cdots \uplus C_n,$$

or m independent sets,

$$V_\Gamma = I_1 \uplus I_2 \uplus \cdots \uplus I_m,$$

respectively.

In the following all groups will be assumed finite. We now focus our attention on a graph associated with a finite group – the so-called commuting graph. Let G be a finite group and X a nonempty subset of G. The *commuting graph* $\mathcal{C}(G, X) = \mathcal{C}(X)$, has X as its vertex set with two distinct elements of X joined by an edge when they *commute* in G. Commuting graphs have been investigated by many authors in various contexts, see for instance [4,5,8]. Clearly, $\mathcal{C}(G)$ is $(0,n)$-partitionable if and only if G can be partitioned into n commuting subsets. This suggests the following definition.

Definition 1.1. Let G be a nonabelian group and $A \subset G$ a commuting subset containing the identity element. We say that G has an *n-abelian partition* (with respect to A), if there exists a partition of G into A and n disjoint commuting subsets $A_1, A_2, \ldots, A_n$ of G, $G = A \uplus A_1 \uplus A_2 \uplus \cdots \uplus A_n$, such that $|A_i| > 1$ for each $i = 1, 2, \ldots, n$.

Note that, if a group G has an n-abelian partition, then $|G| \geqslant 2n+1$ and the commuting graph $\mathcal{C}(G)$ is a $(0, n+1)$-partitionable graph. The following elementary result shows that there does not exist a 1-abelian partition of a group with respect to a commuting subset including the identity element.

Lemma 1.2. *If a nonabelian group has an n-abelian partition with respect to a commuting subset including the identity element, then $n > 1$.*

Proof. Suppose the contrary. Let $G = A \uplus A_1$ be a 1-abelian partition of G with respect to a commuting subset A including the identity element. Then either $|A| \geqslant |G|/2$ or $|A_1| \geqslant |G|/2$. In the first case, $\langle A \rangle = G$, and in the second case, $\langle A_1 \rangle = G$. Thus, in both cases G is abelian, a contradiction. □

Clearly, if G has a commuting subset A (including the identity element) for which every element outside of A has order larger than 2, then G has an n-abelian partition for some integer $n \geqslant 2$. Actually, in this case, we can pair every element outside of A with its inverse. Thus, for some integer $n \geqslant 2$, we have

$$G \setminus A = \{x_i, x_i^{-1} | i = 1, 2, \ldots, n\},$$

and so

$$G = A \uplus \{x_1, x_1^{-1}\} \uplus \{x_2, x_2^{-1}\} \uplus \cdots \uplus \{x_n, x_n^{-1}\},$$

would be an n-abelian partition of G. In particular, this shows that every group of *odd* order has an n-abelian partition with respect to $A = 1$, where $n = (|G| - 1)/2$.

Finally, our discussion in the previous paragraph and next lemma show that the problem of finding an abelian partition of a group reduces to the case of *centerless groups of even order*.

Lemma 1.3. *If G is a nonabelian group with $|Z(G)| > 1$, then G has an n-abelian partition with respect to $Z(G)$, where $n = |G : Z(G)| - 1$.*

Proof. Let $Z = Z(G)$, $n = |G : Z| - 1$ and $T = \{x_0, x_1, \ldots, x_n\}$ is a transversal for Z in G, where $x_0 \in Z$. Clearly, as the cosets of the centre are commuting subsets of G, we have the following n-abelian partition for G with respect to Z:

$$G = Z \uplus Zx_1 \uplus Zx_2 \uplus \cdots \uplus Zx_n,$$

as required. □

For example, the dihedral groups

$$D_{2k} = \langle a, b \mid a^k = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle,$$

where $k > 2$ is an even integer, have a $(k-1)$-abelian partition with respect to $Z(D_{2k}) = \{1, a^{k/2}\}$. Nevertheless, the dihedral groups D_{2k}, where $k > 1$ is an odd integer, have no such a partition.

Lemma 1.4. *For any integer $n \geqslant 2$, there exists a group G which has an n-abelian partition with respect to an abelian subgroup.*

Proof. It follows by considering the dihedral group D_{2k}, where $k = 2n$. Take $A = \langle a \rangle$ and $A_i = Za^ib$, $i = 1, 2, \ldots, n$, where $Z = Z(D_{2k}) = \{1, a^n\}$. □

We are particularly interested in groups that have an abelian partition with respect to an *abelian subgroup*. In fact, in all the examples we know of groups with abelian partitions, they have a partition with respect to an abelian subgroup. Therefore, we present the following conjecture for future work.

Conjecture 1.5. *If G is a group with an abelian partition (with respect to a commuting subset containing the identity element), then G has an abelian partition with respect to an abelian subgroup.*

In the next section, the structure of groups G which have an n-abelian partition with respect to an abelian subgroup, for $n = 2$ and 3, is obtained (Theorems 2.4 and 2.5).

All notation and terminology for groups are standard, however, we introduce some more notation. Following S. M. Belcastro and G. J. Sherman [1], we denote by $\#\mathrm{Cent}(G)$ the number of distinct centralizers in a group G. We shall say that a group G is n-centralizer if $\#\mathrm{Cent}(G) = n$. A *noncommuting set* of a group G (i.e., an independent set in commuting graph $\mathcal{C}(G)$) has the property that no two of its elements commute under the group operation. We denote by $\mathrm{nc}(G)$ the maximum cardinality of any noncommuting set of G (the independence number of $\mathcal{C}(G)$). Finally, the number of distinct conjugacy classes of G is denoted by $\mathrm{k}(G)$. We use $\mathbb{A}_n$ and $\mathbb{S}_n$ to denote an alternating and a symmetric group of degree n, respectively.

2. Main results

If G has an n-abelian partition, then the pigeon-hole principle gives $\mathrm{nc}(G) \leqslant n+1$. Thus, by Corollary 2.2 (a) in [2], we obtain

$$|G| \leqslant \mathrm{nc}(G) \cdot \mathrm{k}(G) \leqslant (n+1)\mathrm{k}(G),$$

which immediately implies that

$$n \geqslant \left\lfloor \frac{|G|}{\mathrm{k}(G)} \right\rfloor - 1. \tag{1}$$

Therefore, we have found a lower bound for n, when $\mathrm{k}(G)$ is known.

Example 2.1. *Let $G = L_2(q)$, where $q \geqslant 4$ is a power of 2. We know that $|G| = q(q^2-1)$ and $\mathrm{k}(L_2(q)) = q+1$. Thus, if G has an n-abelian partition, then by Eq. (1), we get $n \geqslant q^2 - q - 1$. In particular, since $\mathbb{A}_5 \cong L_2(4)$, if $\mathbb{A}_5$ has an n-abelian partition, then $n \geqslant 11$. In fact, $\mathbb{A}_5$ has a 20-abelian partition, as follows:*

$$\mathbb{A}_5 = A \uplus A_1^{\#} \uplus A_2^{\#} \uplus \cdots \uplus A_{20}^{\#},$$

where $A_i^{\#} = A_i \setminus \{1\}$, for every i, and

$A, A_1, \ldots, A_5$ *are Sylow* 5*-subgroups of order* 5*,*
$A_6, A_7, \ldots, A_{15}$ *are Sylow* 3*-subgroups of order* 3*,*
$A_{16}, A_{17}, \ldots, A_{20}$ *are Sylow* 2*-subgroups of order* 4*.*

Example 2.2. *Similarly, if $G_1 = \mathrm{GL}(2,q)$ and $G_2 = \mathrm{GL}(3,q)$, q a prime power, then we have*

$|G_1| = (q^2-1)(q^2-q)$ *and* $\mathrm{k}(G_1) = q^2-1$*, while*
$|G_2| = (q^3-1)(q^3-q)(q^3-q^2)$ *and* $\mathrm{k}(G_2) = q^3-q$*.*

Again, if G_i has an n_i-abelian partition, for $i = 1, 2$, by Eq. (1), we obtain $n_1 \geqslant q(q-1)-1$ and $n_2 \geqslant q^2(q^3-1)(q-1)-1$.

Lemma 2.3. [9, Lemma 4.1] *Let $\{g_1, \ldots, g_m\}$ be a largest noncommuting subset of G. Then $\cap_{i=1}^m C_G(g_i)$ is an abelian subgroup of G.*

Proof. Assume the contrary and choose $a, b \in \cap_{i=1}^m C_G(g_i)$ such that $ab \neq ba$. Then it is easy to see that $\{a, bg_1, bg_2, \ldots, bg_m\}$ is a noncommuting subset of G, a contradiction. □

Before stating our main results we introduce another notation. Given a finite group G, we denote by $\mathrm{cs}(G)$ the set of conjugacy class sizes of G. Itô proved that [7, Theorem 1] *if $\mathrm{cs}(G) = \{1, m\}$, then G is a direct product of a Sylow p-group of G with an abelian group.* In particular, then m is a *power of p*.

Theorem 2.4. *The following conditions on a nonabelian group G are equivalent:*

(1) *G has a 2-abelian partition with respect to an abelian subgroup A.*

(2) *$G = P \times Q$, where $P \in \mathrm{Syl}_2(G)$ with $P/Z(P) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and Q is abelian, and $A = \langle Z(G), t\rangle$, where $t \in G \setminus Z(G)$ and $t^2 \in Z(G)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that G is a nonabelian group, which has a 2-abelian partition $G = A \uplus A_1 \uplus A_2$. First of all, we notice that every noncommuting set of G can have at most three elements. Now fix a noncentral element x of G. Since $C_G(x) < G$, we can choose $y \in G$, such that x and y do not commute. It is well known that a group cannot be written as the union of two proper subgroups, thus $C_G(x) \cup C_G(y) < G$, and so we can choose z in G, such that $B = \{x, y, z\}$ is a noncommuting set of G. Now, we have

$$G = C_x \cup C_y \cup C_z,$$

where $C_x = C_G(x)$, $C_y = C_G(y)$ and $C_z = C_G(z)$. Put $K = C_x \cap C_y \cap C_z$, which is an abelian subgroup of G, by Lemma 2.3. Indeed, by a result of Scorza [10], we have

(a) $[G : C_x] = [G : C_y] = [G : C_z] = 2$,

(b) $K = C_x \cap C_y = C_x \cap C_z = C_y \cap C_z$, and

(c) K is a normal subgroup of G and the factor group G/K is isomorphic to the Klein Four Group.

Thus $|x^G| = 2$, and since $x \in G \setminus Z(G)$ was arbitrary, it follows that $\mathrm{cs}(G) = \{1, 2\}$. By Itô's result [7, Theorem 1], $G = P \times Q$, where $P \in \mathrm{Syl}_2(G)$ is nonabelian and $Q \leqslant Z(G)$.

On the other hand, B is a maximal noncommuting set of G, which forces $C_t \setminus K$ to be a commuting set of G for each $t \in B$, and so the centralizer C_t is abelian, because $C_t = \langle C_t \setminus K\rangle$. This implies that $K = Z(G)$, and so

$$\frac{P}{Z(P)} \cong \frac{P \times Q}{Z(P) \times Q} = \frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

and the proof is complete.

(2) $\Rightarrow$ (1) Let $\{t_1, t_2, t_3, t_4\}$ be a transversal for $Z(G)$ in G, with $t_1 \in Z(G)$. Then, G is a disjoint union:

$$G = Z(G) \cup Z(G)t_2 \cup Z(G)t_3 \cup Z(G)t_4.$$

Put $A = Z(G) \cup Z(G)t_2$, $A_1 = Z(G)t_3$ and $A_2 = Z(G)t_4$. Then A is an abelian group (since $t_2^2 \in Z(G)$), A_1 and A_2 are commuting sets, and $G = A \uplus A_1 \uplus A_2$ is a 2-abelian partition of G. □

We now work to determine which groups have a 3-abelian partition with respect to an abelian subgroup.

Theorem 2.5. *The following conditions on a nonabelian group G are equivalent:*

(1) *G has a 3-abelian partition with respect to an abelian subgroup A.*

(2) *$|Z(G)| \geqslant 2$ and $G/Z(G)$ is isomorphic to one of the following groups:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{S}_3.$$

In the first case, $A = Z(G)$, and in two other cases $A = \langle Z(G), x\rangle$, where x is an element of order 3 outside of $Z(G)$.

Proof. (1) $\Rightarrow$ (2) Suppose that G is a nonabelian group, which has a 3-abelian partition $G = A \uplus A_1 \uplus A_2 \uplus A_3$. First of all, we notice that $\mathrm{nc}(G) = 3$ or 4. It is now easy to see that G is either 3-centralizer or 4-centralizer, respectively. Therefore, by Theorems 2 and 4 in [1], we conclude that G modulo its center is isomorphic to one of the groups: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{S}_3$, as required. Finally, since G is a nonabelian group with at least 7 elements, $|Z(G)| \geqslant 2$.

(2) $\Rightarrow$ (1) Let $Z = Z(G)$. We treat separately the different cases:

(a) $G/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, there are noncentral elements x_1, x_2, and x_3 of G such that $G = Z \uplus Zx_1 \uplus Zx_2 \uplus Zx_3$, which is a 3-abelian partition of G.

(b) $G/Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. In this case, we have

$$G/Z \cong \langle Zx, Zy \mid x^3, y^3, [x,y] \in Z\rangle,$$

which implies that

$$G = Z \cup Zx \cup Zx^2 \cup Zy \cup Zy^2 \cup Zxy \cup Zx^2y^2 \cup Zxy^2 \cup Zx^2y.$$

We put

$A := Z \cup Zx \cup Zx^2 = \langle Z, x\rangle,$

$A_1 := Zy \cup Zy^2 = \langle Z, y\rangle \setminus Z,$

$A_2 := Zxy \cup Zx^2y^2 = \langle Z, xy\rangle \setminus Z,$

$A_3 := Zxy^2 \cup Zx^2y = \langle Z, xy^2\rangle \setminus Z.$

Then $G = A \uplus A_1 \uplus A_2 \uplus A_3$ is a 3-abelian partition of G.

(c) $G/Z \cong \mathbb{S}_3$. In this case, we have $G/Z \cong \langle Zx, Zy \mid x^3, y^2, (xy)^2 \in Z\rangle$, which implies that

$$G = Z \cup Zx \cup Zx^2 \cup Zy \cup Zyx \cup Zyx^2.$$

Put $A := Z \cup Zx \cup Zx^2 = \langle Z, x\rangle$. Then, $G = A \uplus Zy \uplus Zyx \uplus Zyx^2$ is a 3-abelian partition of G.

The proof is complete. □

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